An Introduction To Linear Algebra

Kenneth Kuttler

July 6, 2010

Contents

1 Preliminaries **9**
1.1 The Number Line And Algebra Of The Real Numbers . . . 9
1.2 Ordered fields . . . 10
1.3 The Complex Numbers . . . 12
1.4 Exercises . . . 15
1.5 Completeness of $\mathbb{R}$ . . . 16
1.6 Well Ordering And Archimedian Property . . . 17
1.7 Division And Numbers . . . 19
1.8 Systems Of Equations . . . 22
1.9 Exercises . . . 27
1.10 $\mathbb{F}^n$ . . . 27
1.11 Algebra in $\mathbb{F}^n$ . . . 27
1.12 Exercises . . . 28
1.13 The Inner Product In $\mathbb{F}^n$ . . . 29
1.14 Exercises . . . 31

2 Matrices And Linear Transformations **33**
2.1 Matrices . . . 33
2.1.1 The ij^{th} Entry Of A Product . . . 38
2.1.2 A Cute Application . . . 40
2.1.3 Properties Of Matrix Multiplication . . . 42
2.1.4 Finding The Inverse Of A Matrix . . . 45
2.2 Exercises . . . 49
2.3 Linear Transformations . . . 51
2.4 Subspaces And Spans . . . 53
2.5 An Application To Matrices . . . 57
2.6 Matrices And Calculus . . . 58
2.6.1 The Coriolis Acceleration . . . 59
2.6.2 The Coriolis Acceleration On The Rotating Earth . . . 63
2.7 Exercises . . . 68

3 Determinants **73**
3.1 Basic Techniques And Properties . . . 73
3.2 Exercises . . . 80
3.3 The Mathematical Theory Of Determinants . . . 82
3.3.1 The Function sgn . . . 82
3.3.2 The Definition Of The Determinant . . . 84
3.3.3 A Symmetric Definition . . . 85
3.3.4 Basic Properties Of The Determinant . . . 87

3.3.5 Expansion Using Cofactors . . . 88
3.3.6 A Formula For The Inverse . . . 90
3.3.7 Rank Of A Matrix . . . 92
3.3.8 Summary Of Determinants . . . 94
3.4 The Cayley Hamilton Theorem . . . 95
3.5 Block Multiplication Of Matrices . . . 96
3.6 Exercises . . . 100

4 Row Operations 105
4.1 Elementary Matrices . . . 105
4.2 The Rank Of A Matrix . . . 111
4.3 The Row Reduced Echelon Form . . . 113
4.4 Rank And Existence Of Solutions To Linear Systems . . . 116
4.5 Fredholm Alternative . . . 117
4.6 Exercises . . . 119

5 Some Factorizations 123
5.1 LU Factorization . . . 123
5.2 Finding An LU Factorization . . . 123
5.3 Solving Linear Systems Using An LU Factorization . . . 125
5.4 The PLU Factorization . . . 126
5.5 Justification For The Multiplier Method . . . 127
5.6 Existence For The PLU Factorization . . . 129
5.7 The QR Factorization . . . 130
5.8 Exercises . . . 133

6 Linear Programming 137
6.1 Simple Geometric Considerations . . . 137
6.2 The Simplex Tableau . . . 138
6.3 The Simplex Algorithm . . . 142
6.3.1 Maximums . . . 142
6.3.2 Minimums . . . 144
6.4 Finding A Basic Feasible Solution . . . 151
6.5 Duality . . . 152
6.6 Exercises . . . 156

7 Spectral Theory 159
7.1 Eigenvalues And Eigenvectors Of A Matrix . . . 159
7.2 Some Applications Of Eigenvalues And Eigenvectors . . . 167
7.3 Exercises . . . 169
7.4 Shur's Theorem . . . 176
7.5 Trace And Determinant . . . 184
7.6 Quadratic Forms . . . 185
7.7 Second Derivative Test . . . 186
7.8 The Estimation Of Eigenvalues . . . 190
7.9 Advanced Theorems . . . 192
7.10 Exercises . . . 195

8 Vector Spaces And Fields **203**
8.1 Vector Space Axioms . . . 203
8.2 Subspaces And Bases . . . 204
8.2.1 Basic Definitions . . . 204
8.2.2 A Fundamental Theorem . . . 204
8.2.3 The Basis Of A Subspace . . . 207
8.3 Lots Of Fields . . . 208
8.3.1 Irreducible Polynomials . . . 208
8.3.2 Polynomials And Fields . . . 212
8.3.3 The Algebraic Numbers . . . 217
8.3.4 The Lindemann Weierstrass Theorem And Vector Spaces . . . 220
8.4 Exercises . . . 220

9 Linear Transformations **227**
9.1 Matrix Multiplication As A Linear Transformation . . . 227
9.2 $\mathcal{L}(V,W)$ As A Vector Space . . . 227
9.3 The Matrix Of A Linear Transformation . . . 229
9.3.1 Some Geometrically Defined Linear Transformations . . . 236
9.3.2 Rotations About A Given Vector . . . 239
9.3.3 The Euler Angles . . . 241
9.4 Eigenvalues And Eigenvectors Of Linear Transformations . . . 242
9.5 Exercises . . . 244

10 Linear Transformations Canonical Forms **249**
10.1 A Theorem Of Sylvester, Direct Sums . . . 249
10.2 Direct Sums, Block Diagonal Matrices . . . 252
10.3 The Jordan Canonical Form . . . 255
10.4 Exercises . . . 263
10.5 The Rational Canonical Form . . . 267
10.6 Uniqueness . . . 274
10.7 Exercises . . . 280

11 Markov Chains And Migration Processes **283**
11.1 Regular Markov Matrices . . . 283
11.2 Migration Matrices . . . 287
11.3 Markov Chains . . . 287
11.4 Exercises . . . 292

12 Inner Product Spaces **295**
12.1 General Theory . . . 295
12.2 The Gramm Schmidt Process . . . 297
12.3 Riesz Representation Theorem . . . 300
12.4 The Tensor Product Of Two Vectors . . . 303
12.5 Least Squares . . . 305
12.6 Fredholm Alternative Again . . . 306
12.7 Exercises . . . 306
12.8 The Determinant And Volume . . . 311
12.9 Exercises . . . 314

13 Self Adjoint Operators **315**
13.1 Simultaneous Diagonalization . . . 315
13.2 Schur's Theorem . . . 318
13.3 Spectral Theory Of Self Adjoint Operators . . . 320
13.4 Positive And Negative Linear Transformations . . . 325
13.5 Fractional Powers . . . 327
13.6 Polar Decompositions . . . 330
13.7 An Application To Statistics . . . 333
13.8 The Singular Value Decomposition . . . 335
13.9 Approximation In The Frobenius Norm . . . 337
13.10Least Squares And Singular Value Decomposition . . . 339
13.11The Moore Penrose Inverse . . . 339
13.12Exercises . . . 343

14 Norms For Finite Dimensional Vector Spaces **347**
14.1 The p Norms . . . 353
14.2 The Condition Number . . . 356
14.3 The Spectral Radius . . . 358
14.4 Series And Sequences Of Linear Operators . . . 361
14.5 Iterative Methods For Linear Systems . . . 365
14.6 Theory Of Convergence . . . 371
14.7 Exercises . . . 374

15 Numerical Methods For Finding Eigenvalues **383**
15.1 The Power Method For Eigenvalues . . . 383
15.1.1 The Shifted Inverse Power Method . . . 388
15.1.2 The Explicit Description Of The Method . . . 388
15.1.3 Complex Eigenvalues . . . 395
15.1.4 Rayleigh Quotients And Estimates for Eigenvalues . . . 397
15.2 The QR Algorithm . . . 401
15.2.1 Basic Properties And Definition . . . 401
15.2.2 The Case Of Real Eigenvalues . . . 404
15.2.3 The QR Algorithm In The General Case . . . 409
15.3 Exercises . . . 416

A Some Interesting Topics **419**
A.1 Positive Matrices . . . 419
A.2 Functions Of Matrices . . . 426

B Applications To Differential Equations **433**
B.1 Theory Of Ordinary Differntial Equations . . . 433
B.2 Linear Systems . . . 434
B.3 Local Solutions . . . 435
B.4 First Order Linear Systems . . . 437
B.5 Geometric Theory Of Autonomous Systems . . . 445
B.6 General Geometric Theory . . . 449
B.7 The Stable Manifold . . . 451

C The Fundamental Theorem Of Algebra **457**

D Polynomials **459**
D.1 Symmetric Polynomials In Many Variables 459
D.2 The Fundamental Theorem Of Algebra 464
D.3 Transcendental Numbers 467
Copyright © 2004,

Preliminaries

1.1 The Number Line And Algebra Of The Real Numbers

To begin with, consider the real numbers, denoted by $\mathbb{R}$, as a line extending infinitely far in both directions. In this book, the notation, $\equiv$ indicates something is being defined. Thus the integers are defined as

$$\mathbb{Z} \equiv \{\cdots -1, 0, 1, \cdots\},$$

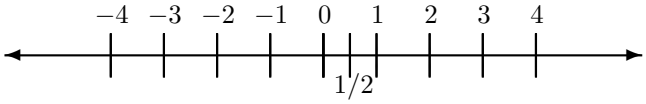
the natural numbers,

$$\mathbb{N} \equiv \{1, 2, \cdots\}$$

and the rational numbers, defined as the numbers which are the quotient of two integers.

$$\mathbb{Q} \equiv \left\{ \frac{m}{n} \text{ such that } m, n \in \mathbb{Z}, n \neq 0 \right\}$$

are each subsets of $\mathbb{R}$ as indicated in the following picture.



As shown in the picture, $\frac{1}{2}$ is half way between the number 0 and the number, 1. By analogy, you can see where to place all the other rational numbers. It is assumed that $\mathbb{R}$ has the following algebra properties, listed here as a collection of assertions called axioms. These properties will not be proved which is why they are called axioms rather than theorems. In general, axioms are statements which are regarded as true. Often these are things which are "self evident" either from experience or from some sort of intuition but this does not have to be the case.

Axiom 1.1.1 $x + y = y + x$, *(commutative law for addition)*

Axiom 1.1.2 $x + 0 = x$, *(additive identity).*

Axiom 1.1.3 *For each* $x \in \mathbb{R}$, *there exists* $-x \in \mathbb{R}$ *such that* $x + (-x) = 0$, *(existence of additive inverse).*

Axiom 1.1.4 $(x + y) + z = x + (y + z)$, *(associative law for addition).*

Axiom 1.1.5 $xy = yx$, (*commutative law for multiplication).*

Axiom 1.1.6 $(xy)\, z = x\,(yz)$, (*associative law for multiplication).*

Axiom 1.1.7 $1x = x$, (*multiplicative identity).*

Axiom 1.1.8 *For each* $x \neq 0$, *there exists* x^{-1} *such that* $xx^{-1} = 1$.(*existence of multiplicative inverse).*

Axiom 1.1.9 $x\,(y+z) = xy + xz$.(*distributive law).*

These axioms are known as the field axioms and any set (there are many others besides $\mathbb{R}$) which has two such operations satisfying the above axioms is called a field. Division and subtraction are defined in the usual way by $x - y \equiv x + (-y)$ and $x/y \equiv x\left(y^{-1}\right)$.

Here is a little proposition which derives some familiar facts.

Proposition 1.1.10 0 *and* 1 *are unique. Also* $-x$ *is unique and* x^{-1} *is unique. Furthermore,* $0x = x0 = 0$ *and* $-x = (-1)\, x$.

Proof: Suppose $0'$ is another additive identity. Then

$$0' = 0' + 0 = 0.$$

Thus 0 is unique. Say $1'$ is another multiplicative identity. Then

$$1 = 1'1 = 1'.$$

Now suppose y acts like the additive inverse of x. Then

$$-x = (-x) + 0 = (-x) + (x + y) = (-x + x) + y = y$$

Finally,

$$0x = (0 + 0)\, x = 0x + 0x$$

and so

$$0 = -(0x) + 0x = -(0x) + (0x + 0x) = (-(0x) + 0x) + 0x = 0x$$

Finally

$$x + (-1)\, x = (1 + (-1))\, x = 0x = 0$$

and so by uniqueness of the additive inverse, $(-1)\, x = -x$. This proves the proposition.

1.2 Ordered fields

The real numbers $\mathbb{R}$ are an example of an ordered field. More generally, here is a definition.

Definition 1.2.1 *Let* F *be a field. It is an ordered field if there exists an order,* $<$ *which satisfies*

1. *For any* $x \neq y$, *either* $x < y$ *or* $y < x$.
2. *If* $x < y$ *and either* $z < w$ *or* $z = w$, *then,* $x + z < y + w$.
3. *If* $0 < x, 0 < y$, *then* $xy > 0$.

With this definition, the familiar properties of order can be proved. The following proposition lists many of these familiar properties. The relation '$a > b$' has the same meaning as '$b < a$'.

Proposition 1.2.2 *The following are obtained.*

1. *If* $x < y$ *and* $y < z$, *then* $x < z$.
2. *If* $x > 0$ *and* $y > 0$, *then* $x + y > 0$.
3. *If* $x > 0$, *then* $-x < 0$.
4. *If* $x \neq 0$, *either* x *or* $-x$ *is* > 0.
5. *If* $x < y$, *then* $-x > -y$.
6. *If* $x \neq 0$, *then* $x^2 > 0$.
7. *If* $0 < x < y$ *then* $x^{-1} > y^{-1}$.

Proof: First consider 1, called the transitive law. Suppose that $x < y$ and $y < z$. Then from the axioms, $x + y < y + z$ and so, adding $-y$ to both sides, it follows

$$x < z$$

Next consider 2. Suppose $x > 0$ and $y > 0$. Then from 2,

$$0 = 0 + 0 < x + y.$$

Next consider 3. It is assumed $x > 0$ so

$$0 = -x + x > 0 + (-x) = -x$$

Now consider 4. If $x < 0$, then

$$0 = x + (-x) < 0 + (-x) = -x.$$

Consider the 5. Since $x < y$, it follows from 2

$$0 = x + (-x) < y + (-x)$$

and so by 4 and Proposition 1.1.10,

$$(-1)(y + (-x)) < 0$$

Also from Proposition 1.1.10 $(-1)(-x) = -(-x) = x$ and so

$$-y + x < 0.$$

Hence

$$-y < -x.$$

Consider 6. If $x > 0$, there is nothing to show. It follows from the definition. If $x < 0$, then by 4, $-x > 0$ and so by Proposition 1.1.10 and the definition of the order,

$$(-x)^2 = (-1)(-1)x^2 > 0$$

By this proposition again, $(-1)(-1) = -(-1) = 1$ and so $x^2 > 0$ as claimed. Note that $1 > 0$ because it equals 1^2.

Finally, consider 7. First, if $x > 0$ then if $x^{-1} < 0$, it would follow $(-1)x^{-1} > 0$ and so $x(-1)x^{-1} = (-1)1 = -1 > 0$. However, this would require

$$0 > 1 = 1^2 > 0$$

from what was just shown. Therefore, $x^{-1} > 0$. Now the assumption implies $y + (-1)x > 0$ and so multiplying by x^{-1},

$$yx^{-1} + (-1)xx^{-1} = yx^{-1} + (-1) > 0$$

Now multiply by y^{-1}, which by the above satisfies $y^{-1} > 0$, to obtain

$$x^{-1} + (-1)y^{-1} > 0$$

and so

$$x^{-1} > y^{-1}.$$

This proves the proposition. □

In an ordered field the symbols $\leq$ and $\geq$ have the usual meanings. Thus $a \leq b$ means $a < b$ or else $a = b$, etc.

1.3 The Complex Numbers

Just as a real number should be considered as a point on the line, a complex number is considered a point in the plane which can be identified in the usual way using the Cartesian coordinates of the point. Thus (a, b) identifies a point whose x coordinate is a and whose y coordinate is b. In dealing with complex numbers, such a point is written as $a + ib$ and multiplication and addition are defined in the most obvious way subject to the convention that $i^2 = -1$. Thus,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and

$$\begin{aligned}(a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(bc + ad).\end{aligned}$$

Every non zero complex number, $a+ib$, with $a^2+b^2 \neq 0$, has a unique multiplicative inverse.

$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

You should prove the following theorem.

Theorem 1.3.1 *The complex numbers with multiplication and addition defined as above form a field satisfying all the field axioms listed on Page 9.*

The field of complex numbers is denoted as $\mathbb{C}$. An important construction regarding complex numbers is the complex conjugate denoted by a horizontal line above the number. It is defined as follows.

$$\overline{a + ib} \equiv a - ib.$$

What it does is reflect a given complex number across the x axis. Algebraically, the following formula is easy to obtain.

$$\left(\overline{a + ib}\right)(a + ib) = a^2 + b^2.$$

Definition 1.3.2 *Define the absolute value of a complex number as follows.*

$$|a+ib| \equiv \sqrt{a^2+b^2}.$$

Thus, denoting by z the complex number, $z = a+ib$,

$$|z| = \left(z\overline{z}\right)^{1/2}.$$

With this definition, it is important to note the following. Be sure to verify this. It is not too hard but you need to do it.

Remark 1.3.3 *: Let $z = a+ib$ and $w = c+id$. Then $|z-w| = \sqrt{(a-c)^2+(b-d)^2}$. Thus the distance between the point in the plane determined by the ordered pair, (a,b) and the ordered pair (c,d) equals $|z-w|$ where z and w are as just described.*

For example, consider the distance between $(2,5)$ and $(1,8)$. From the distance formula this distance equals $\sqrt{(2-1)^2+(5-8)^2} = \sqrt{10}$. On the other hand, letting $z = 2+i5$ and $w = 1+i8$, $z-w = 1-i3$ and so $(z-w)\left(\overline{z-w}\right) = (1-i3)(1+i3) = 10$ so $|z-w| = \sqrt{10}$, the same thing obtained with the distance formula.

Complex numbers, are often written in the so called polar form which is described next. Suppose $x+iy$ is a complex number. Then

$$x+iy = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right).$$

Now note that

$$\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1$$

and so

$$\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

is a point on the unit circle. Therefore, there exists a unique angle, $\theta \in [0, 2\pi)$ such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

The polar form of the complex number is then

$$r\left(\cos\theta + i\sin\theta\right)$$

where θ is this angle just described and $r = \sqrt{x^2+y^2}$.

A fundamental identity is the formula of De Moivre which follows.

Theorem 1.3.4 *Let $r > 0$ be given. Then if n is a positive integer,*

$$\left[r\left(\cos t + i\sin t\right)\right]^n = r^n\left(\cos nt + i\sin nt\right).$$

Proof: It is clear the formula holds if $n = 1$. Suppose it is true for n.

$$\left[r\left(\cos t + i\sin t\right)\right]^{n+1} = \left[r\left(\cos t + i\sin t\right)\right]^n\left[r\left(\cos t + i\sin t\right)\right]$$

which by induction equals

$$= r^{n+1}\left(\cos nt + i\sin nt\right)\left(\cos t + i\sin t\right)$$

$$= r^{n+1}\left(\left(\cos nt\cos t - \sin nt\sin t\right) + i\left(\sin nt\cos t + \cos nt\sin t\right)\right)$$

$$= r^{n+1}\left(\cos\left(n+1\right)t + i\sin\left(n+1\right)t\right)$$

by the formulas for the cosine and sine of the sum of two angles.

Corollary 1.3.5 *Let z be a non zero complex number. Then there are always exactly k k^{th} roots of z in $\mathbb{C}$.*

Proof: Let $z = x + iy$ and let $z = |z|\left(\cos t + i\sin t\right)$ be the polar form of the complex number. By De Moivre's theorem, a complex number,

$$r\left(\cos\alpha + i\sin\alpha\right),$$

is a k^{th} root of z if and only if

$$r^k\left(\cos k\alpha + i\sin k\alpha\right) = |z|\left(\cos t + i\sin t\right).$$

This requires $r^k = |z|$ and so $r = |z|^{1/k}$ and also both $\cos\left(k\alpha\right) = \cos t$ and $\sin\left(k\alpha\right) = \sin t$. This can only happen if

$$k\alpha = t + 2l\pi$$

for l an integer. Thus

$$\alpha = \frac{t+2l\pi}{k}, l \in \mathbb{Z}$$

and so the k^{th} roots of z are of the form

$$|z|^{1/k}\left(\cos\left(\frac{t+2l\pi}{k}\right) + i\sin\left(\frac{t+2l\pi}{k}\right)\right),\ l \in \mathbb{Z}.$$

Since the cosine and sine are periodic of period 2π, there are exactly k distinct numbers which result from this formula.

Example 1.3.6 *Find the three cube roots of i.*

First note that $i = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$. Using the formula in the proof of the above corollary, the cube roots of i are

$$1\left(\cos\left(\frac{\left(\pi/2\right)+2l\pi}{3}\right) + i\sin\left(\frac{\left(\pi/2\right)+2l\pi}{3}\right)\right)$$

where $l = 0, 1, 2$. Therefore, the roots are

$$\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right), \cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right),$$

and

$$\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right).$$

Thus the cube roots of i are $\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)$, $\frac{-\sqrt{3}}{2} + i\left(\frac{1}{2}\right)$, and $-i$.

The ability to find k^{th} roots can also be used to factor some polynomials.

Example 1.3.7 *Factor the polynomial* $x^3 - 27$.

First find the cube roots of 27. By the above proceedure using De Moivre's theorem, these cube roots are $3, 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$. Therefore, $x^3 + 27 =$

$$(x-3)\left(x - 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - 3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right).$$

Note also $\left(x - 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - 3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right) = x^2 + 3x + 9$ and so

$$x^3 - 27 = (x-3)\left(x^2 + 3x + 9\right)$$

where the quadratic polynomial, $x^2 + 3x + 9$ cannot be factored without using complex numbers.

The real and complex numbers both are fields satisfying the axioms on Page 9 and it is usually one of these two fields which is used in linear algebra. The numbers are often called scalars. However, it turns out that all algebraic notions work for any field and there are many others. For this reason, I will often refer to the field of scalars as $\mathbb{F}$ although $\mathbb{F}$ will usually be either the real or complex numbers. If there is any doubt, assume it is the field of complex numbers which is meant.

1.4 Exercises

1. Let $z = 5 + i9$. Find z^{-1}.
2. Let $z = 2 + i7$ and let $w = 3 - i8$. Find $zw, z + w, z^2$, and w/z.
3. Give the complete solution to $x^4 + 16 = 0$.
4. Graph the complex cube roots of 8 in the complex plane. Do the same for the four fourth roots of 16.
5. If z is a complex number, show there exists ω a complex number with $|\omega| = 1$ and $\omega z = |z|$.
6. De Moivre's theorem says $[r(\cos t + i\sin t)]^n = r^n(\cos nt + i\sin nt)$ for n a positive integer. Does this formula continue to hold for all integers, n, even negative integers? Explain.
7. You already know formulas for $\cos(x+y)$ and $\sin(x+y)$ and these were used to prove De Moivre's theorem. Now using De Moivre's theorem, derive a formula for $\sin(5x)$ and one for $\cos(5x)$. **Hint:** Use the binomial theorem.
8. If z and w are two complex numbers and the polar form of z involves the angle θ while the polar form of w involves the angle ϕ, show that in the polar form for zw the angle involved is $\theta + \phi$. Also, show that in the polar form of a complex number, z, $r = |z|$.
9. Factor $x^3 + 8$ as a product of linear factors.
10. Write $x^3 + 27$ in the form $(x+3)\left(x^2 + ax + b\right)$ where $x^2 + ax + b$ cannot be factored any more using only real numbers.
11. Completely factor $x^4 + 16$ as a product of linear factors.

12. Factor $x^4 + 16$ as the product of two quadratic polynomials each of which cannot be factored further without using complex numbers.

13. If z, w are complex numbers prove $\overline{zw} = \overline{z}\,\overline{w}$ and then show by induction that $\overline{z_1 \cdots z_m} = \overline{z_1} \cdots \overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.

14. Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where all the a_k are real numbers. Suppose also that $p(z) = 0$ for some $z \in \mathbb{C}$. Show it follows that $p(\overline{z}) = 0$ also.

15. I claim that $1 = -1$. Here is why.

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?

16. De Moivre's theorem is really a grand thing. I plan to use it now for rational exponents, not just integers.

$$1 = 1^{(1/4)} = (\cos 2\pi + i \sin 2\pi)^{1/4} = \cos(\pi/2) + i \sin(\pi/2) = i.$$

Therefore, squaring both sides it follows $1 = -1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?

17. Show that $\mathbb{C}$ cannot be considered an ordered field. **Hint:** Consider $i^2 = -1$. Recall that $1 > 0$ by Proposition 1.2.2.

18. Say $a + ib < x + iy$ if $a < x$ or if $a = x$, then $b < y$. This is called the lexicographic order. Show that any two different complex numbers can be compared with this order. What goes wrong in terms of the other requirements for an ordered field.

19. With the order of Problem 18, consider for $n \in \mathbb{N}$ the complex number $1 - \frac{1}{n}$. Show that with the lexicographic order just described, each of $1 - in$ is an upper bound to all these numbers. Therefore, this is a set which is "bounded above" but has no least upper bound with respect to the lexicographic order on $\mathbb{C}$.

1.5 Completeness of $\mathbb{R}$

Recall the following important definition from calculus, completeness of **R**.

Definition 1.5.1 *A non empty set, $S \subseteq \mathbb{R}$ is bounded above (below) if there exists $x \in \mathbb{R}$ such that $x \geq (\leq)\, s$ for all $s \in S$. If S is a nonempty set in $\mathbb{R}$ which is bounded above, then a number, l which has the property that l is an upper bound and that every other upper bound is no smaller than l is called a least upper bound, l.u.b.(S) or often* $\sup(S)$*. If S is a nonempty set bounded below, define the greatest lower bound, g.l.b.(S) or* $\inf(S)$ *similarly. Thus g is the g.l.b.(S) means g is a lower bound for S and it is the largest of all lower bounds. If S is a nonempty subset of $\mathbb{R}$ which is not bounded above, this information is expressed by saying* $\sup(S) = +\infty$ *and if S is not bounded below,* $\inf(S) = -\infty$.

Every existence theorem in calculus depends on some form of the completeness axiom.

Axiom 1.5.2 *(completeness) Every nonempty set of real numbers which is bounded above has a least upper bound and every nonempty set of real numbers which is bounded below has a greatest lower bound.*

It is this axiom which distinguishes Calculus from Algebra. A fundamental result about sup and inf is the following.

Proposition 1.5.3 *Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

Proof: Consider the first claim. If the indicated set equals $\emptyset$, then $\sup(S) - \delta$ is an upper bound for S which is smaller than $\sup(S)$, contrary to the definition of $\sup(S)$ as the least upper bound. In the second claim, if the indicated set equals $\emptyset$, then $\inf(S) + \delta$ would be a lower bound which is larger than $\inf(S)$ contrary to the definition of $\inf(S)$.

1.6 Well Ordering And Archimedian Property

Definition 1.6.1 *A set is well ordered if every nonempty subset S, contains a smallest element z having the property that $z \leq x$ for all $x \in S$.*

Axiom 1.6.2 *Any set of integers larger than a given number is well ordered.*

In particular, the natural numbers defined as

$$\mathbb{N} \equiv \{1, 2, \cdots\}$$

is well ordered.

The above axiom implies the principle of mathematical induction.

Theorem 1.6.3 *(Mathematical induction) A set $S \subseteq \mathbb{Z}$, having the property that $a \in S$ and $n + 1 \in S$ whenever $n \in S$ contains all integers $x \in \mathbb{Z}$ such that $x \geq a$.*

Proof: Let $T \equiv ([a, \infty) \cap \mathbb{Z}) \setminus S$. Thus T consists of all integers larger than or equal to a which are not in S. The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T, denoted as b. It must be the case that $b > a$ since by definition, $a \notin T$. Then the integer, $b - 1 \geq a$ and $b - 1 \notin S$ because if $b - 1 \in S$, then $b - 1 + 1 = b \in S$ by the assumed property of S. Therefore, $b - 1 \in ([a, \infty) \cap \mathbb{Z}) \setminus S = T$ which contradicts the choice of b as the smallest element of T. ($b - 1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that everything in $[a, \infty) \cap \mathbb{Z}$ is also in S.

Example 1.6.4 *Show that for all $n \in \mathbb{N}$, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.*

If $n = 1$ this reduces to the statement that $\frac{1}{2} < \frac{1}{\sqrt{3}}$ which is obviously true. Suppose then that the inequality holds for n. Then

$$\begin{aligned} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} &< \frac{1}{\sqrt{2n+1}} \frac{2n+1}{2n+2} \\ &= \frac{\sqrt{2n+1}}{2n+2}. \end{aligned}$$

The theorem will be proved if this last expression is less than $\frac{1}{\sqrt{2n+3}}$. This happens if and only if

$$\left(\frac{1}{\sqrt{2n+3}}\right)^2 = \frac{1}{2n+3} > \frac{2n+1}{(2n+2)^2}$$

which occurs if and only if $(2n+2)^2 > (2n+3)(2n+1)$ and this is clearly true which may be seen from expanding both sides. This proves the inequality.

Definition 1.6.5 *The Archimedian property states that whenever $x \in \mathbb{R}$, and $a > 0$, there exists $n \in \mathbb{N}$ such that $na > x$.*

Proposition 1.6.6 $\mathbb{R}$ *has the Archimedian property.*

Proof: Suppose it is not true. Then there exists $x \in \mathbb{R}$ and $a > 0$ such that $na \leq x$ for all $n \in \mathbb{N}$. Let $S = \{na : n \in \mathbb{N}\}$. By assumption, this is bounded above by x. By completeness, it has a least upper bound y. By Proposition 1.5.3 there exists $n \in \mathbb{N}$ such that

$$y - a < na \leq y.$$

Then $y = y - a + a < na + a = (n+1)a \leq y$, a contradiction. This proves the proposition.

Theorem 1.6.7 *Suppose $x < y$ and $y - x > 1$. Then there exists an integer, $l \in \mathbb{Z}$, such that $x < l < y$. If x is an integer, there is no integer y satisfying $x < y < x + 1$.*

Proof: Let x be the smallest positive integer. Not surprisingly, $x = 1$ but this can be proved. If $x < 1$ then $x^2 < x$ contradicting the assertion that x is the smallest natural number. Therefore, 1 is the smallest natural number. This shows there is no integer, y, satisfying $x < y < x + 1$ since otherwise, you could subtract x and conclude $0 < y - x < 1$ for some integer $y - x$.

Now suppose $y - x > 1$ and let

$$S \equiv \{w \in \mathbb{N} : w \geq y\}.$$

The set S is nonempty by the Archimedian property. Let k be the smallest element of S. Therefore, $k - 1 < y$. Either $k - 1 \leq x$ or $k - 1 > x$. If $k - 1 \leq x$, then

$$y - x \leq y - (k-1) = \overbrace{y - k}^{\leq 0} + 1 \leq 1$$

contrary to the assumption that $y - x > 1$. Therefore, $x < k - 1 < y$ and this proves the theorem with $l = k - 1$.

It is the next theorem which gives the density of the rational numbers. This means that for any real number, there exists a rational number arbitrarily close to it.

Theorem 1.6.8 *If $x < y$ then there exists a rational number r such that $x < r < y$.*

Proof: Let $n \in \mathbb{N}$ be large enough that

$$n(y - x) > 1.$$

Thus $(y - x)$ added to itself n times is larger than 1. Therefore,

$$n(y - x) = ny + n(-x) = ny - nx > 1.$$

It follows from Theorem 1.6.7 there exists $m \in \mathbb{Z}$ such that

$$nx < m < ny$$

and so take $r = m/n$.

Definition 1.6.9 *A set, $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if whenever $a < b$, $S \cap (a,b) \neq \emptyset$.*

Thus the above theorem says $\mathbb{Q}$ is "dense" in $\mathbb{R}$.

Theorem 1.6.10 *Suppose $0 < a$ and let $b \geq 0$. Then there exists a unique integer p and real number r such that $0 \leq r < a$ and $b = pa + r$.*

Proof: Let $S \equiv \{n \in \mathbb{N} : an > b\}$. By the Archimedian property this set is nonempty. Let $p+1$ be the smallest element of S. Then $pa \leq b$ because $p+1$ is the smallest in S. Therefore,

$$r \equiv b - pa \geq 0.$$

If $r \geq a$ then $b - pa \geq a$ and so $b \geq (p+1)\, a$ contradicting $p+1 \in S$. Therefore, $r < a$ as desired.

To verify uniqueness of p and r, suppose p_i and r_i, $i = 1, 2$, both work and $r_2 > r_1$. Then a little algebra shows

$$p_1 - p_2 = \frac{r_2 - r_1}{a} \in (0,1)\,.$$

Thus $p_1 - p_2$ is an integer between 0 and 1, contradicting Theorem 1.6.7. The case that $r_1 > r_2$ cannot occur either by similar reasoning. Thus $r_1 = r_2$ and it follows that $p_1 = p_2$. This proves the theorem.

This theorem is called the Euclidean algorithm when a and b are integers.

1.7 Division And Numbers

First recall Theorem 1.6.10, the Euclidean algorithm.

Theorem 1.7.1 *Suppose $0 < a$ and let $b \geq 0$. Then there exists a unique integer p and real number r such that $0 \leq r < a$ and $b = pa + r$.*

The following definition describes what is meant by a prime number and also what is meant by the word "divides".

Definition 1.7.2 *The number, a divides the number, b if in Theorem 1.6.10, $r = 0$. That is there is zero remainder. The notation for this is $a|b$, read a divides b and a is called a factor of b. A prime number is one which has the property that the only numbers which divide it are itself and 1. The greatest common divisor of two positive integers, m, n is that number, p which has the property that p divides both m and n and also if q divides both m and n, then q divides p. Two integers are relatively prime if their greatest common divisor is one. The greatest common divisor of m and n is denoted as (m,n).*

There is a phenomenal and amazing theorem which relates the greatest common divisor to the smallest number in a certain set. Suppose m, n are two positive integers. Then if x, y are integers, so is $xm + yn$. Consider all integers which are of this form. Some are positive such as $1m + 1n$ and some are not. The set S in the following theorem consists of exactly those integers of this form which are positive. Then the greatest common divisor of m and n will be the smallest number in S. This is what the following theorem says.

Theorem 1.7.3 *Let m, n be two positive integers and define*

$$S \equiv \{xm + yn \in \mathbb{N} : x, y \in \mathbb{Z}\ \}\,.$$

Then the smallest number in S is the greatest common divisor, denoted by (m,n).

Proof: First note that both m and n are in S so it is a nonempty set of positive integers. By well ordering, there is a smallest element of S, called $p = x_0 m + y_0 n$. Either p divides m or it does not. If p does not divide m, then by Theorem 1.6.10,

$$m = pq + r$$

where $0 < r < p$. Thus $m = (x_0 m + y_0 n) q + r$ and so, solving for r,

$$r = m(1 - x_0) + (-y_0 q) n \in S.$$

However, this is a contradiction because p was the smallest element of S. Thus $p|m$. Similarly $p|n$.

Now suppose q divides both m and n. Then $m = qx$ and $n = qy$ for integers, x and y. Therefore,

$$p = mx_0 + ny_0 = x_0 qx + y_0 qy = q(x_0 x + y_0 y)$$

showing $q|p$. Therefore, $p = (m, n)$. This proves the theorem.

There is a relatively simple algorithm for finding (m, n) which will be discussed now. Suppose $0 < m < n$ where m, n are integers. Also suppose the greatest common divisor is $(m, n) = d$. Then by the Euclidean algorithm, there exist integers q, r such that

$$n = qm + r, \ r < m \tag{1.1}$$

Now d divides n and m so there are numbers k, l such that $dk = m, dl = n$. From the above equation,

$$r = n - qm = dl - qdk = d(l - qk)$$

Thus d divides both m and r. If k divides both m and r, then from the equation of 1.1 it follows k also divides n. Therefore, k divides d by the definition of the greatest common divisor. Thus d is the greatest common divisor of m and r but $m + r < m + n$. This yields another pair of positive integers for which d is still the greatest common divisor but the sum of these integers is strictly smaller than the sum of the first two. Now you can do the same thing to these integers. Eventually the process must end because the sum gets strictly smaller each time it is done. It ends when there are not two positive integers produced. That is, one is a multiple of the other. At this point, the greatest common divisor is the smaller of the two numbers.

Procedure 1.7.4 *To find the greatest common divisor of m, n where $0 < m < n$, replace the pair $\{m, n\}$ with $\{m, r\}$ where $n = qm + r$ for $r < m$. This new pair of numbers has the same greatest common divisor. Do the process to this pair and continue doing this till you obtain a pair of numbers where one is a multiple of the other. Then the smaller is the sought for greatest common divisor.*

Example 1.7.5 *Find the greatest common divisor of* 165 *and* 385.

Use the Euclidean algorithm to write

$$385 = 2(165) + 55$$

Thus the next two numbers are 55 and 165. Then

$$165 = 3 \times 55$$

and so the greatest common divisor of the first two numbers is 55.

Example 1.7.6 *Find the greatest common divisor of 1237 and 4322.*

Use the Euclidean algorithm

$$4322 = 3\,(1237) + 611$$

Now the two new numbers are 1237,611. Then

$$1237 = 2\,(611) + 15$$

The two new numbers are 15,611. Then

$$611 = 40\,(15) + 11$$

The two new numbers are 15,11. Then

$$15 = 1\,(11) + 4$$

The two new numbers are 11,4

$$2\,(4) + 3$$

The two new numbers are $4, 3$. Then

$$4 = 1\,(3) + 1$$

The two new numbers are $3, 1$. Then

$$3 = 3 \times 1$$

and so 1 is the greatest common divisor. Of course you could see this right away when the two new numbers were 15 and 11. Recall the process delivers numbers which have the same greatest common divisor.

This amazing theorem will now be used to prove a fundamental property of prime numbers which leads to the fundamental theorem of arithmetic, the major theorem which says every integer can be factored as a product of primes.

Theorem 1.7.7 *If p is a prime and $p|ab$ then either $p|a$ or $p|b$.*

Proof: Suppose p does not divide a. Then since p is prime, the only factors of p are 1 and p so follows $(p, a) = 1$ and therefore, there exists integers, x and y such that

$$1 = ax + yp.$$

Multiplying this equation by b yields

$$b = abx + ybp.$$

Since $p|ab$, $ab = pz$ for some integer z. Therefore,

$$b = abx + ybp = pzx + ybp = p\,(xz + yb)$$

and this shows p divides b.

Theorem 1.7.8 *(Fundamental theorem of arithmetic) Let $a \in \mathbb{N} \setminus \{1\}$. Then $a = \prod_{i=1}^{n} p_i$ where p_i are all prime numbers. Furthermore, this prime factorization is unique except for the order of the factors.*

Proof: If a equals a prime number, the prime factorization clearly exists. In particular the prime factorization exists for the prime number 2. Assume this theorem is true for all $a \leq n-1$. If n is a prime, then it has a prime factorization. On the other hand, if n is not a prime, then there exist two integers k and m such that $n = km$ where each of k and m are less than n. Therefore, each of these is no larger than $n-1$ and consequently, each has a prime factorization. Thus so does n. It remains to argue the prime factorization is unique except for order of the factors.

Suppose

$$\prod_{i=1}^{n} p_i = \prod_{j=1}^{m} q_j$$

where the p_i and q_j are all prime, there is no way to reorder the q_k such that $m = n$ and $p_i = q_i$ for all i, and $n + m$ is the smallest positive integer such that this happens. Then by Theorem 1.7.7, $p_1 | q_j$ for some j. Since these are prime numbers this requires $p_1 = q_j$. Reordering if necessary it can be assumed that $q_j = q_1$. Then dividing both sides by $p_1 = q_1$,

$$\prod_{i=1}^{n-1} p_{i+1} = \prod_{j=1}^{m-1} q_{j+1}.$$

Since $n + m$ was as small as possible for the theorem to fail, it follows that $n - 1 = m - 1$ and the prime numbers, $q_2, \cdots, q_m$ can be reordered in such a way that $p_k = q_k$ for all $k = 2, \cdots, n$. Hence $p_i = q_i$ for all i because it was already argued that $p_1 = q_1$, and this results in a contradiction, proving the theorem.

1.8 Systems Of Equations

Sometimes it is necessary to solve systems of equations. For example the problem could be to find x and y such that

$$x + y = 7 \text{ and } 2x - y = 8. \tag{1.2}$$

The set of ordered pairs, (x, y) which solve both equations is called the solution set. For example, you can see that $(5, 2) = (x, y)$ is a solution to the above system. To solve this, note that the solution set does not change if any equation is replaced by a non zero multiple of itself. It also does not change if one equation is replaced by itself added to a multiple of the other equation. For example, x and y solve the above system if and only if x and y solve the system

$$x + y = 7, \overbrace{2x - y + (-2)(x + y) = 8 + (-2)(7)}^{-3y=-6}. \tag{1.3}$$

The second equation was replaced by -2 times the first equation added to the second. Thus the solution is $y = 2$, from $-3y = -6$ and now, knowing $y = 2$, it follows from the other equation that $x + 2 = 7$ and so $x = 5$.

Why exactly does the replacement of one equation with a multiple of another added to it not change the solution set? The two equations of 1.2 are of the form

$$E_1 = f_1, E_2 = f_2 \tag{1.4}$$

where E_1 and E_2 are expressions involving the variables. The claim is that if a is a number, then 1.4 has the same solution set as

$$E_1 = f_1, \; E_2 + aE_1 = f_2 + af_1. \tag{1.5}$$

Why is this?

If (x, y) solves 1.4 then it solves the first equation in 1.5. Also, it satisfies $aE_1 = af_1$ and so, since it also solves $E_2 = f_2$ it must solve the second equation in 1.5. If (x, y) solves 1.5 then it solves the first equation of 1.4. Also $aE_1 = af_1$ and it is given that the second equation of 1.5 is verified. Therefore, $E_2 = f_2$ and it follows (x, y) is a solution of the second equation in 1.4. This shows the solutions to 1.4 and 1.5 are exactly the same which means they have the same solution set. Of course the same reasoning applies with no change if there are many more variables than two and many more equations than two. It is still the case that when one equation is replaced with a multiple of another one added to itself, the solution set of the whole system does not change.

The other thing which does not change the solution set of a system of equations consists of listing the equations in a different order. Here is another example.

Example 1.8.1 *Find the solutions to the system,*

$$\begin{array}{c} x + 3y + 6z = 25 \\ 2x + 7y + 14z = 58 \\ 2y + 5z = 19 \end{array} \tag{1.6}$$

To solve this system replace the second equation by (-2) times the first equation added to the second. This yields. the system

$$\begin{array}{c} x + 3y + 6z = 25 \\ y + 2z = 8 \\ 2y + 5z = 19 \end{array} \tag{1.7}$$

Now take (-2) times the second and add to the third. More precisely, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{array}{c} x + 3y + 6z = 25 \\ y + 2z = 8 \\ z = 3 \end{array} \tag{1.8}$$

At this point, you can tell what the solution is. This system has the same solution as the original system and in the above, $z = 3$. Then using this in the second equation, it follows $y + 6 = 8$ and so $y = 2$. Now using this in the top equation yields $x + 6 + 18 = 25$ and so $x = 1$.

This process is not really much different from what you have always done in solving a single equation. For example, suppose you wanted to solve $2x + 5 = 3x - 6$. You did the same thing to both sides of the equation thus preserving the solution set until you obtained an equation which was simple enough to give the answer. In this case, you would add $-2x$ to both sides and then add 6 to both sides. This yields $x = 11$.

In 1.8 you could have continued as follows. Add (-2) times the bottom equation to the middle and then add (-6) times the bottom to the top. This yields

$$\begin{array}{c} x + 3y = 19 \\ y = 6 \\ z = 3 \end{array}$$

Now add (-3) times the second to the top. This yields

$$\begin{array}{c} x = 1 \\ y = 6 \\ z = 3 \end{array},$$

a system which has the same solution set as the original system.

It is foolish to write the variables every time you do these operations. It is easier to write the system 1.6 as the following "augmented matrix"

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{pmatrix}.$$

It has exactly the same information as the original system but here it is understood there is an x column, $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, a y column, $\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$ and a z column, $\begin{pmatrix} 6 \\ 14 \\ 5 \end{pmatrix}$. The rows correspond to the equations in the system. Thus the top row in the augmented matrix corresponds to the equation,

$$x + 3y + 6z = 25.$$

Now when you replace an equation with a multiple of another equation added to itself, you are just taking a row of this augmented matrix and replacing it with a multiple of another row added to it. Thus the first step in solving 1.6 would be to take (-2) times the first row of the augmented matrix above and add it to the second row,

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{pmatrix}.$$

Note how this corresponds to 1.7. Next take (-2) times the second row and add to the third,

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

which is the same as 1.8. You get the idea I hope. Write the system as an augmented matrix and follow the proceedure of either switching rows, multiplying a row by a non zero number, or replacing a row by a multiple of another row added to it. Each of these operations leaves the solution set unchanged. These operations are called row operations.

Definition 1.8.2 *The row operations consist of the following*

1. *Switch two rows.*
2. *Multiply a row by a nonzero number.*
3. *Replace a row by a multiple of another row added to it.*

Example 1.8.3 *Give the complete solution to the system of equations,* $5x + 10y - 7z = -2$, $2x + 4y - 3z = -1$, *and* $3x + 6y + 5z = 9$.

The augmented matrix for this system is

$$\begin{pmatrix} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{pmatrix}$$

Multiply the second row by 2, the first row by 5, and then take (-1) times the first row and add to the second. Then multiply the first row by $1/5$. This yields

$$\begin{pmatrix} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{pmatrix}$$

Now, combining some row operations, take (-3) times the first row and add this to 2 times the last row and replace the last row with this. This yields.

$$\begin{pmatrix} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{pmatrix}.$$

Putting in the variables, the last two rows say $z = 1$ and $z = 21$. This is impossible so the last system of equations determined by the above augmented matrix has no solution. However, it has the same solution set as the first system of equations. This shows there is no solution to the three given equations. When this happens, the system is called inconsistent.

This should not be surprising that something like this can take place. It can even happen for one equation in one variable. Consider for example, $x = x+1$. There is clearly no solution to this.

Example 1.8.4 *Give the complete solution to the system of equations,* $3x - y - 5z = 9$, $y - 10z = 0$, *and* $-2x + y = -6$.

The augmented matrix of this system is

$$\begin{pmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{pmatrix}$$

Replace the last row with 2 times the top row added to 3 times the bottom row. This gives

$$\begin{pmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{pmatrix}$$

Next take -1 times the middle row and add to the bottom.

$$\begin{pmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Take the middle row and add to the top and then divide the top row which results by 3.

$$\begin{pmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This says $y = 10z$ and $x = 3 + 5z$. Apparently z can equal any number. Therefore, the solution set of this system is $x = 3 + 5t, y = 10t$, and $z = t$ where t is completely arbitrary. The system has an infinite set of solutions and this is a good description of the solutions. This is what it is all about, finding the solutions to the system.

Definition 1.8.5 *Since $z = t$ where t is arbitrary, the variable z is called a* ***free variable.***

The phenomenon of an infinite solution set occurs in equations having only one variable also. For example, consider the equation $x = x$. It doesn't matter what x equals.

Definition 1.8.6 *A system of linear equations is a list of equations,*

$$\sum_{j=1}^{n} a_{ij}x_j = f_j,\ i = 1, 2, 3, \cdots, m$$

where a_{ij} are numbers, f_j is a number, and it is desired to find $(x_1, \cdots, x_n)$ solving each of the equations listed.

As illustrated above, such a system of linear equations may have a unique solution, no solution, or infinitely many solutions. It turns out these are the only three cases which can occur for linear systems. Furthermore, you do exactly the same things to solve any linear system. You write the augmented matrix and do row operations until you get a simpler system in which it is possible to see the solution. All is based on the observation that the row operations do not change the solution set. You can have more equations than variables, fewer equations than variables, etc. It doesn't matter. You always set up the augmented matrix and go to work on it. These things are all the same.

Example 1.8.7 *Give the complete solution to the system of equations, $-41x + 15y = 168$, $109x - 40y = -447$, $-3x + y = 12$, and $2x + z = -1$.*

The augmented matrix is

$$\begin{pmatrix} -41 & 15 & 0 & 168 \\ 109 & -40 & 0 & -447 \\ -3 & 1 & 0 & 12 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

To solve this multiply the top row by 109, the second row by 41, add the top row to the second row, and multiply the top row by 1/109. This yields

$$\begin{pmatrix} -41 & 15 & 0 & 168 \\ 0 & -5 & 0 & -15 \\ -3 & 1 & 0 & 12 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

Now take 2 times the third row and replace the fourth row by this added to 3 times the fourth row.

$$\begin{pmatrix} -41 & 15 & 0 & 168 \\ 0 & -5 & 0 & -15 \\ -3 & 1 & 0 & 12 \\ 0 & 2 & 3 & 21 \end{pmatrix}.$$

Take (-41) times the third row and replace the first row by this added to 3 times the first row. Then switch the third and the first rows.

$$\begin{pmatrix} 123 & -41 & 0 & -492 \\ 0 & -5 & 0 & -15 \\ 0 & 4 & 0 & 12 \\ 0 & 2 & 3 & 21 \end{pmatrix}.$$

Take $-1/2$ times the third row and add to the bottom row. Then take 5 times the third row and add to four times the second. Finally take 41 times the third row and add to 4 times the top row. This yields

$$\begin{pmatrix} 492 & 0 & 0 & -1476 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 12 \\ 0 & 0 & 3 & 15 \end{pmatrix}$$

It follows $x = \frac{-1476}{492} = -3, y = 3$ and $z = 5$.

You should practice solving systems of equations. Here are some exercises.

1.9 Exercises

1. Give the complete solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

2. Give the complete solution to the system of equations, $2x + z = 511$, $x + 6z = 27$, and $y = 1$.

3. Consider the system $-5x + 2y - z = 0$ and $-5x - 2y - z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Thus x and z can equal anything. But when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, it doesn't work. Why?

4. Give the complete solution to the system of equations, $-9x+15y = 66$, $-11x+18y = 79$,$-x + y = 4$, and $z = 3$.

1.10 $\mathbb{F}^n$

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of n complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of n real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

Definition 1.10.1 *Define* $\mathbb{F}^n \equiv \{(x_1, \cdots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \cdots, n\}$. $(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ *if and only if for all* $j = 1, \cdots, n$, $x_j = y_j$. *When* $(x_1, \cdots, x_n) \in \mathbb{F}^n$, *it is conventional to denote* $(x_1, \cdots, x_n)$ *by the single bold face letter,* $\mathbf{x}$. *The numbers,* x_j *are called the coordinates. The set*

$$\{(0, \cdots, 0, t, 0, \cdots, 0) : t \in \mathbb{F}\}$$

for t *in the* i^{th} *slot is called the* i^{th} *coordinate axis. The point* $\mathbf{0} \equiv (0, \cdots, 0)$ *is called the origin.*

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don't match up. In particular, the first entries are not equal.

1.11 Algebra in $\mathbb{F}^n$

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 1.11.1 *If* $\mathbf{x} \in \mathbb{F}^n$ *and* $a \in \mathbb{F}$, *also called a scalar, then* $a\mathbf{x} \in \mathbb{F}^n$ *is defined by*

$$a\mathbf{x} = a(x_1, \cdots, x_n) \equiv (ax_1, \cdots, ax_n). \tag{1.9}$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ *then* $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ *and is defined by*

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \\ &\equiv (x_1 + y_1, \cdots, x_n + y_n) \end{aligned} \tag{1.10}$$

With this definition, the algebraic properties satisfy the conclusions of the following theorem.

Theorem 1.11.2 *For* $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ *and* α, β *scalars, (real numbers), the following hold.*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \tag{1.11}$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \tag{1.12}$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \tag{1.13}$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \tag{1.14}$$

the existence of an additive inverse, Also

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}, \tag{1.15}$$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}, \tag{1.16}$$

$$\alpha(\beta\mathbf{v}) = \alpha\beta(\mathbf{v}), \tag{1.17}$$

$$1\mathbf{v} = \mathbf{v}. \tag{1.18}$$

In the above $\mathbf{0} = (0, \cdots, 0)$.

You should verify these properties all hold. For example, consider 1.15

$$\begin{aligned} \alpha(\mathbf{v} + \mathbf{w}) &= \alpha(v_1 + w_1, \cdots, v_n + w_n) \\ &= (\alpha(v_1 + w_1), \cdots, \alpha(v_n + w_n)) \\ &= (\alpha v_1 + \alpha w_1, \cdots, \alpha v_n + \alpha w_n) \\ &= (\alpha v_1, \cdots, \alpha v_n) + (\alpha w_1, \cdots, \alpha w_n) \\ &= \alpha\mathbf{v} + \alpha\mathbf{w}. \end{aligned}$$

As usual subtraction is defined as $\mathbf{x} - \mathbf{y} \equiv \mathbf{x} + (-\mathbf{y})$.

1.12 Exercises

1. Verify all the properties 1.11-1.18.
2. Compute $5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7)$.
3. Draw a picture of the points in $\mathbb{R}^2$ which are determined by the following ordered pairs.
   (a) $(1, 2)$

(b) $(-2,-2)$

(c) $(-2,3)$

(d) $(2,-5)$

4. Does it make sense to write $(1,2)+(2,3,1)$? Explain.

5. Draw a picture of the points in $\mathbb{R}^3$ which are determined by the following ordered triples.

(a) $(1,2,0)$

(b) $(-2,-2,1)$

(c) $(-2,3,-2)$

1.13 The Inner Product In $\mathbb{F}^n$

The inner product is also called the dot product or scalar product.

Definition 1.13.1 *Let* $\mathbf{a},\mathbf{b}\in\mathbb{F}^n$ *define* $\mathbf{a}\cdot\mathbf{b}$ *as*

$$\mathbf{a}\cdot\mathbf{b}\equiv\sum_{k=1}^{n}a_k\bar{b}_k.$$

With this definition, there are several important properties satisfied by the dot product. In the statement of these properties, α and β will denote scalars and $\mathbf{a},\mathbf{b},\mathbf{c}$ will denote vectors or in other words, points in $\mathbb{F}^n$.

Proposition 1.13.2 *The dot product satisfies the following properties.*

$$\mathbf{a}\cdot\mathbf{b}=\overline{\mathbf{b}\cdot\mathbf{a}} \tag{1.19}$$

$$\mathbf{a}\cdot\mathbf{a}\geq 0 \text{ and equals zero if and only if } \mathbf{a}=\mathbf{0} \tag{1.20}$$

$$(\alpha\mathbf{a}+\beta\mathbf{b})\cdot\mathbf{c}=\alpha(\mathbf{a}\cdot\mathbf{c})+\beta(\mathbf{b}\cdot\mathbf{c}) \tag{1.21}$$

$$\mathbf{c}\cdot(\alpha\mathbf{a}+\beta\mathbf{b})=\overline{\alpha}(\mathbf{c}\cdot\mathbf{a})+\overline{\beta}(\mathbf{c}\cdot\mathbf{b}) \tag{1.22}$$

$$|\mathbf{a}|^2=\mathbf{a}\cdot\mathbf{a} \tag{1.23}$$

You should verify these properties. Also be sure you understand that 1.22 follows from the first three and is therefore redundant. It is listed here for the sake of convenience.

Example 1.13.3 *Find* $(1,2,0,-1)\cdot(0,i,2,3)$.

This equals $0+2(-i)+0+-3=-3-2i$

The Cauchy Schwarz inequality takes the following form in terms of the inner product. I will prove it all over again, using only the above axioms for the dot product.

Theorem 1.13.4 *The dot product satisfies the inequality*

$$|\mathbf{a}\cdot\mathbf{b}|\leq|\mathbf{a}||\mathbf{b}|. \tag{1.24}$$

Furthermore equality is obtained if and only if one of $\mathbf{a}$ *or* $\mathbf{b}$ *is a scalar multiple of the other.*

Proof: First define $\theta \in \mathbb{C}$ such that

$$\overline{\theta}(\mathbf{a}\cdot\mathbf{b}) = |\mathbf{a}\cdot\mathbf{b}|, |\theta| = 1,$$

and define a function of $t \in \mathbb{R}$

$$f(t) = (\mathbf{a}+t\theta\mathbf{b})\cdot(\mathbf{a}+t\theta\mathbf{b}).$$

Then by 1.20, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Also from 1.21,1.22,1.19, and 1.23

$$\begin{aligned} f(t) &= \mathbf{a}\cdot(\mathbf{a}+t\theta\mathbf{b}) + t\theta\mathbf{b}\cdot(\mathbf{a}+t\theta\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a} + t\overline{\theta}(\mathbf{a}\cdot\mathbf{b}) + t\theta(\mathbf{b}\cdot\mathbf{a}) + t^2|\theta|^2\mathbf{b}\cdot\mathbf{b} \\ &= |\mathbf{a}|^2 + 2t\,\mathrm{Re}\,\overline{\theta}(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2 t^2 \\ &= |\mathbf{a}|^2 + 2t|\mathbf{a}\cdot\mathbf{b}| + |\mathbf{b}|^2 t^2 \end{aligned}$$

Now if $|\mathbf{b}|^2 = 0$ it must be the case that $\mathbf{a}\cdot\mathbf{b} = 0$ because otherwise, you could pick large negative values of t and violate $f(t) \geq 0$. Therefore, in this case, the Cauchy Schwarz inequality holds. In the case that $|\mathbf{b}| \neq 0$, $y = f(t)$ is a polynomial which opens up and therefore, if it is always nonnegative, the quadratic formula requires that

$$\overbrace{4|\mathbf{a}\cdot\mathbf{b}|^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2}^{\text{The discriminant}} \leq 0$$

since otherwise the function, $f(t)$ would have two real zeros and would necessarily have a graph which dips below the t axis. This proves 1.24.

It is clear from the axioms of the inner product that equality holds in 1.24 whenever one of the vectors is a scalar multiple of the other. It only remains to verify this is the only way equality can occur. If either vector equals zero, then equality is obtained in 1.24 so it can be assumed both vectors are non zero. Then if equality is achieved, it follows $f(t)$ has exactly one real zero because the discriminant vanishes. Therefore, for some value of $t, \mathbf{a}+t\theta\mathbf{b} = \mathbf{0}$ showing that $\mathbf{a}$ is a multiple of $\mathbf{b}$. This proves the theorem.

You should note that the entire argument was based only on the properties of the dot product listed in 1.19 - 1.23. This means that whenever something satisfies these properties, the Cauchy Schwartz inequality holds. There are many other instances of these properties besides vectors in $\mathbb{F}^n$.

The Cauchy Schwartz inequality allows a proof of the triangle inequality for distances in $\mathbb{F}^n$ in much the same way as the triangle inequality for the absolute value.

Theorem 1.13.5 *(Triangle inequality) For* $\mathbf{a}, \mathbf{b} \in \mathbb{F}^n$

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \tag{1.25}$$

and equality holds if and only if one of the vectors is a nonnegative scalar multiple of the other. Also

$$||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a}-\mathbf{b}| \tag{1.26}$$

Proof: By properties of the dot product and the Cauchy Schwartz inequality,

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \\ &= (\mathbf{a}\cdot\mathbf{a}) + (\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{a}) + (\mathbf{b}\cdot\mathbf{b}) \\ &= |\mathbf{a}|^2 + 2\,\mathrm{Re}(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2 \\ &\leq |\mathbf{a}|^2 + 2|\mathbf{a}\cdot\mathbf{b}| + |\mathbf{b}|^2 \\ &\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \\ &= (|\mathbf{a}| + |\mathbf{b}|)^2. \end{aligned}$$

Taking square roots of both sides you obtain 1.25.

It remains to consider when equality occurs. If either vector equals zero, then that vector equals zero times the other vector and the claim about when equality occurs is verified. Therefore, it can be assumed both vectors are nonzero. To get equality in the second inequality above, Theorem 1.13.4 implies one of the vectors must be a multiple of the other. Say $\mathbf{b} = \alpha\mathbf{a}$. Also, to get equality in the first inequality, $(\mathbf{a} \cdot \mathbf{b})$ must be a nonnegative real number. Thus

$$0 \leq (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \alpha\mathbf{a}) = \overline{\alpha} \left|\mathbf{a}\right|^2.$$

Therefore, α must be a real number which is nonnegative.

To get the other form of the triangle inequality,

$$\mathbf{a} = \mathbf{a} - \mathbf{b} + \mathbf{b}$$

so

$$\begin{aligned} \left|\mathbf{a}\right| &= \left|\mathbf{a} - \mathbf{b} + \mathbf{b}\right| \\ &\leq \left|\mathbf{a} - \mathbf{b}\right| + \left|\mathbf{b}\right|. \end{aligned}$$

Therefore,

$$\left|\mathbf{a}\right| - \left|\mathbf{b}\right| \leq \left|\mathbf{a} - \mathbf{b}\right| \tag{1.27}$$

Similarly,

$$\left|\mathbf{b}\right| - \left|\mathbf{a}\right| \leq \left|\mathbf{b} - \mathbf{a}\right| = \left|\mathbf{a} - \mathbf{b}\right|. \tag{1.28}$$

It follows from 1.27 and 1.28 that 1.26 holds. This is because $||\mathbf{a}| - |\mathbf{b}||$ equals the left side of either 1.27 or 1.28 and either way, $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} - \mathbf{b}|$. This proves the theorem.

1.14 Exercises

1. Show that $(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{4}\left[\left|\mathbf{a} + \mathbf{b}\right|^2 - \left|\mathbf{a} - \mathbf{b}\right|^2\right]$.

2. Prove from the axioms of the dot product the parallelogram identity, $\left|\mathbf{a} + \mathbf{b}\right|^2 + \left|\mathbf{a} - \mathbf{b}\right|^2 = 2\left|\mathbf{a}\right|^2 + 2\left|\mathbf{b}\right|^2$.

3. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, define $\mathbf{a} \cdot \mathbf{b} \equiv \sum_{k=1}^n \beta_k a_k b_k$ where $\beta_k > 0$ for each k. Show this satisfies the axioms of the dot product. What does the Cauchy Schwarz inequality say in this case.

4. In Problem 3 above, suppose you only know $\beta_k \geq 0$. Does the Cauchy Schwarz inequality still hold? If so, prove it.

5. Let f, g be continuous functions and define

$$f \cdot g \equiv \int_0^1 f(t)\,\overline{g(t)}\,dt$$

   show this satisfies the axioms of a dot product if you think of continuous functions in the place of a vector in $\mathbb{F}^n$. What does the Cauchy Schwarz inequality say in this case?

6. Show that if f is a real valued continuous function,

$$\left(\int_a^b f(t)\,dt\right)^2 \leq (b-a)^{1/2} \int_a^b f(t)^2\,dt.$$

Matrices And Linear Transformations

2.1 Matrices

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out such rectangular arrays of numbers are important from many other different points of view. Numbers are also called scalars. In this book numbers will always be either real or complex numbers.

A matrix is a rectangular array of numbers. Several of them are referred to as matrices. For example, here is a matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{pmatrix}$$

This matrix is a 3×4 matrix because there are three rows and four columns. The first row is $(1\ 2\ 3\ 4)$, the second row is $(5\ 2\ 8\ 7)$ and so forth. The first column is $\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$. The convention in dealing with matrices is to always list the rows first and then the columns. Also, you can remember the columns are like columns in a Greek temple. They stand up right while the rows just lay there like rows made by a tractor in a plowed field. Elements of the matrix are identified according to position in the matrix. For example, 8 is in position $2,3$ because it is in the second row and the third column. You might remember that you always list the rows before the columns by using the phrase **Row**man **C**atholic. The symbol, (a_{ij}) refers to a matrix in which the i denotes the row and the j denotes the column. Using this notation on the above matrix, $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

There are various operations which are done on matrices. They can sometimes be added, multiplied by a scalar and sometimes multiplied. To illustrate scalar multiplication, consider the following example.

$$3\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{pmatrix}.$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar. If A is an $m \times n$ matrix, $-A$ is defined to equal $(-1)\,A$.

Two matrices which are the same size can be added. When this is done, the result is the

matrix which is obtained by adding corresponding entries. Thus

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 2 & 8 \\ 6 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 5 & 12 \\ 11 & -2 \end{pmatrix}.$$

Two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

because they are different sizes. As noted above, you write (c_{ij}) for the matrix C whose ij^{th} entry is c_{ij}. In doing arithmetic with matrices you must define what happens in terms of the c_{ij} sometimes called the entries of the matrix or the components of the matrix.

The above discussion stated for general matrices is given in the following definition.

Definition 2.1.1 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Then $A + B = C$ where*

$$C = (c_{ij})$$

for $c_{ij} = a_{ij} + b_{ij}$. Also if x is a scalar,

$$xA = (c_{ij})$$

where $c_{ij} = xa_{ij}$. The number A_{ij} will typically refer to the ij^{th} entry of the matrix, A. The zero matrix, denoted by 0 will be the matrix consisting of all zeros.

Do not be upset by the use of the subscripts, ij. The expression $c_{ij} = a_{ij} + b_{ij}$ is just saying that you add corresponding entries to get the result of summing two matrices as discussed above.

Note there are 2×3 zero matrices, 3×4 zero matrices, etc. In fact for every size there is a zero matrix.

With this definition, the following properties are all obvious but you should verify all of these properties are valid for A, B, and C, $m \times n$ matrices and 0 an $m \times n$ zero matrix,

$$A + B = B + A, \tag{2.1}$$

the commutative law of addition,

$$(A + B) + C = A + (B + C), \tag{2.2}$$

the associative law for addition,

$$A + 0 = A, \tag{2.3}$$

the existence of an additive identity,

$$A + (-A) = 0, \tag{2.4}$$

the existence of an additive inverse. Also, for α, β scalars, the following also hold.

$$\alpha (A + B) = \alpha A + \alpha B, \tag{2.5}$$

$$(\alpha + \beta) A = \alpha A + \beta A, \tag{2.6}$$

$$\alpha (\beta A) = \alpha\beta (A), \tag{2.7}$$

$$1A = A. \tag{2.8}$$

The above properties, 2.1 - 2.8 are known as the vector space axioms and the fact that the $m \times n$ matrices satisfy these axioms is what is meant by saying this set of matrices forms a vector space. You may need to study these later.

Definition 2.1.2 *Matrices which are $n \times 1$ or $1 \times n$ are especially called vectors and are often denoted by a bold letter. Thus*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a $n \times 1$ matrix also called a column vector while a $1 \times n$ matrix of the form $(x_1 \cdots x_n)$ is referred to as a row vector.

All the above is fine, but the real reason for considering matrices is that they can be multiplied. This is where things quit being banal.

First consider the problem of multiplying an $m \times n$ matrix by an $n \times 1$ column vector. Consider the following example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = ?$$

The way I like to remember this is as follows. Slide the vector, placing it on top the two rows as shown

$$\begin{pmatrix} {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} \\ 1 & 2 & 3 \\ {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} \\ 4 & 5 & 6 \end{pmatrix},$$

multiply the numbers on the top by the numbers on the bottom and add them up to get a single number for each row of the matrix. These numbers are listed in the same order giving, in this case, a 2×1 matrix. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \times 1 + 8 \times 2 + 9 \times 3 \\ 7 \times 4 + 8 \times 5 + 9 \times 6 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}.$$

In more general terms,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{pmatrix}.$$

Another way to think of this is

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. Motivated by this example, here is the definition of how to multiply an $m \times n$ matrix by an $n \times 1$ matrix. (vector)

Definition 2.1.3 *Let $A = A_{ij}$ be an $m \times n$ matrix and let $\mathbf{v}$ be an $n \times 1$ matrix,*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then $A\mathbf{v}$ is an $m \times 1$ matrix and the i^{th} component of this matrix is

$$(A\mathbf{v})_i = \sum_{j=1}^{n} A_{ij} v_j.$$

Thus

$$A\mathbf{v} = \begin{pmatrix} \sum_{j=1}^{n} A_{1j} v_j \\ \vdots \\ \sum_{j=1}^{n} A_{mj} v_j \end{pmatrix}. \tag{2.9}$$

In other words, if

$$A = (\mathbf{a}_1, \cdots, \mathbf{a}_n)$$

where the $\mathbf{a}_k$ are the columns,

$$A\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{a}_k$$

This follows from 2.9 and the observation that the j^{th} column of A is

$$\begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix}$$

so 2.9 reduces to

$$v_1 \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} + v_2 \begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{pmatrix} + \cdots + v_n \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{pmatrix}$$

Note also that multiplication by an $m \times n$ matrix takes an $n \times 1$ matrix, and produces an $m \times 1$ matrix.

Here is another example.

Example 2.1.4 *Compute*

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

First of all this is of the form $(3 \times 4)\,(4 \times 1)$ and so the result should be a (3×1). Note how the inside numbers cancel. To get the entry in the second row and first and only column, compute

$$\begin{aligned} \sum_{k=1}^{4} a_{2k} v_k &= a_{21} v_1 + a_{22} v_2 + a_{23} v_3 + a_{24} v_4 \\ &= 0 \times 1 + 2 \times 2 + 1 \times 0 + (-2) \times 1 = 2. \end{aligned}$$

You should do the rest of the problem and verify

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 5 \end{pmatrix}.$$

With this done, the next task is to multiply an $m \times n$ matrix times an $n \times p$ matrix. Before doing so, the following may be helpful.

$$(m \times \overbrace{\quad n)\,(n}^{\text{these must match}} \times p \quad) = m \times p$$

If the two middle numbers don't match, you can't multiply the matrices!

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then B is of the form

$$B = (\mathbf{b}_1, \cdots, \mathbf{b}_p)$$

where $\mathbf{b}_k$ is an $n \times 1$ matrix. Then an $m \times p$ matrix, AB is defined as follows:

$$AB \equiv (A\mathbf{b}_1, \cdots, A\mathbf{b}_p) \tag{2.10}$$

where $A\mathbf{b}_k$ is an $m \times 1$ matrix. Hence AB as just defined is an $m \times p$ matrix. For example,

Example 2.1.5 *Multiply the following.*

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

The first thing you need to check before doing anything else is whether it is possible to do the multiplication. The first matrix is a 2×3 and the second matrix is a 3×3. Therefore, is it possible to multiply these matrices. According to the above discussion it should be a 2×3 matrix of the form

$$\left(\overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}^{\text{First column}}, \overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}^{\text{Second column}}, \overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}^{\text{Third column}} \right)$$

You know how to multiply a matrix times a vector and so you do so to obtain each of the three columns. Thus

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{pmatrix}.$$

Here is another example.

Example 2.1.6 *Multiply the following.*

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

First check if it is possible. This is of the form $(3\times 3)(2\times 3)$. The inside numbers do not match and so you can't do this multiplication. This means that anything you write will be absolute nonsense because it is impossible to multiply these matrices in this order. Aren't they the same two matrices considered in the previous example? Yes they are. It is just that here they are in a different order. This shows something you must always remember about matrix multiplication.

Order Matters!

Matrix multiplication is not commutative. This is very different than multiplication of numbers!

2.1.1 The ij^{th} Entry Of A Product

It is important to describe matrix multiplication in terms of entries of the matrices. What is the ij^{th} entry of AB? It would be the i^{th} entry of the j^{th} column of AB. Thus it would be the i^{th} entry of $A\mathbf{b}_j$. Now

$$\mathbf{b}_j = \begin{pmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{pmatrix}$$

and from the above definition, the i^{th} entry is

$$\sum_{k=1}^{n} A_{ik}B_{kj}. \tag{2.11}$$

In terms of pictures of the matrix, you are doing

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{np} \end{pmatrix}$$

Then as explained above, the j^{th} column is of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix}$$

which is a $m\times 1$ matrix or column vector which equals

$$\begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} B_{1j} + \begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{pmatrix} B_{2j} + \cdots + \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{pmatrix} B_{nj}.$$

The second entry of this $m\times 1$ matrix is

$$A_{21}B_{1j} + A_{22}B_{2j} + \cdots + A_{2n}B_{nj} = \sum_{k=1}^{m} A_{2k}B_{kj}.$$

Similarly, the i^{th} entry of this $m \times 1$ matrix is

$$A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} = \sum_{k=1}^{m} A_{ik}B_{kj}.$$

This shows the following definition for matrix multiplication in terms of the ij^{th} entries of the product coincides with Definition 2.1.3.

This motivates the definition for matrix multiplication which identifies the ij^{th} entries of the product.

Definition 2.1.7 *Let $A = (A_{ij})$ be an $m \times n$ matrix and let $B = (B_{ij})$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}. \tag{2.12}$$

Two matrices, A and B are said to be conformable in a particular order if they can be multiplied in that order. Thus if A is an $r \times s$ matrix and B is a $s \times p$ then A and B are conformable in the order, AB.

Example 2.1.8 *Multiply if possible* $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{pmatrix}$.

First check to see if this is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it must be possible to do this and the result should be a 3×3 matrix. The answer is of the form

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{pmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{pmatrix},$$

a 3×3 matrix as desired. In terms of the ij^{th} entries and the above definition, the entry in the third row and second column of the product should equal

$$\begin{aligned} \sum_j a_{3k}b_{k2} &= a_{31}b_{12} + a_{32}b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42. \end{aligned}$$

You should try a few more such examples to verify the above definition in terms of the ij^{th} entries works for other entries.

Example 2.1.9 *Multiply if possible* $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

This is not possible because it is of the form $(3 \times 2)(3 \times 3)$ and the middle numbers don't match.

Example 2.1.10 *Multiply if possible* $\begin{pmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix}$.

This is possible because in this case it is of the form $(3 \times 3)(3 \times 2)$ and the middle numbers do match. When the multiplication is done it equals

$$\begin{pmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{pmatrix}.$$

Check this and be sure you come up with the same answer.

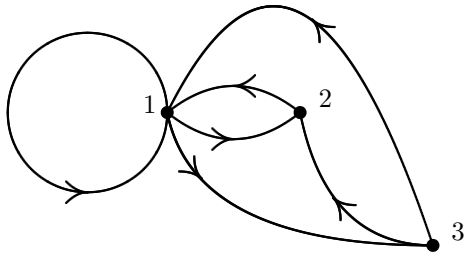
Example 2.1.11 *Multiply if possible* $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 \end{pmatrix}$.

In this case you are trying to do $(3 \times 1)(1 \times 4)$. The inside numbers match so you can do it. Verify

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

2.1.2 A Cute Application

Consider the following graph illustrated in the picture.



There are three locations in this graph, labelled 1,2, and 3. The directed lines represent a way of going from one location to another. Thus there is one way to go from location 1 to location 1. There is one way to go from location 1 to location 3. It is not possible to go from location 2 to location 3 although it is possible to go from location 3 to location 2. Lets refer to moving along one of these directed lines as a step. The following 3×3 matrix is a numerical way of writing the above graph. This is sometimes called a digraph.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Thus a_{ij}, the entry in the i^{th} row and j^{th} column represents the number of ways to go from location i to location j in one step.

Problem: Find the number of ways to go from i to j using exactly k steps.

Denote the answer to the above problem by a_{ij}^k. We don't know what it is right now unless $k = 1$ when it equals a_{ij} described above. However, if we did know what it was, we could find a_{ij}^{k+1} as follows.

$$a_{ij}^{k+1} = \sum_r a_{ir}^k a_{rj}$$

This is because if you go from i to j in $k+1$ steps, you first go from i to r in k steps and then for each of these ways there are a_{rj} ways to go from there to j. Thus $a_{ir}^k a_{rj}$ gives the number of ways to go from i to j in $k+1$ steps such that the k^{th} step leaves you at location r. Adding these gives the above sum. Now you recognize this as the ij^{th} entry of the product of two matrices. Thus

$$a_{ij}^2 = \sum_r a_{ir} a_{rj}$$

$$a_{ij}^3 = \sum_r a_{ir}^2 a_{rj}$$

and so forth. From the above definition of matrix multiplication, this shows that if A is the matrix associated with the directed graph as above, then a_{ij}^k is just the ij^{th} entry of A^k where A^k is just what you would think it should be, A multiplied by itself k times.

Thus in the above example, to find the number of ways of going from 1 to 3 in two steps you would take that matrix and multiply it by itself and then take the entry in the first row and third column. Thus

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

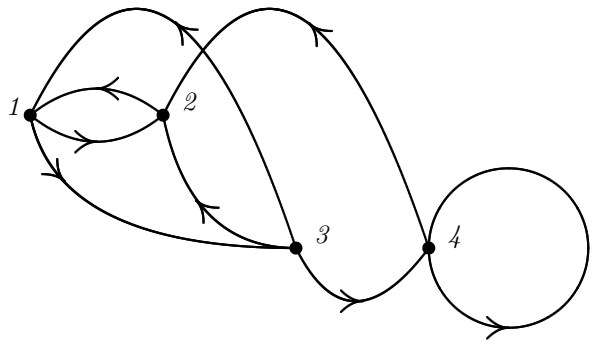
and you see there is exactly one way to go from 1 to 3 in two steps. You can easily see this is true from looking at the graph also. Note there are three ways to go from 1 to 1 in 2 steps. Can you find them from the graph? What would you do if you wanted to consider 5 steps?

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}^5 = \begin{pmatrix} 28 & 19 & 13 \\ 13 & 9 & 6 \\ 19 & 13 & 9 \end{pmatrix}$$

There are 19 ways to go from 1 to 2 in five steps. Do you think you could list them all by looking at the graph? I don't think you could do it without wasting a lot of time.

Of course there is nothing sacred about having only three locations. Everything works just as well with any number of locations. In general if you have n locations, you would need to use a $n \times n$ matrix.

Example 2.1.12 *Consider the following directed graph.*



Write the matrix which is associated with this directed graph and find the number of ways to go from 2 to 4 in three steps.

Here you need to use a 4×4 matrix. The one you need is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then to find the answer, you just need to multiply this matrix by itself three times and look at the entry in the second row and fourth column.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

There is exactly one way to go from 2 to 4 in three steps.

How many ways would there be of going from 2 to 4 in five steps?

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}^5 = \begin{pmatrix} 5 & 9 & 5 & 4 \\ 5 & 4 & 1 & 3 \\ 9 & 10 & 4 & 6 \\ 4 & 6 & 3 & 3 \end{pmatrix}$$

There are three ways. Note there are 10 ways to go from 3 to 2 in five steps.

This is an interesting application of the concept of the ij^{th} entry of the product matrices.

2.1.3 Properties Of Matrix Multiplication

As pointed out above, sometimes it is possible to multiply matrices in one order but not in the other order. What if it makes sense to multiply them in either order? Will they be equal then?

Example 2.1.13 *Compare* $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

The first product is

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix},$$

the second product is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix},$$

and you see these are not equal. Therefore, you cannot conclude that $AB = BA$ for matrix multiplication. However, there are some properties which do hold.

Proposition 2.1.14 *If all multiplications and additions make sense, the following hold for matrices, A, B, C and a, b scalars.*

$$A(aB + bC) = a(AB) + b(AC) \tag{2.13}$$

$$(B + C)A = BA + CA \tag{2.14}$$

$$A(BC) = (AB)C \tag{2.15}$$

Proof: Using the repeated index summation convention and the above definition of matrix multiplication,

$$\begin{aligned} (A(aB + bC))_{ij} &= \sum_k A_{ik} (aB + bC)_{kj} \\ &= \sum_k A_{ik} (aB_{kj} + bC_{kj}) \\ &= a \sum_k A_{ik} B_{kj} + b \sum_k A_{ik} C_{kj} \\ &= a(AB)_{ij} + b(AC)_{ij} \\ &= (a(AB) + b(AC))_{ij} \end{aligned}$$

showing that $A(B + C) = AB + AC$ as claimed. Formula 2.14 is entirely similar.

Consider 2.15, the associative law of multiplication. Before reading this, review the definition of matrix multiplication in terms of entries of the matrices.

$$\begin{aligned} (A(BC))_{ij} &= \sum_k A_{ik} (BC)_{kj} \\ &= \sum_k A_{ik} \sum_l B_{kl} C_{lj} \\ &= \sum_l (AB)_{il} C_{lj} \\ &= ((AB)C)_{ij}. \end{aligned}$$

This proves 2.15.

Another important operation on matrices is that of taking the transpose. The following example shows what is meant by this operation, denoted by placing a T as an exponent on the matrix.

$$\begin{pmatrix} 1 & 1+2i \\ 3 & 1 \\ 2 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 2 \\ 1+2i & 1 & 6 \end{pmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 3 was in the second row and the first column and it ended up in the first row and second column. This motivates the following definition of the transpose of a matrix.

Definition 2.1.15 *Let A be an $m \times n$ matrix. Then A^T denotes the $n \times m$ matrix which is defined as follows.*

$$\left(A^T\right)_{ij} = A_{ji}$$

The transpose of a matrix has the following important property.

Lemma 2.1.16 *Let A be an $m \times n$ matrix and let B be a $n \times p$ matrix. Then*

$$(AB)^T = B^T A^T \tag{2.16}$$

and if α and β are scalars,

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T \tag{2.17}$$

Proof: From the definition,

$$\begin{aligned}
\left((AB)^T\right)_{ij} &= (AB)_{ji} \\
&= \sum_k A_{jk} B_{ki} \\
&= \sum_k \left(B^T\right)_{ik} \left(A^T\right)_{kj} \\
&= \left(B^T A^T\right)_{ij}
\end{aligned}$$

2.17 is left as an exercise and this proves the lemma.

Definition 2.1.17 *An $n \times n$ matrix, A is said to be symmetric if $A = A^T$. It is said to be skew symmetric if $A^T = -A$.*

Example 2.1.18 *Let*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{pmatrix}.$$

Then A is symmetric.

Example 2.1.19 *Let*

$$A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}$$

Then A is skew symmetric.

There is a special matrix called I and defined by

$$I_{ij} = \delta_{ij}$$

where δ_{ij} is the Kroneker symbol defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

It is called the identity matrix because it is a multiplicative identity in the following sense.

Lemma 2.1.20 *Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Then $AI_n = A$. If I_m is the $m \times m$ identity matrix, it also follows that $I_m A = A$.*

Proof:

$$\begin{aligned}(AI_n)_{ij} &= \sum_k A_{ik}\delta_{kj} \\ &= A_{ij}\end{aligned}$$

and so $AI_n = A$. The other case is left as an exercise for you.

Definition 2.1.21 *An $n \times n$ matrix, A has an inverse, A^{-1} if and only if $AA^{-1} = A^{-1}A = I$ where $I = (\delta_{ij})$ for*

$$\delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Such a matrix is called invertible.

2.1.4 Finding The Inverse Of A Matrix

A little later a formula is given for the inverse of a matrix. However, it is not a good way to find the inverse for a matrix. There is a much easier way and it is this which is presented here. It is also important to note that not all matrices have inverses.

Example 2.1.22 *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Does A have an inverse?*

One might think A would have an inverse because it does not equal zero. However,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and if A^{-1} existed, this could not happen because you could write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A^{-1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = A^{-1}\left(A\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) =$$

$$= (A^{-1}A)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = I\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

a contradiction. Thus the answer is that A does not have an inverse.

Example 2.1.23 *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Show $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ is the inverse of A.*

To check this, multiply

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

showing that this matrix is indeed the inverse of A.

In the last example, how would you find A^{-1}? You wish to find a matrix, $\begin{pmatrix} x & z \\ y & w \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This requires the solution of the systems of equations,

$$x + y = 1, x + 2y = 0$$

and

$$z + w = 0, z + 2w = 1.$$

Writing the augmented matrix for these two systems gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \tag{2.18}$$

for the first system and

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \tag{2.19}$$

for the second. Lets solve the first system. Take (-1) times the first row and add to the second to get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Now take (-1) times the second row and add to the first to get

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}.$$

Putting in the variables, this says $x = 2$ and $y = -1$.

Now solve the second system, 2.19 to find z and w. Take (-1) times the first row and add to the second to get

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Now take (-1) times the second row and add to the first to get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Putting in the variables, this says $z = -1$ and $w = 1$. Therefore, the inverse is

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Didn't the above seem rather repetitive? Note that exactly the same row operations were used in both systems. In each case, the end result was something of the form $(I|\mathbf{v})$ where I is the identity and $\mathbf{v}$ gave a column of the inverse. In the above, $\begin{pmatrix} x \\ y \end{pmatrix}$, the first column of the inverse was obtained first and then the second column $\begin{pmatrix} z \\ w \end{pmatrix}$.

This is the reason for the following simple procedure for finding the inverse of a matrix. This procedure is called the Gauss Jordan procedure. It produces the inverse if the matrix has one. Actually it produces a right inverse. Later it will be shown this is really the inverse.

Procedure 2.1.24 *Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix,*

$$(A|I)$$

and then do row operations until you obtain an $n \times 2n$ matrix of the form

$$(I|B) \tag{2.20}$$

if possible. When this has been done, $B = A^{-1}$. The matrix, A has no inverse exactly when it is impossible to do row operations and end up with one like 2.20.

Example 2.1.25 *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. *Find* A^{-1}.

Form the augmented matrix,

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Now do row operations untill the $n \times n$ matrix on the left becomes the identity matrix. This yields after some computations,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and so the inverse of A is the matrix on the right,

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Checking the answer is easy. Just multiply the matrices and see if it works.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Always check your answer because if you are like some of us, you will usually have made a mistake.

Example 2.1.26 *Let* $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$. *Find* A^{-1}.

Set up the augmented matrix, $(A|I)$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Next take (-1) times the first row and add to the second followed by (-3) times the first row added to the last. This yields

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{pmatrix}.$$

Then take 5 times the second row and add to -2 times the last row.

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{pmatrix}$$

Next take the last row and add to (-7) times the top row. This yields

$$\begin{pmatrix} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{pmatrix}.$$

Now take $(-7/5)$ times the second row and add to the top.

$$\begin{pmatrix} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{pmatrix}.$$

Finally divide the top row by -7, the second row by -10 and the bottom row by 14 which yields

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{pmatrix}.$$

Therefore, the inverse is

$$\begin{pmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{pmatrix}$$

Example 2.1.27 *Let* $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. *Find* A^{-1}.

Write the augmented matrix, $(A|I)$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{pmatrix}$$

and proceed to do row operations attempting to obtain $\left(I|A^{-1}\right)$. Take (-1) times the top row and add to the second. Then take (-2) times the top row and add to the bottom.

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{pmatrix}$$

Next add (-1) times the second row to the bottom row.

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}$$

At this point, you can see there will be no inverse because you have obtained a row of zeros in the left half of the augmented matrix, $(A|I)$. Thus there will be no way to obtain I on the left. In other words, the three systems of equations you must solve to find the inverse

have no solution. In particular, there is no solution for the first column of A^{-1} which must solve

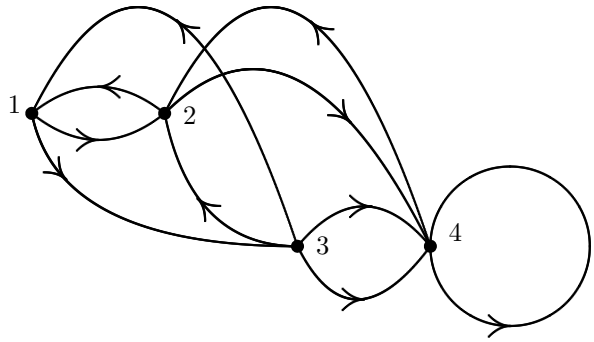
$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

because a sequence of row operations leads to the impossible equation, $0x + 0y + 0z = -1$.

2.2 Exercises

1. In 2.1 - 2.8 describe $-A$ and 0.
2. ♠ Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix.
3. ♠ Show every skew symmetric matrix has all zeros down the main diagonal. The main diagonal consists of every entry of the matrix which is of the form a_{ii}. It runs from the upper left down to the lower right.
4. ♠ Using only the properties 2.1 - 2.8 show $-A$ is unique.
5. ♠ Using only the properties 2.1 - 2.8 show 0 is unique.
6. ♠ Using only the properties 2.1 - 2.8 show $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero for $m \times n$ matrices.
7. ♠ Using only the properties 2.1 - 2.8 and previous problems show $(-1)\, A = -A$.
8. Prove 2.17.
9. ♠ Prove that $I_m A = A$ where A is an $m \times n$ matrix.
10. ♠ Let A and be a real $m \times n$ matrix and let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Show $(A\mathbf{x}, \mathbf{y})_{\mathbb{R}^m} = \left(\mathbf{x}, A^T\mathbf{y}\right)_{\mathbb{R}^n}$ where $(\cdot,\cdot)_{\mathbb{R}^k}$ denotes the dot product in $\mathbb{R}^k$.
11. ♠ Use the result of Problem 10 to verify directly that $(AB)^T = B^T A^T$ without making any reference to subscripts.
12. Let $\mathbf{x} = (-1, -1, 1)$ and $\mathbf{y} = (0, 1, 2)$. Find $\mathbf{x}^T\mathbf{y}$ and $\mathbf{x}\mathbf{y}^T$ if possible.
13. ♠ Give an example of matrices, A, B, C such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.
14. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{pmatrix}$. Find if possible.
    (a) AB
    (b) BA
    (c) AC
    (d) CA
    (e) CB
    (f) BC

15. ♠ Consider the following digraph.



Write the matrix associated with this digraph and find the number of ways to go from 3 to 4 in three steps.

16. ♠ Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

17. Show $(AB)^{-1} = B^{-1}A^{-1}$.

18. ♠ Show that if A is an invertible $n \times n$ matrix, then so is A^T and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

19. Show that if A is an $n \times n$ invertible matrix and $\mathbf{x}$ is a $n \times 1$ matrix such that $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ an $n \times 1$ matrix, then $\mathbf{x} = A^{-1}\mathbf{b}$.

20. ♠ Give an example of a matrix, A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

21. ♠ Give an example of matrices, A, B such that neither A nor B equals zero and yet $AB = 0$.

22. Write $\begin{pmatrix} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{pmatrix}$ in the form $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

23. Give another example other than the one given in this section of two square matrices, A and B such that $AB \neq BA$.

24. ♠ Suppose A and B are square matrices of the same size. Which of the following are correct?

(a) $(A - B)^2 = A^2 - 2AB + B^2$

(b) $(AB)^2 = A^2B^2$

(c) $(A + B)^2 = A^2 + 2AB + B^2$

(d) $(A + B)^2 = A^2 + AB + BA + B^2$

(e) $A^2B^2 = A(AB)B$

(f) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

(g) $(A + B)(A - B) = A^2 - B^2$

(h) None of the above. They are all wrong.

(i) All of the above. They are all right.

25. Let $A = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.

26. Prove that if A^{-1} exists and $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.

27. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.

28. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.

29. ♠ Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.

30. Let
$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.

2.3 Linear Transformations

By 2.13, if A is an $m \times n$ matrix, then for $\mathbf{v}, \mathbf{u}$ vectors in $\mathbb{F}^n$ and a, b scalars,

$$A\left(\overbrace{a\mathbf{u} + b\mathbf{v}}^{\in \mathbb{F}^n}\right) = aA\mathbf{u} + bA\mathbf{v} \in \mathbb{F}^m \tag{2.21}$$

Definition 2.3.1 *A function, $A : \mathbb{F}^n \to \mathbb{F}^m$ is called a linear transformation if for all $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$ and a, b scalars, 2.21 holds.*

From 2.21, matrix multiplication defines a linear transformation as just defined. It turns out this is the only type of linear transformation available. Thus if A is a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$, there is always a matrix which produces A. Before showing this, here is a simple definition.

Definition 2.3.2 *A vector, $\mathbf{e}_i \in \mathbb{F}^n$ is defined as follows:*

$$\mathbf{e}_i \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

where the 1 is in the i^{th} position and there are zeros everywhere else. Thus

$$\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T.$$

Of course the $\mathbf{e}_i$ for a particular value of i in $\mathbb{F}^n$ would be different than the $\mathbf{e}_i$ for that same value of i in $\mathbb{F}^m$ for $m \neq n$. One of them is longer than the other. However, which one is meant will be determined by the context in which they occur.

These vectors have a significant property.

Lemma 2.3.3 *Let $\mathbf{v} \in \mathbb{F}^n$. Thus $\mathbf{v}$ is a list of numbers arranged vertically, $v_1, \cdots, v_n$. Then*

$$\mathbf{e}_i^T \mathbf{v} = v_i. \tag{2.22}$$

Also, if A is an $m \times n$ matrix, then letting $\mathbf{e}_i \in \mathbb{F}^m$ and $\mathbf{e}_j \in \mathbb{F}^n$,

$$\mathbf{e}_i^T A \mathbf{e}_j = A_{ij} \tag{2.23}$$

Proof: First note that $\mathbf{e}_i^T$ is a $1 \times n$ matrix and $\mathbf{v}$ is an $n \times 1$ matrix so the above multiplication in 2.22 makes perfect sense. It equals

$$(0, \cdots, 1, \cdots 0) \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix} = v_i$$

as claimed.

Consider 2.23. From the definition of matrix multiplication using the repeated index summation convention, and noting that $(\mathbf{e}_j)_k = \delta_{kj}$

$$\mathbf{e}_i^T A \mathbf{e}_j = \mathbf{e}_i^T \begin{pmatrix} A_{1k} (\mathbf{e}_j)_k \\ \vdots \\ A_{ik} (\mathbf{e}_j)_k \\ \vdots \\ A_{mk} (\mathbf{e}_j)_k \end{pmatrix} = \mathbf{e}_i^T \begin{pmatrix} A_{1j} \\ \vdots \\ A_{ij} \\ \vdots \\ A_{mj} \end{pmatrix} = A_{ij}$$

by the first part of the lemma. This proves the lemma.

Theorem 2.3.4 *Let $L : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix, A such that*

$$A\mathbf{x} = L\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{F}^n$. The ik^{th} entry of this matrix is given by

$$\mathbf{e}_i^T L \mathbf{e}_k \tag{2.24}$$

Proof: By the lemma,

$$(L\mathbf{x})_i = \mathbf{e}_i^T L\mathbf{x} = \mathbf{e}_i^T x_k L\mathbf{e}_k = \left(\mathbf{e}_i^T L\mathbf{e}_k\right) x_k.$$

Let $A_{ik} = \mathbf{e}_i^T L\mathbf{e}_k$, to prove the existence part of the theorem.

To verify uniqueness, suppose $B\mathbf{x} = A\mathbf{x} = L\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Then in particular, this is true for $\mathbf{x} = \mathbf{e}_j$ and then multiply on the left by $\mathbf{e}_i^T$ to obtain

$$B_{ij} = \mathbf{e}_i^T B\mathbf{e}_j = \mathbf{e}_i^T A\mathbf{e}_j = A_{ij}$$

showing $A = B$. This proves uniqueness.

Corollary 2.3.5 *A linear transformation, $L : \mathbb{F}^n \to \mathbb{F}^m$ is completely determined by the vectors $\{L\mathbf{e}_1, \cdots, L\mathbf{e}_n\}$.*

Proof: This follows immediately from the above theorem. The unique matrix determining the linear transformation which is given in 2.24 depends only on these vectors.

This theorem shows that any linear transformation defined on $\mathbb{F}^n$ can always be considered as a matrix. Therefore, the terms "linear transformation" and "matrix" are often used interchangeably. For example, to say a matrix is one to one, means the linear transformation determined by the matrix is one to one.

Example 2.3.6 *Find the linear transformation, $L : \mathbb{R}^2 \to \mathbb{R}^2$ which has the property that $L\mathbf{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $L\mathbf{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. From the above theorem and corollary, this linear transformation is that determined by matrix multiplication by the matrix*

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

2.4 Subspaces And Spans

Definition 2.4.1 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ be vectors in $\mathbb{F}^n$. A linear combination is any expression of the form*

$$\sum_{i=1}^{p} c_i \mathbf{x}_i$$

where the c_i are scalars. The set of all linear combinations of these vectors is called $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$. If $V \subseteq \mathbb{F}^n$, then V is called a subspace if whenever α, β are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors of V, it follows $\alpha\mathbf{u} + \beta\mathbf{v} \in V$. That is, it is "closed under the algebraic operations of vector addition and scalar multiplication". A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is called linearly independent if whenever

$$\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$$

it follows that all the scalars, c_k equal zero. A set of vectors, $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_i, i = 1, \cdots, n$, not all zero such that $\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$.

Lemma 2.4.2 *A set of vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.*

Proof: Suppose first that $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent. If $\mathbf{x}_k = \sum_{j \neq k} c_j \mathbf{x}_j$, then

$$\mathbf{0} = 1\mathbf{x}_k + \sum_{j \neq k} (-c_j)\, \mathbf{x}_j,$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ linearly independent? If it is not there exist scalars, c_i, not all zero such that

$$\sum_{i=1}^{p} c_i \mathbf{x}_i = \mathbf{0}.$$

Say $c_k \neq 0$. Then you can solve for $\mathbf{x}_k$ as

$$\mathbf{x}_k = \sum_{j \neq k} (-c_j)\, / c_k \mathbf{x}_j$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

Theorem 2.4.3 *(Exchange Theorem) Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be a linearly independent set of vectors such that each $\mathbf{x}_i$ is in span$(\mathbf{y}_1, \cdots, \mathbf{y}_s)$. Then $r \leq s$.*

Proof 1: Define span$\{\mathbf{y}_1, \cdots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \cdots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^{s} c_i \mathbf{y}_i. \tag{2.25}$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = 0$ and so $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = \mathbf{0}$, $1\mathbf{x}_1 + \sum_{i=2}^{r} 0\mathbf{x}_i = \mathbf{x}_1 = \mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve (2.25) for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \text{span}\left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s}^{\text{s-1 vectors here}}\right).$$

Define $\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s\}$$

Therefore, span$\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \cdots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors, $\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ to obtain $\mathbf{v} \in$ span$\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$. The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that span$\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$ where the vectors, $\mathbf{z}_1, \cdots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ and $l + p = s$. This has now been done

for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l+1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \cdots, \mathbf{x}_l\}$ and since $\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$, there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^{l} c_i \mathbf{x}_i + \sum_{j=1}^{p} d_j \mathbf{z}_j. \tag{2.26}$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (2.26) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\text{span}\left\{\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_p}^{\text{p-1 vectors here}}\right\} = V.$$

Continue this way, eventually obtaining

$$\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_s\} = V.$$

But then $\mathbf{x}_r \in \text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_s\}$ contrary to the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Proof 2: Suppose $r > s$. Let $\mathbf{z}_k$ denote a vector of $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$. Thus there exists j as small as possible such that

$$\text{span}(\mathbf{y}_1, \cdots, \mathbf{y}_s) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_m, \mathbf{z}_1, \cdots, \mathbf{z}_j)$$

where $m + j = s$. It is given that $m = 0$, corresponding to no vectors of $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ and $j = s$, corresponding to all the $\mathbf{y}_k$ results in the above equation holding. If $j > 0$ then $m < s$ and so

$$\mathbf{x}_{m+1} = \sum_{k=1}^{m} a_k \mathbf{x}_k + \sum_{i=1}^{j} b_i \mathbf{z}_i$$

Not all the b_i can equal 0 and so you can solve for one of them in terms of $\mathbf{x}_{m+1}, \mathbf{x}_m, \cdots, \mathbf{x}_1$, and the other $\mathbf{z}_k$. Therefore, there exists

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{j-1}\} \subseteq \{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$$

such that

$$\text{span}(\mathbf{y}_1, \cdots, \mathbf{y}_s) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_{m+1}, \mathbf{z}_1, \cdots, \mathbf{z}_{j-1})$$

contradicting the choice of j. Hence $j = 0$ and

$$\text{span}(\mathbf{y}_1, \cdots, \mathbf{y}_s) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s)$$

It follows that

$$\mathbf{x}_{s+1} \in \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s)$$

contrary to the assumption the $\mathbf{x}_k$ are linearly independent. Therefore, $r \leq s$ as claimed. This proves the theorem.

Definition 2.4.4 *A finite set of vectors, $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is a basis for $\mathbb{F}^n$ if $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = \mathbb{F}^n$ and $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent.*

Corollary 2.4.5 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases*[1] *of* $\mathbb{F}^n$. *Then* $r = s = n$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$\mathbf{e}_i = \overbrace{(0, \cdots, 0, 1, 0 \cdots, 0)}^{1 \text{ is in the } i^{th} \text{ slot}}$$

for $i = 1, 2, \cdots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

Lemma 2.4.6 *Let* $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ *be a set of vectors. Then* $V \equiv \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_r)$ *is a subspace.*

Proof: Suppose α, β are two scalars and let $\sum_{k=1}^r c_k \mathbf{v}_k$ and $\sum_{k=1}^r d_k \mathbf{v}_k$ are two elements of V. What about

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k?$$

Is it also in V?

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k = \sum_{k=1}^r (\alpha c_k + \beta d_k) \mathbf{v}_k \in V$$

so the answer is yes. This proves the lemma.

Definition 2.4.7 *A finite set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is a basis for a subspace,* V *of* $\mathbb{F}^n$ *if* $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = V$ *and* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is linearly independent.*

Corollary 2.4.8 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases for* V. *Then* $r = s$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Therefore, this proves the corollary.

Definition 2.4.9 *Let* V *be a subspace of* $\mathbb{F}^n$. *Then* $\dim(V)$ *read as the dimension of* V *is the number of vectors in a basis.*

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 2.4.10 *Suppose* $\mathbf{v} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ *and* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ *is linearly independent. Then* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}\}$ *is also linearly independent.*

Proof: Suppose $\sum_{i=1}^k c_i \mathbf{u}_i + d\mathbf{v} = \mathbf{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$,

$$\mathbf{v} = -\sum_{i=1}^k \left(\frac{c_i}{d}\right) \mathbf{u}_i$$

contrary to assumption. Therefore, $d = 0$. But then $\sum_{i=1}^k c_i \mathbf{u}_i = 0$ and the linear independence of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ implies each $c_i = 0$ also. This proves the lemma.

Theorem 2.4.11 *Let* V *be a nonzero subspace of* $\mathbb{F}^n$. *Then* V *has a basis.*

[1] This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq 0$. If $\text{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V. Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\text{span}\{\mathbf{v}_1\}$. By Lemma 2.4.10 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 2.4.12 *Let V be a subspace of $\mathbb{F}^n$ and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a linearly independent set of vectors in V. Then either it is a basis for V or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_s$ such that $\{\mathbf{v}_1, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_s\}$ is a basis for V.*

Proof: This follows immediately from the proof of Theorem 2.4.11. You do exactly the same argument except you start with $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ rather than $\{\mathbf{v}_1\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.

Theorem 2.4.13 *Let V be a subspace of $\mathbb{F}^n$ and suppose $\text{span}(\mathbf{u}_1 \cdots, \mathbf{u}_p) = V$ where the $\mathbf{u}_i$ are nonzero vectors. Then there exist vectors, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ such that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is a basis for V.*

Proof: Let r be the smallest positive integer with the property that for some set, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$,

$$\text{span}(\mathbf{v}_1 \cdots, \mathbf{v}_r) = V.$$

Then $r \leq p$ and it must be the case that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_k$ would be a linear combination of the others. But then you could delete this vector from $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ and the resulting list of $r-1$ vectors would still span V contrary to the definition of r. This proves the theorem.

2.5 An Application To Matrices

The following is a theorem of major significance.

Theorem 2.5.1 *Suppose A is an $n \times n$ matrix. Then A is one to one if and only if A is onto. Also, if B is an $n \times n$ matrix and $AB = I$, then it follows $BA = I$.*

Proof: First suppose A is one to one. Consider the vectors, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ where $\mathbf{e}_k$ is the column vector which is all zeros except for a 1 in the k^{th} position. This set of vectors is linearly independent because if

$$\sum_{k=1}^{n} c_k A\mathbf{e}_k = \mathbf{0},$$

then since A is linear,

$$A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right) = \mathbf{0}$$

and since A is one to one, it follows

$$\sum_{k=1}^{n} c_k \mathbf{e}_k = \mathbf{0}$$

which implies each $c_k = 0$ because the $\mathbf{e}_k$ are clearly linearly independent.

Therefore, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ must be a basis for $\mathbb{F}^n$ because if not there would exist a vector, $\mathbf{y} \notin \text{span}(A\mathbf{e}_1, \cdots, A\mathbf{e}_n)$ and then by Lemma 2.4.10, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n, \mathbf{y}\}$ would be an independent set of vectors having $n+1$ vectors in it, contrary to the exchange theorem. It follows that for $\mathbf{y} \in \mathbb{F}^n$ there exist constants, c_i such that

$$\mathbf{y} = \sum_{k=1}^{n} c_k A\mathbf{e}_k = A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right)$$

showing that, since $\mathbf{y}$ was arbitrary, A is onto.

Next suppose A is onto. This means the span of the columns of A equals $\mathbb{F}^n$. If these columns are not linearly independent, then by Lemma 2.4.2 on Page 53, one of the columns is a linear combination of the others and so the span of the columns of A equals the span of the $n-1$ other columns. This violates the exchange theorem because $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ would be a linearly independent set of vectors contained in the span of only $n-1$ vectors. Therefore, the columns of A must be independent and this is equivalent to saying that $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. This implies A is one to one because if $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ and so $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

Now suppose $AB = I$. Why is $BA = I$? Since $AB = I$ it follows B is one to one since otherwise, there would exist, $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$ and then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \neq I\mathbf{x}$. Therefore, from what was just shown, B is also onto. In addition to this, A must be one to one because if $A\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = B\mathbf{x}$ for some $\mathbf{x}$ and then $\mathbf{x} = AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$ showing $\mathbf{y} = \mathbf{0}$. Now from what is given to be so, it follows $(AB)A = A$ and so using the associative law for matrix multiplication,

$$A(BA) - A = A(BA - I) = 0.$$

But this means $(BA - I)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ since otherwise, A would not be one to one. Hence $BA = I$ as claimed. This proves the theorem.

This theorem shows that if an $n \times n$ matrix, B acts like an inverse when multiplied on one side of A it follows that $B = A^{-1}$and it will act like an inverse on both sides of A.

The conclusion of this theorem pertains to square matrices only. For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \tag{2.27}$$

Then

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.6 Matrices And Calculus

The study of moving coordinate systems gives a non trivial example of the usefulness of the ideas involving linear transformations and matrices. To begin with, here is the concept of the product rule extended to matrix multiplication.

Definition 2.6.1 *Let $A(t)$ be an $m \times n$ matrix. Say $A(t) = (A_{ij}(t))$. Suppose also that $A_{ij}(t)$ is a differentiable function for all i, j. Then define $A'(t) \equiv \left(A'_{ij}(t)\right)$. That is, $A'(t)$ is the matrix which consists of replacing each entry by its derivative. Such an $m \times n$ matrix in which the entries are differentiable functions is called a differentiable matrix.*

The next lemma is just a version of the product rule.

Lemma 2.6.2 *Let $A(t)$ be an $m \times n$ matrix and let $B(t)$ be an $n \times p$ matrix with the property that all the entries of these matrices are differentiable functions. Then*

$$(A(t)B(t))' = A'(t)B(t) + A(t)B'(t).$$

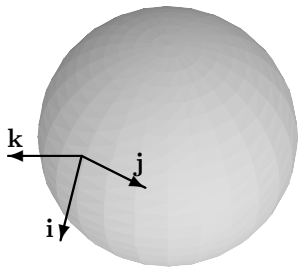
Proof: $(A(t)B(t))' = \left(C'_{ij}(t)\right)$ where $C_{ij}(t) = A_{ik}(t)B_{kj}(t)$ and the repeated index summation convention is being used. Therefore,

$$\begin{aligned} C'_{ij}(t) &= A'_{ik}(t)B_{kj}(t) + A_{ik}(t)B'_{kj}(t) \\ &= (A'(t)B(t))_{ij} + (A(t)B'(t))_{ij} \\ &= (A'(t)B(t) + A(t)B'(t))_{ij} \end{aligned}$$

Therefore, the ij^{th} entry of $A(t)B(t)$ equals the ij^{th} entry of $A'(t)B(t) + A(t)B'(t)$ and this proves the lemma.

2.6.1 The Coriolis Acceleration

Imagine a point on the surface of the earth. Now consider unit vectors, one pointing South, one pointing East and one pointing directly away from the center of the earth.



Denote the first as $\mathbf{i}$, the second as $\mathbf{j}$ and the third as $\mathbf{k}$. If you are standing on the earth you will consider these vectors as fixed, but of course they are not. As the earth turns, they change direction and so each is in reality a function of t. Nevertheless, it is with respect to these apparently fixed vectors that you wish to understand acceleration, velocities, and displacements.

In general, let $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$ be the usual fixed vectors in space and let $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$ be an orthonormal basis of vectors for each t, like the vectors described in the first paragraph. It is assumed these vectors are C^1 functions of t. Letting the positive x axis extend in the direction of $\mathbf{i}(t)$, the positive y axis extend in the direction of $\mathbf{j}(t)$, and the positive z axis extend in the direction of $\mathbf{k}(t)$, yields a moving coordinate system. Now let $\mathbf{u}$ be a vector and let t_0 be some reference time. For example you could let $t_0 = 0$. Then define the components of $\mathbf{u}$ with respect to these vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ at time t_0 as

$$\mathbf{u} \equiv u^1\mathbf{i}(t_0) + u^2\mathbf{j}(t_0) + u^3\mathbf{k}(t_0).$$

Let $\mathbf{u}(t)$ be defined as the vector which has the same components with respect to $\mathbf{i},\mathbf{j},\mathbf{k}$ but at time t. Thus

$$\mathbf{u}(t) \equiv u^1\mathbf{i}(t) + u^2\mathbf{j}(t) + u^3\mathbf{k}(t).$$

and the vector has changed although the components have not.

This is exactly the situation in the case of the apparently fixed basis vectors on the earth if $\mathbf{u}$ is a position vector from the given spot on the earth's surface to a point regarded as fixed with the earth due to its keeping the same coordinates relative to the coordinate axes which are fixed with the earth. Now define a linear transformation $Q(t)$ mapping $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$Q(t)\mathbf{u} \equiv u^1\mathbf{i}(t) + u^2\mathbf{j}(t) + u^3\mathbf{k}(t)$$

where

$$\mathbf{u} \equiv u^1\mathbf{i}(t_0) + u^2\mathbf{j}(t_0) + u^3\mathbf{k}(t_0)$$

Thus letting $\mathbf{v}$ be a vector defined in the same manner as $\mathbf{u}$ and α,β, scalars,

$$Q(t)(\alpha\mathbf{u}+\beta\mathbf{v}) \equiv \left(\alpha u^1+\beta v^1\right)\mathbf{i}(t) + \left(\alpha u^2+\beta v^2\right)\mathbf{j}(t) + \left(\alpha u^3+\beta v^3\right)\mathbf{k}(t)$$

$$\begin{aligned} &= \left(\alpha u^1\mathbf{i}(t) + \alpha u^2\mathbf{j}(t) + \alpha u^3\mathbf{k}(t)\right) + \left(\beta v^1\mathbf{i}(t) + \beta v^2\mathbf{j}(t) + \beta v^3\mathbf{k}(t)\right) \\ &= \alpha\left(u^1\mathbf{i}(t) + u^2\mathbf{j}(t) + u^3\mathbf{k}(t)\right) + \beta\left(v^1\mathbf{i}(t) + v^2\mathbf{j}(t) + v^3\mathbf{k}(t)\right) \\ &\equiv \alpha Q(t)\mathbf{u} + \beta Q(t)\mathbf{v} \end{aligned}$$

showing that $Q(t)$ is a linear transformation. Also, $Q(t)$ preserves all distances because, since the vectors, $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$ form an orthonormal set,

$$|Q(t)\mathbf{u}| = \left(\sum_{i=1}^{3}\left(u^i\right)^2\right)^{1/2} = |\mathbf{u}|.$$

Lemma 2.6.3 *Suppose $Q(t)$ is a real, differentiable $n\times n$ matrix which preserves distances. Then $Q(t)Q(t)^T = Q(t)^TQ(t) = I$. Also, if $\mathbf{u}(t) \equiv Q(t)\mathbf{u}$, then there exists a vector, $\mathbf{\Omega}(t)$ such that*

$$\mathbf{u}'(t) = \mathbf{\Omega}(t)\times\mathbf{u}(t).$$

The symbol $\times$ refers to the cross product from calculus.

Proof: Recall that $(\mathbf{z}\cdot\mathbf{w}) = \frac{1}{4}\left(|\mathbf{z}+\mathbf{w}|^2 - |\mathbf{z}-\mathbf{w}|^2\right)$. Therefore,

$$\begin{aligned} (Q(t)\mathbf{u}\cdot Q(t)\mathbf{w}) &= \frac{1}{4}\left(|Q(t)(\mathbf{u}+\mathbf{w})|^2 - |Q(t)(\mathbf{u}-\mathbf{w})|^2\right) \\ &= \frac{1}{4}\left(|\mathbf{u}+\mathbf{w}|^2 - |\mathbf{u}-\mathbf{w}|^2\right) \\ &= (\mathbf{u}\cdot\mathbf{w}). \end{aligned}$$

This implies

$$\left(Q(t)^TQ(t)\mathbf{u}\cdot\mathbf{w}\right) = (\mathbf{u}\cdot\mathbf{w})$$

for all $\mathbf{u},\mathbf{w}$. Therefore, $Q(t)^TQ(t)\mathbf{u} = \mathbf{u}$ and so $Q(t)^TQ(t) = Q(t)Q(t)^T = I$. This proves the first part of the lemma.

It follows from the product rule, Lemma 2.6.2 that

$$Q'(t)Q(t)^T + Q(t)Q'(t)^T = 0$$

and so

$$Q'(t)\,Q(t)^T = -\left(Q'(t)\,Q(t)^T\right)^T. \tag{2.28}$$

From the definition, $Q(t)\,\mathbf{u} = \mathbf{u}(t)$,

$$\mathbf{u}'(t) = Q'(t)\,\mathbf{u} = Q'(t)\,\overbrace{Q(t)^T\,\mathbf{u}(t)}^{=\mathbf{u}}.$$

Then writing the matrix of $Q'(t)\,Q(t)^T$ with respect to fixed in space orthonormal basis vectors, $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$, where these are the usual basis vectors for $\mathbb{R}^3$, it follows from 2.28 that the matrix of $Q'(t)\,Q(t)^T$ is of the form

$$\begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}$$

for some time dependent scalars, ω_i. Therefore,

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}'(t) = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}(t)$$

where the u^i are the components of the vector $\mathbf{u}(t)$ in terms of the fixed vectors $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$. Therefore,

$$\mathbf{u}'(t) = \boldsymbol{\Omega}(t) \times \mathbf{u}(t) = Q'(t)\,Q(t)^T\,\mathbf{u}(t) \tag{2.29}$$

where

$$\boldsymbol{\Omega}(t) = \omega_1(t)\,\mathbf{i}^* + \omega_2(t)\,\mathbf{j}^* + \omega_3(t)\,\mathbf{k}^*.$$

because

$$\boldsymbol{\Omega}(t) \times \mathbf{u}(t) \equiv \begin{vmatrix} \mathbf{i}^* & \mathbf{j}^* & \mathbf{k}^* \\ w_1 & w_2 & w_3 \\ u^1 & u^2 & u^3 \end{vmatrix} \equiv$$

$$\mathbf{i}^*\left(w_2u^3 - w_3u^2\right) + \mathbf{j}^*\left(w_3u^1 - w_1^3\right) + \mathbf{k}^*\left(w_1u^2 - w_2u^1\right).$$

This proves the lemma and yields the existence part of the following theorem.

Theorem 2.6.4 *Let $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$ be as described. Then there exists a unique vector $\boldsymbol{\Omega}(t)$ such that if $\mathbf{u}(t)$ is a vector whose components are constant with respect to $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$, then*

$$\mathbf{u}'(t) = \boldsymbol{\Omega}(t) \times \mathbf{u}(t).$$

Proof: It only remains to prove uniqueness. Suppose $\boldsymbol{\Omega}_1$ also works. Then $\mathbf{u}(t) = Q(t)\,\mathbf{u}$ and so $\mathbf{u}'(t) = Q'(t)\,\mathbf{u}$ and

$$Q'(t)\,\mathbf{u} = \boldsymbol{\Omega} \times Q(t)\,\mathbf{u} = \boldsymbol{\Omega}_1 \times Q(t)\,\mathbf{u}$$

for all $\mathbf{u}$. Therefore,

$$(\boldsymbol{\Omega} - \boldsymbol{\Omega}_1) \times Q(t)\,\mathbf{u} = \mathbf{0}$$

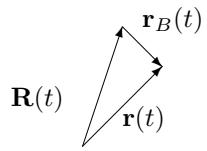
for all $\mathbf{u}$ and since $Q(t)$ is one to one and onto, this implies $(\boldsymbol{\Omega} - \boldsymbol{\Omega}_1) \times \mathbf{w} = \mathbf{0}$ for all $\mathbf{w}$ and thus $\boldsymbol{\Omega} - \boldsymbol{\Omega}_1 = \mathbf{0}$. This proves the theorem.

Now let $\mathbf{R}(t)$ be a position vector and let

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}_B(t)$$

where

$$\mathbf{r}_B(t) \equiv x(t)\,\mathbf{i}(t) + y(t)\,\mathbf{j}(t) + z(t)\,\mathbf{k}(t).$$



In the example of the earth, $\mathbf{R}(t)$ is the position vector of a point $\mathbf{p}(t)$ on the earth's surface and $\mathbf{r}_B(t)$ is the position vector of another point from $\mathbf{p}(t)$, thus regarding $\mathbf{p}(t)$ as the origin. $\mathbf{r}_B(t)$ is the position vector of a point as perceived by the observer on the earth with respect to the vectors he thinks of as fixed. Similarly, $\mathbf{v}_B(t)$ and $\mathbf{a}_B(t)$ will be the velocity and acceleration relative to $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$, and so $\mathbf{v}_B = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ and $\mathbf{a}_B = x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}$. Then

$$\mathbf{v} \equiv \mathbf{r}' = \mathbf{R}' + x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} + x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}'.$$

By , 2.29, if $\mathbf{e} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{e}' = \boldsymbol{\Omega} \times \mathbf{e}$ because the components of these vectors with respect to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant. Therefore,

$$\begin{aligned} x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}' &= x\boldsymbol{\Omega} \times \mathbf{i} + y\boldsymbol{\Omega} \times \mathbf{j} + z\boldsymbol{\Omega} \times \mathbf{k} \\ &= \Omega(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \end{aligned}$$

and consequently,

$$\mathbf{v} = \mathbf{R}' + x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} + \boldsymbol{\Omega} \times \mathbf{r}_B = \mathbf{R}' + x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} + \boldsymbol{\Omega} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Now consider the acceleration. Quantities which are relative to the moving coordinate system and quantities which are relative to a fixed coordinate system are distinguished by using the subscript, B on those relative to the moving coordinates system.

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= \mathbf{R}'' + x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k} + \overbrace{x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'}^{\boldsymbol{\Omega} \times \mathbf{v}_B} + \boldsymbol{\Omega}' \times \mathbf{r}_B \\ &\quad + \boldsymbol{\Omega} \times \left(\overbrace{x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}}^{\mathbf{v}_B} + \overbrace{x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}'}^{\boldsymbol{\Omega} \times \mathbf{r}_B(t)} \right) \\ &= \mathbf{R}'' + \mathbf{a}_B + \boldsymbol{\Omega}' \times \mathbf{r}_B + 2\boldsymbol{\Omega} \times \mathbf{v}_B + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B). \end{aligned}$$

The acceleration $\mathbf{a}_B$ is that perceived by an observer who is moving with the moving coordinate system and for whom the moving coordinate system is fixed. The term $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B)$ is called the centripetal acceleration. Solving for $\mathbf{a}_B$,

$$\mathbf{a}_B = \mathbf{a} - \mathbf{R}'' - \boldsymbol{\Omega}' \times \mathbf{r}_B - 2\boldsymbol{\Omega} \times \mathbf{v}_B - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B). \tag{2.30}$$

Here the term $-(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B))$ is called the centrifugal acceleration, it being an acceleration felt by the observer relative to the moving coordinate system which he regards as fixed, and the term $-2\boldsymbol{\Omega} \times \mathbf{v}_B$ is called the Coriolis acceleration, an acceleration experienced by the observer as he moves relative to the moving coordinate system. The mass multiplied by the Coriolis acceleration defines the Coriolis force.

There is a ride found in some amusement parks in which the victims stand next to a circular wall covered with a carpet or some rough material. Then the whole circular room begins to revolve faster and faster. At some point, the bottom drops out and the victims

are held in place by friction. The force they feel is called centrifugal force and it causes centrifugal acceleration. It is not necessary to move relative to coordinates fixed with the revolving wall in order to feel this force and it is is pretty predictable. However, if the nauseated victim moves relative to the rotating wall, he will feel the effects of the Coriolis force and this force is really strange. The difference between these forces is that the Coriolis force is caused by movement relative to the moving coordinate system and the centrifugal force is not.

2.6.2 The Coriolis Acceleration On The Rotating Earth

Now consider the earth. Let $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$, be the usual basis vectors fixed in space with $\mathbf{k}^*$ pointing in the direction of the north pole from the center of the earth and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors described earlier with $\mathbf{i}$ pointing South, $\mathbf{j}$ pointing East, and $\mathbf{k}$ pointing away from the center of the earth at some point of the rotating earth's surface, $\mathbf{p}$. Letting $\mathbf{R}(t)$ be the position vector of the point $\mathbf{p}$, from the center of the earth, observe the coordinates of $\mathbf{R}(t)$ are constant with respect to $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$. Also, since the earth rotates from West to East and the speed of a point on the surface of the earth relative to an observer fixed in space is $\omega |\mathbf{R}| \sin\phi$ where ω is the angular speed of the earth about an axis through the poles, it follows from the geometric definition of the cross product that

$$\mathbf{R}' = \omega \mathbf{k}^* \times \mathbf{R}$$

Therefore, the vector of Theorem 2.6.4 is $\mathbf{\Omega} = \omega \mathbf{k}^*$ and so

$$\mathbf{R}'' = \overbrace{\mathbf{\Omega}' \times \mathbf{R}}^{=\mathbf{0}} + \mathbf{\Omega} \times \mathbf{R}' = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R})$$

since $\mathbf{\Omega}$ does not depend on t. Formula 2.30 implies

$$\mathbf{a}_B = \mathbf{a} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) - 2\mathbf{\Omega} \times \mathbf{v}_B - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_B). \tag{2.31}$$

In this formula, you can totally ignore the term $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_B)$ because it is so small whenever you are considering motion near some point on the earth's surface. To see this, note $\omega \overbrace{(24)(3600)}^{\text{seconds in a day}} = 2\pi$, and so $\omega = 7.2722 \times 10^{-5}$ in radians per second. If you are using seconds to measure time and feet to measure distance, this term is therefore, no larger than

$$\left(7.2722 \times 10^{-5}\right)^2 |\mathbf{r}_B|.$$

Clearly this is not worth considering in the presence of the acceleration due to gravity which is approximately 32 feet per second squared near the surface of the earth.

If the acceleration $\mathbf{a}$, is due to gravity, then

$$\mathbf{a}_B = \mathbf{a} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) - 2\mathbf{\Omega} \times \mathbf{v}_B =$$

$$\overbrace{-\frac{GM(\mathbf{R}+\mathbf{r}_B)}{|\mathbf{R}+\mathbf{r}_B|^3} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R})}^{\equiv \mathbf{g}} - 2\mathbf{\Omega} \times \mathbf{v}_B \equiv \mathbf{g} - 2\mathbf{\Omega} \times \mathbf{v_B}.$$

Note that

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) = (\mathbf{\Omega} \cdot \mathbf{R})\mathbf{\Omega} - |\mathbf{\Omega}|^2 \mathbf{R}$$

and so $\mathbf{g}$, the acceleration relative to the moving coordinate system on the earth is not directed exactly toward the center of the earth except at the poles and at the equator,

although the components of acceleration which are in other directions are very small when compared with the acceleration due to the force of gravity and are often neglected. Therefore, if the only force acting on an object is due to gravity, the following formula describes the acceleration relative to a coordinate system moving with the earth's surface.

$$\mathbf{a}_B = \mathbf{g} - 2(\boldsymbol{\Omega} \times \mathbf{v}_B)$$

While the vector, $\boldsymbol{\Omega}$ is quite small, if the relative velocity, $\mathbf{v}_B$ is large, the Coriolis acceleration could be significant. This is described in terms of the vectors $\mathbf{i}(t), \mathbf{j}(t), \mathbf{k}(t)$ next.

Letting (ρ, θ, ϕ) be the usual spherical coordinates of the point $\mathbf{p}(t)$ on the surface taken with respect to $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$ the usual way with ϕ the polar angle, it follows the $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$ coordinates of this point are

$$\begin{pmatrix} \rho \sin(\phi) \cos(\theta) \\ \rho \sin(\phi) \sin(\theta) \\ \rho \cos(\phi) \end{pmatrix}.$$

It follows,

$$\mathbf{i} = \cos(\phi) \cos(\theta)\, \mathbf{i}^* + \cos(\phi) \sin(\theta)\, \mathbf{j}^* - \sin(\phi)\, \mathbf{k}^*$$

$$\mathbf{j} = -\sin(\theta)\, \mathbf{i}^* + \cos(\theta)\, \mathbf{j}^* + 0\mathbf{k}^*$$

and

$$\mathbf{k} = \sin(\phi) \cos(\theta)\, \mathbf{i}^* + \sin(\phi) \sin(\theta)\, \mathbf{j}^* + \cos(\phi)\, \mathbf{k}^*.$$

It is necessary to obtain $\mathbf{k}^*$ in terms of the vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Thus the following equation needs to be solved for a, b, c to find $\mathbf{k}^* = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$\overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{\mathbf{k}^*} = \begin{pmatrix} \cos(\phi) \cos(\theta) & -\sin(\theta) & \sin(\phi) \cos(\theta) \\ \cos(\phi) \sin(\theta) & \cos(\theta) & \sin(\phi) \sin(\theta) \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.32}$$

The first column is $\mathbf{i}$, the second is $\mathbf{j}$ and the third is $\mathbf{k}$ in the above matrix. The solution is $a = -\sin(\phi), b = 0$, and $c = \cos(\phi)$.

Now the Coriolis acceleration on the earth equals

$$2(\boldsymbol{\Omega} \times \mathbf{v}_B) = 2\omega \left(\overbrace{-\sin(\phi)\, \mathbf{i} + 0\mathbf{j} + \cos(\phi)\, \mathbf{k}}^{\mathbf{k}^*} \right) \times (x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}).$$

This equals

$$2\omega \left[(-y' \cos\phi)\, \mathbf{i} + (x' \cos\phi + z' \sin\phi)\, \mathbf{j} - (y' \sin\phi)\, \mathbf{k} \right]. \tag{2.33}$$

Remember ϕ is fixed and pertains to the fixed point, $\mathbf{p}(t)$ on the earth's surface. Therefore, if the acceleration, $\mathbf{a}$ is due to gravity,

$$\mathbf{a}_B = \mathbf{g} - 2\omega \left[(-y' \cos\phi)\, \mathbf{i} + (x' \cos\phi + z' \sin\phi)\, \mathbf{j} - (y' \sin\phi)\, \mathbf{k} \right]$$

where $\mathbf{g} = -\frac{GM(\mathbf{R}+\mathbf{r}_B)}{|\mathbf{R}+\mathbf{r}_B|^3} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R})$ as explained above. The term $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R})$ is pretty small and so it will be neglected. However, the Coriolis force will not be neglected.

Example 2.6.5 *Suppose a rock is dropped from a tall building. Where will it stike?*

Assume $\mathbf{a} = -g\mathbf{k}$ and the $\mathbf{j}$ component of $\mathbf{a}_B$ is approximately

$$-2\omega\left(x'\cos\phi + z'\sin\phi\right).$$

The dominant term in this expression is clearly the second one because x' will be small. Also, the $\mathbf{i}$ and $\mathbf{k}$ contributions will be very small. Therefore, the following equation is descriptive of the situation.

$$\mathbf{a}_B = -g\mathbf{k} - 2z'\omega\sin\phi\mathbf{j}.$$

$z' = -gt$ approximately. Therefore, considering the $\mathbf{j}$ component, this is

$$2gt\omega\sin\phi.$$

Two integrations give $\left(\omega gt^3/3\right)\sin\phi$ for the $\mathbf{j}$ component of the relative displacement at time t.

This shows the rock does not fall directly towards the center of the earth as expected but slightly to the east.

Example 2.6.6 *In 1851 Foucault set a pendulum vibrating and observed the earth rotate out from under it. It was a very long pendulum with a heavy weight at the end so that it would vibrate for a long time without stopping*[2]*. This is what allowed him to observe the earth rotate out from under it. Clearly such a pendulum will take 24 hours for the plane of vibration to appear to make one complete revolution at the north pole. It is also reasonable to expect that no such observed rotation would take place on the equator. Is it possible to predict what will take place at various latitudes?*

Using 2.33, in 2.31,

$$\mathbf{a}_B = \mathbf{a} - \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{R}\right)$$
$$-2\omega\left[\left(-y'\cos\phi\right)\mathbf{i} + \left(x'\cos\phi + z'\sin\phi\right)\mathbf{j} - \left(y'\sin\phi\right)\mathbf{k}\right].$$

Neglecting the small term, $\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{R}\right)$, this becomes

$$= -g\mathbf{k} + \mathbf{T}/m - 2\omega\left[\left(-y'\cos\phi\right)\mathbf{i} + \left(x'\cos\phi + z'\sin\phi\right)\mathbf{j} - \left(y'\sin\phi\right)\mathbf{k}\right]$$

where $\mathbf{T}$, the tension in the string of the pendulum, is directed towards the point at which the pendulum is supported, and m is the mass of the pendulum bob. The pendulum can be thought of as the position vector from $(0,0,l)$ to the surface of the sphere $x^2+y^2+(z-l)^2 = l^2$. Therefore,

$$\mathbf{T} = -T\frac{x}{l}\mathbf{i} - T\frac{y}{l}\mathbf{j} + T\frac{l-z}{l}\mathbf{k}$$

and consequently, the differential equations of relative motion are

$$x'' = -T\frac{x}{ml} + 2\omega y'\cos\phi$$

$$y'' = -T\frac{y}{ml} - 2\omega\left(x'\cos\phi + z'\sin\phi\right)$$

and

$$z'' = T\frac{l-z}{ml} - g + 2\omega y'\sin\phi.$$

If the vibrations of the pendulum are small so that for practical purposes, $z'' = z = 0$, the last equation may be solved for T to get

$$gm - 2\omega y'\sin\left(\phi\right)m = T.$$

[2]There is such a pendulum in the Eyring building at BYU and to keep people from touching it, there is a little sign which says Warning! 1000 ohms.

Therefore, the first two equations become

$$x'' = -\left(gm - 2\omega m y' \sin\phi\right)\frac{x}{ml} + 2\omega y' \cos\phi$$

and

$$y'' = -\left(gm - 2\omega m y' \sin\phi\right)\frac{y}{ml} - 2\omega\left(x' \cos\phi + z' \sin\phi\right).$$

All terms of the form xy' or $y'y$ can be neglected because it is assumed x and y remain small. Also, the pendulum is assumed to be long with a heavy weight so that x' and y' are also small. With these simplifying assumptions, the equations of motion become

$$x'' + g\frac{x}{l} = 2\omega y' \cos\phi$$

and

$$y'' + g\frac{y}{l} = -2\omega x' \cos\phi.$$

These equations are of the form

$$x'' + a^2 x = by',\ y'' + a^2 y = -bx' \tag{2.34}$$

where $a^2 = \frac{g}{l}$ and $b = 2\omega\cos\phi$. Then it is fairly tedious but routine to verify that for each constant, c,

$$x = c\sin\left(\frac{bt}{2}\right)\sin\left(\frac{\sqrt{b^2+4a^2}}{2}t\right),\ y = c\cos\left(\frac{bt}{2}\right)\sin\left(\frac{\sqrt{b^2+4a^2}}{2}t\right) \tag{2.35}$$

yields a solution to 2.34 along with the initial conditions,

$$x(0) = 0, y(0) = 0, x'(0) = 0, y'(0) = \frac{c\sqrt{b^2+4a^2}}{2}. \tag{2.36}$$

It is clear from experiments with the pendulum that the earth does indeed rotate out from under it causing the plane of vibration of the pendulum to appear to rotate. The purpose of this discussion is not to establish these self evident facts but to predict how long it takes for the plane of vibration to make one revolution. Therefore, there will be some instant in time at which the pendulum will be vibrating in a plane determined by $\mathbf{k}$ and $\mathbf{j}$. (Recall $\mathbf{k}$ points away from the center of the earth and $\mathbf{j}$ points East.) At this instant in time, defined as $t = 0$, the conditions of 2.36 will hold for some value of c and so the solution to 2.34 having these initial conditions will be those of 2.35 by uniqueness of the initial value problem. Writing these solutions differently,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c\begin{pmatrix} \sin\left(\frac{bt}{2}\right) \\ \cos\left(\frac{bt}{2}\right) \end{pmatrix}\sin\left(\frac{\sqrt{b^2+4a^2}}{2}t\right)$$

This is very interesting! The vector, $c\begin{pmatrix} \sin\left(\frac{bt}{2}\right) \\ \cos\left(\frac{bt}{2}\right) \end{pmatrix}$ always has magnitude equal to $|c|$ but its direction changes very slowly because b is very small. The plane of vibration is determined by this vector and the vector $\mathbf{k}$. The term $\sin\left(\frac{\sqrt{b^2+4a^2}}{2}t\right)$ changes relatively fast and takes values between -1 and 1. This is what describes the actual observed vibrations of the pendulum. Thus the plane of vibration will have made one complete revolution when $t = T$ for

$$\frac{bT}{2} \equiv 2\pi.$$

Therefore, the time it takes for the earth to turn out from under the pendulum is

$$T = \frac{4\pi}{2\omega \cos\phi} = \frac{2\pi}{\omega} \sec\phi.$$

Since ω is the angular speed of the rotating earth, it follows $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$ in radians per hour. Therefore, the above formula implies

$$T = 24 \sec\phi.$$

I think this is really amazing. You could actually determine latitude, not by taking readings with instuments using the North Star but by doing an experiment with a big pendulum. You would set it vibrating, observe T in hours, and then solve the above equation for ϕ. Also note the pendulum would not appear to change its plane of vibration at the equator because $\lim_{\phi \to \pi/2} \sec\phi = \infty$.

The Coriolis acceleration is also responsible for the phenomenon of the next example.

Example 2.6.7 *It is known that low pressure areas rotate counterclockwise as seen from above in the Northern hemisphere but clockwise in the Southern hemisphere. Why?*

Neglect accelerations other than the Coriolis acceleration and the following acceleration which comes from an assumption that the point $\mathbf{p}(t)$ is the location of the lowest pressure.

$$\mathbf{a} = -a(r_B)\,\mathbf{r}_B$$

where $r_B = r$ will denote the distance from the fixed point $\mathbf{p}(t)$ on the earth's surface which is also the lowest pressure point. Of course the situation could be more complicated but this will suffice to expain the above question. Then the acceleration observed by a person on the earth relative to the apparantly fixed vectors, $\mathbf{i}, \mathbf{k}, \mathbf{j}$, is

$$\mathbf{a}_B = -a(r_B)\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) - 2\omega\left[-y'\cos(\phi)\,\mathbf{i} + (x'\cos(\phi) + z'\sin(\phi))\,\mathbf{j} - (y'\sin(\phi)\,\mathbf{k})\right]$$

Therefore, one obtains some differential equations from $\mathbf{a}_B = x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}$ by matching the components. These are

$$\begin{aligned} x'' + a(r_B)\,x &= 2\omega y' \cos\phi \\ y'' + a(r_B)\,y &= -2\omega x' \cos\phi - 2\omega z' \sin(\phi) \\ z'' + a(r_B)\,z &= 2\omega y' \sin\phi \end{aligned}$$

Now remember, the vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed relative to the earth and so are constant vectors. Therefore, from the properties of the determinant and the above differential equations,

$$(\mathbf{r}_B' \times \mathbf{r}_B)' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ x & y & z \end{vmatrix}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'' & y'' & z'' \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a(r_B)\,x + 2\omega y'\cos\phi & -a(r_B)\,y - 2\omega x'\cos\phi - 2\omega z'\sin(\phi) & -a(r_B)\,z + 2\omega y'\sin\phi \\ x & y & z \end{vmatrix}$$

Then the $\mathbf{k}^{th}$ component of this cross product equals

$$\omega\cos(\phi)\left(y^2 + x^2\right)' + 2\omega x z' \sin(\phi).$$

The first term will be negative because it is assumed $\mathbf{p}(t)$ is the location of low pressure causing y^2+x^2 to be a decreasing function. If it is assumed there is not a substantial motion in the $\mathbf{k}$ direction, so that z is fairly constant and the last term can be neglected, then the $\mathbf{k}^{th}$ component of $(\mathbf{r}'_B \times \mathbf{r}_B)'$ is negative provided $\phi \in \left(0, \frac{\pi}{2}\right)$ and positive if $\phi \in \left(\frac{\pi}{2}, \pi\right)$. Beginning with a point at rest, this implies $\mathbf{r}'_B \times \mathbf{r}_B = \mathbf{0}$ initially and then the above implies its $\mathbf{k}^{th}$ component is negative in the upper hemisphere when $\phi < \pi/2$ and positive in the lower hemisphere when $\phi > \pi/2$. Using the right hand and the geometric definition of the cross product, this shows clockwise rotation in the lower hemisphere and counter clockwise rotation in the upper hemisphere.

Note also that as ϕ gets close to $\pi/2$ near the equator, the above reasoning tends to break down because $\cos(\phi)$ becomes close to zero. Therefore, the motion towards the low pressure has to be more pronounced in comparison with the motion in the $\mathbf{k}$ direction in order to draw this conclusion.

2.7 Exercises

1. Show the map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix and $\mathbf{x}$ is an $m \times 1$ column vector is a linear transformation.

2. ♠Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.

3. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.

4. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $-\pi/3$.

5. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$.

6. ♠Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/12$. **Hint:** Note that $\pi/12 = \pi/3 - \pi/4$.

7. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.

8. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$ and then reflects across the x axis.

9. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$ and then reflects across the x axis.

10. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.

11. ♠Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/4$.

12. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/4$.

13. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/6$.

14. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/6$.

15. ♠Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $5\pi/12$. **Hint:** Note that $5\pi/12 = 2\pi/3 - \pi/4$.

16. Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1, -2, 3)^T$.

17. ♠Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1, 5, 3)^T$.

18. Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1, 0, 3)^T$.

19. Give an example of a 2×2 matrix A which has all its entries nonzero and satisfies $A^2 = A$. Such a matrix is called idempotent.

20. Find $\ker(A)$ for
$$A = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}.$$
Recall $\ker(A)$ is just the set of solutions to $A\mathbf{x} = \mathbf{0}$.

21. If A is a linear transformation, and $A\mathbf{x}_p = \mathbf{b}$. Show that the general solution to the equation $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x}_p + \mathbf{y}$ where $\mathbf{y} \in \ker(A)$. By this I mean to show that whenever $A\mathbf{z} = \mathbf{b}$ there exists $\mathbf{y} \in \ker(A)$ such that $\mathbf{x}_p + \mathbf{y} = \mathbf{z}$. For the definition of $\ker(A)$ see Problem 20.

22. Using Problem 20, find the general solution to the following linear system.
$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 18 \\ 7 \end{pmatrix}$$

23. Using Problem 20, find the general solution to the following linear system.
$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 13 \\ 7 \end{pmatrix}$$

24. Show that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{v}) \equiv \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is also a linear transformation.

25. If $\mathbf{u} = (1, 2, 3)^T$, as in Example 9.3.20 and $T_{\mathbf{u}}$ is given in the above problem, find the matrix, $A_{\mathbf{u}}$ which satisfies $A_{\mathbf{u}}\mathbf{x} = T(\mathbf{x})$.

26. ♠ ↑Suppose V is a subspace of $\mathbb{F}^n$ and $T : V \to \mathbb{F}^p$ is a nonzero linear transformation. Show that there exists a basis for $\mathrm{Im}(T) \equiv T(V)$
$$\{T\mathbf{v}_1, \cdots, T\mathbf{v}_m\}$$

and that in this situation,

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$$

is linearly independent.

27. ♠ ↑In the situation of Problem 26 where V is a subspace of $\mathbb{F}^n$, show that there exists $\{\mathbf{z}_1, \cdots, \mathbf{z}_r\}$ a basis for $\ker(T)$. (Recall Theorem 2.4.11. Since $\ker(T)$ is a subspace, it has a basis.) Now for an arbitrary $Tv \in T(V)$, explain why

$$Tv = a_1 T\mathbf{v}_1 + \cdots + a_m T\mathbf{v}_m$$

and why this implies

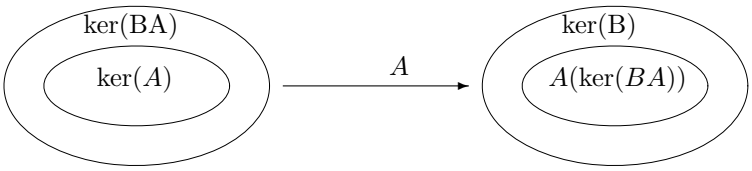
$$v - (a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m) \in \ker(T).$$

Then explain why $V = \operatorname{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_m, \mathbf{z}_1, \cdots, \mathbf{z}_r\}$.

28. ♠ ↑In the situation of the above problem, show $\{\mathbf{v}_1, \cdots, \mathbf{v}_m, \mathbf{z}_1, \cdots, \mathbf{z}_r\}$ is a basis for V and therefore, $\dim(V) = \dim(\ker(T)) + \dim(T(V))$

29. ♠ ↑Let A be a linear transformation from V to W and let B be a linear transformation from W to U where V, W, U are all finite dimensional vector spaces over a field F, written as $A \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}(W, U)$ where V, W, U are all finite dimensional vector spaces. Explain why

$$A(\ker(BA)) \subseteq \ker(B),\ \ker(A) \subseteq \ker(BA).$$



30. ♠ ↑Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis of $\ker(A)$ and let $\{A\mathbf{y}_1, \cdots, A\mathbf{y}_m\}$ be a basis of $A(\ker(BA))$. Let $\mathbf{z} \in \ker(BA)$. Explain why

$$Az \in \operatorname{span}\{A\mathbf{y}_1, \cdots, A\mathbf{y}_m\}$$

and why there exist scalars a_i such that

$$A(z - (a_1\mathbf{y}_1 + \cdots + a_m\mathbf{y}_m)) = 0$$

and why it follows $z - (a_1\mathbf{y}_1 + \cdots + a_m\mathbf{y}_m) \in \operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$. Now explain why

$$\ker(BA) \subseteq \operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_1, \cdots, \mathbf{y}_m\}$$

and so

$$\dim(\ker(BA)) \leq \dim(\ker(B)) + \dim(\ker(A)).$$

This important inequality is due to Sylvester. Show that equality holds if and only if $A(\ker BA) = \ker(B)$.

31. ♠Generalize the result of the previous problem to any finite product of linear mappings.

32. ♠If $W \subseteq V$ for W, V two subspaces of $\mathbb{F}^n$ and if $\dim(W) = \dim(V)$, show $W = V$.

33. ♠Let V be a subspace of $\mathbb{F}^n$and let $V_1, \cdots, V_m$ be subspaces, each contained in V. Then
$$V = V_1 \oplus \cdots \oplus V_m \tag{2.37}$$
if every $v \in V$ can be written in a unique way in the form
$$v = v_1 + \cdots + v_m$$
where each $v_i \in V_i$. This is called a direct sum. If this uniqueness condition does not hold, then one writes
$$V = V_1 + \cdots + V_m$$
and ths symbol means all vectors of the form
$$v_1 + \cdots + v_m,\ v_j \in V_j \text{ for each } j.$$
Show 2.37 is equivalent to saying that if
$$0 = v_1 + \cdots + v_m,\ v_j \in V_j \text{ for each } j,$$
then each $v_j = 0$. Next show that in the situation of 2.37, if $\beta_i = \left\{u_1^i, \cdots, u_{m_i}^i\right\}$ is a basis for V_i, then $\{\beta_1, \cdots, \beta_m\}$ is a basis for V.

34. ♠↑Suppose you have finitely many linear mappings $L_1, L_2, \cdots, L_m$ which map V to V where V is a subspace of $\mathbb{F}^n$ and suppose they commute. That is, $L_iL_j = L_jL_i$ for all i, j. Also suppose L_k is one to one on $\ker(L_j)$ whenever $j \neq k$. Letting P denote the product of these linear transformations, $P = L_1L_2 \cdots L_m$, first show
$$\ker(L_1) + \cdots + \ker(L_m) \subseteq \ker(P)$$
Next show $L_j : \ker(L_i) \to \ker(L_i)$. Then show
$$\ker(L_1) + \cdots + \ker(L_m) = \ker(L_1) \oplus \cdots \oplus \ker(L_m).$$
Using Sylvester's theorem, and the result of Problem 32, show
$$\ker(P) = \ker(L_1) \oplus \cdots \oplus \ker(L_m)$$
Hint: By Sylvester's theorem and the above problem,
$$\begin{aligned} \dim(\ker(P)) &\leq \sum_i \dim(\ker(L_i)) \\ &= \dim(\ker(L_1) \oplus \cdots \oplus \ker(L_m)) \leq \dim(\ker(P)) \end{aligned}$$
Now consider Problem 32.

35. ♠Let $\mathcal{M}(\mathbb{F}^n, \mathbb{F}^n)$ denote the set of all $n \times n$ matrices having entries in $\mathbb{F}$. With the usual operations of matrix addition and scalar multiplications, explain why $\mathcal{M}(\mathbb{F}^n, \mathbb{F}^n)$ can be considered as $\mathbb{F}^{n^2}$. Give a basis for $\mathcal{M}(\mathbb{F}^n, \mathbb{F}^n)$. If $A \in \mathcal{M}(\mathbb{F}^n, \mathbb{F}^n)$, explain why there exists a monic polynomial of the form
$$\lambda^k + a_k\lambda^k + \cdots + a_1\lambda + a_0$$
such that
$$A^k + a_kA^k + \cdots + a_1A + a_0I = 0$$
The minimial polynomial of A is the polynomial like the above, for which $p(A) = 0$ which has smallest degree. I will discuss the uniqueness of this polynomial later. **Hint:** Consider the matrices $I, A, A^2, \cdots, A^{n^2}$. There are $n^2 + 1$ of these matrices. Can they be linearly independent? Now consider all polynomials and pick one of smallest degree and then divide by the leading coefficient.

36. ♠↑Suppose the field of scalars is $\mathbb{C}$ and A is an $n \times n$ matrix. From the preceding problem, and the fundamental theorem of algebra, this minimal polynomial factors

$$(\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k}$$

where r_j is the algebraic multiplicity of λ_j. Thus

$$(A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \cdots (A - \lambda_k I)^{r_k} = 0$$

and so, letting $P = (A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \cdots (A - \lambda_k I)^{r_k}$ and $L_j = (A - \lambda_j I)^{r_j}$ apply the result of Problem 34 to verify that

$$\mathbb{C}^n = \ker(L_1) \oplus \cdots \oplus \ker(L_k)$$

and that $A : \ker(L_j) \to \ker(L_j)$. In this context, $\ker(L_j)$ is called the generalized eigenspace for λ_j. You need to verify the conditions of the result of this problem hold.

37. ♠In the context of Problem 36, show there exists a nonzero vector $\mathbf{x}$ such that

$$(A - \lambda_j I)\mathbf{x} = \mathbf{0}.$$

This is called an eigenvector and the λ_j is called an eigenvalue. **Hint:**There must exist a vector $\mathbf{y}$ such that

$$(A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \cdots (A - \lambda_j I)^{r_j - 1} \cdots (A - \lambda_k I)^{r_k} \mathbf{y} = \mathbf{z} \neq \mathbf{0}$$

Why? Now what happens if you do $(A - \lambda_j I)$ to $\mathbf{z}$?

38. Suppose $Q(t)$ is an orthogonal matrix. This means $Q(t)$ is a real $n \times n$ matrix which satisfies

$$Q(t) Q(t)^T = I$$

Suppose also the entries of $Q(t)$ are differentiable. Show $\left(Q^T\right)' = -Q^T Q' Q^T$.

39. Remember the Coriolis force was $2\mathbf{\Omega} \times \mathbf{v}_B$ where $\mathbf{\Omega}$ was a particular vector which came from the matrix, $Q(t)$ as described above. Show that

$$Q(t) = \begin{pmatrix} \mathbf{i}(t) \cdot \mathbf{i}(t_0) & \mathbf{j}(t) \cdot \mathbf{i}(t_0) & \mathbf{k}(t) \cdot \mathbf{i}(t_0) \\ \mathbf{i}(t) \cdot \mathbf{j}(t_0) & \mathbf{j}(t) \cdot \mathbf{j}(t_0) & \mathbf{k}(t) \cdot \mathbf{j}(t_0) \\ \mathbf{i}(t) \cdot \mathbf{k}(t_0) & \mathbf{j}(t) \cdot \mathbf{k}(t_0) & \mathbf{k}(t) \cdot \mathbf{k}(t_0) \end{pmatrix}.$$

There will be no Coriolis force exactly when $\mathbf{\Omega} = \mathbf{0}$ which corresponds to $Q'(t) = 0$. When will $Q'(t) = 0$?

40. An illustration used in many beginning physics books is that of firing a rifle horizontally and dropping an identical bullet from the same height above the perfectly flat ground followed by an assertion that the two bullets will hit the ground at exactly the same time. Is this true on the rotating earth assuming the experiment takes place over a large perfectly flat field so the curvature of the earth is not an issue? Explain. What other irregularities will occur? Recall the Coriolis force is $2\omega \left[(-y' \cos\phi)\,\mathbf{i} + (x' \cos\phi + z' \sin\phi)\,\mathbf{j} - (y' \sin\phi)\,\mathbf{k}\right]$ where $\mathbf{k}$ points away from the center of the earth, $\mathbf{j}$ points East, and $\mathbf{i}$ points South.

Determinants

3.1 Basic Techniques And Properties

Let A be an $n \times n$ matrix. The determinant of A, denoted as $\det(A)$ is a number. If the matrix is a 2×2 matrix, this number is very easy to find.

Definition 3.1.1 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then*

$$\det(A) \equiv ad - cb.$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Example 3.1.2 *Find* $\det \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$.

From the definition this is just $(2)(6) - (-1)(4) = 16$.

Having defined what is meant by the determinant of a 2×2 matrix, what about a 3×3 matrix?

Example 3.1.3 *Find the determinant of*

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

Here is how it is done by "expanding along the first column".

$$(-1)^{1+1} 1 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + (-1)^{2+1} 4 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + (-1)^{3+1} 3 \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 0.$$

What is going on here? Take the 1 in the upper left corner and cross out the row and the column containing the 1. Then take the determinant of the resulting 2×2 matrix. Now multiply this determinant by 1 and then multiply by $(-1)^{1+1}$ because this 1 is in the first row and the first column. This gives the first term in the above sum. Now go to the 4. Cross out the row and the column which contain 4 and take the determinant of the 2×2 matrix which remains. Multiply this by 4 and then by $(-1)^{2+1}$ because the 4 is in the first column and the second row. Finally consider the 3 on the bottom of the first column. Cross out the row and column containing this 3 and take the determinant of what is left. Then

multiply this by 3 and by $(-1)^{3+1}$ because this 3 is in the third row and the first column. This is the pattern used to evaluate the determinant by expansion along the first column.

You could also expand the determinant along the second row as follows.

$$(-1)^{2+1}4\begin{vmatrix}2&3\\2&1\end{vmatrix}+(-1)^{2+2}3\begin{vmatrix}1&3\\3&1\end{vmatrix}+(-1)^{2+3}2\begin{vmatrix}1&2\\3&2\end{vmatrix}=0.$$

It follows exactly the same pattern and you see it gave the same answer. You pick a row or column and corresponding to each number in that row or column, you cross out the row and column containing it, take the determinant of what is left, multiply this by the number and by $(-1)^{i+j}$ assuming the number is in the i^{th} row and the j^{th} column. Then adding these gives the value of the determinant.

What about a 4×4 matrix?

Example 3.1.4 *Find* $\det(A)$ *where*

$$A=\begin{pmatrix}1&2&3&4\\5&4&2&3\\1&3&4&5\\3&4&3&2\end{pmatrix}$$

As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. $\det(A)=$

$$3(-1)^{1+3}\begin{vmatrix}5&4&3\\1&3&5\\3&4&2\end{vmatrix}+2(-1)^{2+3}\begin{vmatrix}1&2&4\\1&3&5\\3&4&2\end{vmatrix}+$$

$$4(-1)^{3+3}\begin{vmatrix}1&2&4\\5&4&3\\3&4&2\end{vmatrix}+3(-1)^{4+3}\begin{vmatrix}1&2&4\\5&4&3\\1&3&5\end{vmatrix}.$$

Now you know how to expand each of these 3×3 matrices along a row or a column. If you do so, you will get -12 assuming you make no mistakes. You could expand this matrix along any row or any column and assuming you make no mistakes, you will always get the same thing which is defined to be the determinant of the matrix, A. This method of evaluating a determinant by expanding along a row or a column is called the method of Laplace expansion.

Note that each of the four terms above involves three terms consisting of determinants of 2×2 matrices and each of these will need 2 terms. Therefore, there will be $4\times 3\times 2=24$ terms to evaluate in order to find the determinant using the method of Laplace expansion. Suppose now you have a 10×10 matrix. I hope you see that from the above pattern there will be $10!=3,628,800$ terms involved in the evaluation of such a determinant by Laplace expansion along a row or column. This is a lot of terms.

In addition to the difficulties just discussed, I think you should regard the above claim that you always get the same answer by picking any row or column with considerable skepticism. It is incredible and not at all obvious. However, it requires a little effort to establish it. This is done in the section on the theory of the determinant which follows. The above examples motivate the following incredible theorem and definition.

Definition 3.1.5 *Let* $A=(a_{ij})$ *be an* $n\times n$ *matrix. Then a new matrix called the cofactor matrix,* $\operatorname{cof}(A)$ *is defined by* $\operatorname{cof}(A)=(c_{ij})$ *where to obtain* c_{ij} *delete the* i^{th} *row and the* j^{th} *column of* A*, take the determinant of the* $(n-1)\times(n-1)$ *matrix which results, (This is called the* ij^{th} *minor of* A*.) and then multiply this number by* $(-1)^{i+j}$*. To make the formulas easier to remember,* $\operatorname{cof}(A)_{ij}$ *will denote the* ij^{th} *entry of the cofactor matrix.*

Theorem 3.1.6 *Let A be an $n \times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}. \tag{3.1}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Notwithstanding the difficulties involved in using the method of Laplace expansion, certain types of matrices are very easy to deal with.

Definition 3.1.7 *A matrix M, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii}, as shown.*

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

You should verify the following using the above theorem on Laplace expansion.

Corollary 3.1.8 *Let M be an upper (lower) triangular matrix. Then* $\det(M)$ *is obtained by taking the product of the entries on the main diagonal.*

Example 3.1.9 *Let*

$$A = \begin{pmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Find $\det(A)$.

From the above corollary, it suffices to take the product of the diagonal elements. Thus $\det(A) = 1 \times 2 \times 3 \times -1 = -6$. Without using the corollary, you could expand along the first column. This gives

$$1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

and now expand this along the first column to get this equals

$$1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix}$$

Next expand the last along the first column which reduces to the product of the main diagonal elements as claimed. This example also demonstrates why the above corollary is true.

There are many properties satisfied by determinants. Some of the most important are listed in the following theorem.

Theorem 3.1.10 *If two rows or two columns in an $n \times n$ matrix, A, are switched, the determinant of the resulting matrix equals (-1) times the determinant of the original matrix. If A is an $n \times n$ matrix in which two rows are equal or two columns are equal then* $\det(A) = 0$. *Suppose the i^{th} row of A equals $(xa_1 + yb_1, \cdots, xa_n + yb_n)$. Then*

$$\det(A) = x \det(A_1) + y \det(A_2)$$

where the i^{th} row of A_1 is $(a_1, \cdots, a_n)$ and the i^{th} row of A_2 is $(b_1, \cdots, b_n)$, all other rows of A_1 and A_2 coinciding with those of A. In other words, det *is a linear function of each row A. The same is true with the word "row" replaced with the word "column". In addition to this, if A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A) \det(B),$$

and if A is an $n \times n$ matrix, then

$$\det(A) = \det\left(A^T\right).$$

This theorem implies the following corollary which gives a way to find determinants. As I pointed out above, the method of Laplace expansion will not be practical for any matrix of large size.

Corollary 3.1.11 *Let A be an $n \times n$ matrix and let B be the matrix obtained by replacing the i^{th} row (column) of A with the sum of the i^{th} row (column) added to a multiple of another row (column). Then* $\det(A) = \det(B)$. *If B is the matrix obtained from A be replacing the i^{th} row (column) of A by a times the i^{th} row (column) then $a \det(A) = \det(B)$.*

Here is an example which shows how to use this corollary to find a determinant.

Example 3.1.12 *Find the determinant of the matrix,*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{pmatrix}$$

Replace the second row by (-5) times the first row added to it. Then replace the third row by (-4) times the first row added to it. Finally, replace the fourth row by (-2) times the first row added to it. This yields the matrix,

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{pmatrix}$$

and from the above corollary, it has the same determinant as A. Now using the corollary some more, $\det(B) = \left(\frac{-1}{3}\right) \det(C)$ where

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{pmatrix}.$$

The second row was replaced by (-3) times the third row added to the second row and then the last row was multiplied by (-3). Now replace the last row with 2 times the third added to it and then switch the third and second rows. Then $\det(C) = -\det(D)$ where

$$D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{pmatrix}$$

You could do more row operations or you could note that this can be easily expanded along the first column followed by expanding the 3×3 matrix which results along its first column. Thus

$$\det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(C) = -1485$ and $\det(A) = \det(B) = \left(\frac{-1}{3}\right)(-1485) = 495$.

The theorem about expanding a matrix along any row or column also provides a way to give a formula for the inverse of a matrix. Recall the definition of the inverse of a matrix in Definition 2.1.21 on Page 45.

Theorem 3.1.13 *A^{-1} exists if and only if $\det(A) \neq 0$. If $\det(A) \neq 0$, then $A^{-1} = \left(a_{ij}^{-1}\right)$ where*

$$a_{ij}^{-1} = \det(A)^{-1} \operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ the ij^{th} cofactor of A.

Proof: By Theorem 3.1.6 and letting $(a_{ir}) = A$, if $\det(A) \neq 0$,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Theorem 3.1.10. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Now

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ki}^{T}$$

which is the kr^{th} entry of $\operatorname{cof}(A)^T A$. Therefore,

$$\frac{\operatorname{cof}(A)^T}{\det(A)} A = I. \tag{3.2}$$

Using the other formula in Theorem 3.1.6, and similar reasoning,

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

Now

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} = \sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)^{T}_{jk}$$

which is the rk^{th} entry of $A \operatorname{cof}(A)^{T}$. Therefore,

$$A \frac{\operatorname{cof}(A)^{T}}{\det(A)} = I, \tag{3.3}$$

and it follows from 3.2 and 3.3 that $A^{-1} = \left(a_{ij}^{-1}\right)$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

In other words,

$$A^{-1} = \frac{\operatorname{cof}(A)^{T}}{\det(A)}.$$

Now suppose A^{-1} exists. Then by Theorem 3.1.10,

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A) \det\left(A^{-1}\right)$$

so $\det(A) \neq 0$. This proves the theorem.

Theorem 3.1.13 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A.

Example 3.1.14 *Find the inverse of the matrix,*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

First find the determinant of this matrix. Using Corollary 3.1.11 on Page 76, the determinant of this matrix equals the determinant of the matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{pmatrix}$$

which equals 12. The cofactor matrix of A is

$$\begin{pmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{pmatrix}.$$

Each entry of A was replaced by its cofactor. Therefore, from the above theorem, the inverse of A should equal

$$\frac{1}{12} \begin{pmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{pmatrix}^{T} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}.$$

This way of finding inverses is especially useful in the case where it is desired to find the inverse of a matrix whose entries are functions.

Example 3.1.15 *Suppose*

$$A(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$$

Find $A(t)^{-1}$.

First note $\det(A(t)) = e^t$. The cofactor matrix is

$$C(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{pmatrix}$$

and so the inverse is

$$\frac{1}{e^t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{pmatrix}^T = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}.$$

This formula for the inverse also implies a famous procedure known as Cramer's rule. Cramer's rule gives a formula for the solutions, $\mathbf{x}$, to a system of equations, $A\mathbf{x} = \mathbf{y}$.

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det(A)} \operatorname{cof}(A)_{ji}\, y_j.$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

Procedure 3.1.16 *Suppose A is an $n \times n$ matrix and it is desired to solve the system $A\mathbf{x} = \mathbf{y}, \mathbf{y} = (y_1, \cdots, y_n)^T$ for $\mathbf{x} = (x_1, \cdots, x_n)^T$. Then Cramer's rule says*

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is obtained from A by replacing the i^{th} column of A with the column $(y_1, \cdots, y_n)^T$.

The following theorem is of fundamental importance and ties together many of the ideas presented above. It is proved in the next section.

Theorem 3.1.17 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *A is one to one.*
2. *A is onto.*
3. $\det(A) \neq 0$.

3.2 Exercises

1. Find the determinants of the following matrices.

   (a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{pmatrix}$ (The answer is 31.)

   (b) $\begin{pmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{pmatrix}$ (The answer is 375.)

   (c) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$, (The answer is -2.)

2. If A^{-1} exist, what is the relationship between $\det(A)$ and $\det\left(A^{-1}\right)$. Explain your answer.

3. Let A be an $n \times n$ matrix where n is odd. Suppose also that A is skew symmetric. This means $A^T = -A$. Show that $\det(A) = 0$.

4. Is it true that $\det(A+B) = \det(A) + \det(B)$? If this is so, explain why it is so and if it is not so, give a counter example.

5. Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

6. Show $\det(aA) = a^n \det(A)$ where here A is an $n \times n$ matrix and a is a scalar.

7. Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

8. Let A and B be two $n \times n$ matrices. $A \sim B$ (A is similar to B) means there exists an invertible matrix, S such that $A = S^{-1}BS$. Show that if $A \sim B$, then $B \sim A$. Show also that $A \sim A$ and that if $A \sim B$ and $B \sim C$, then $A \sim C$.

9. In the context of Problem 8 show that if $A \sim B$, then $\det(A) = \det(B)$.

10. Let A be an $n \times n$ matrix and let $\mathbf{x}$ be a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar, λ. When this occurs, the vector, $\mathbf{x}$ is called an eigenvector and the scalar, λ is called an eigenvalue. It turns out that not every number is an eigenvalue. Only certain ones are. Why? **Hint:** Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$. Explain why this shows that $(\lambda I - A)$ is not one to one and not onto. Now use Theorem 3.1.17 to argue $\det(\lambda I - A) = 0$. What sort of equation is this? How many solutions does it have?

11. Suppose $\det(\lambda I - A) = 0$. Show using Theorem 3.1.17 there exists $\mathbf{x} \neq \mathbf{0}$ such that $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

12. Let $F(t) = \det \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. Verify

$$F'(t) = \det \begin{pmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{pmatrix} + \det \begin{pmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{pmatrix}.$$

Now suppose

$$F(t) = \det \begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix}.$$

Use Laplace expansion and the first part to verify $F'(t) =$

$$\det \begin{pmatrix} a'(t) & b'(t) & c'(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix} + \det \begin{pmatrix} a(t) & b(t) & c(t) \\ d'(t) & e'(t) & f'(t) \\ g(t) & h(t) & i(t) \end{pmatrix}$$
$$+ \det \begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g'(t) & h'(t) & i'(t) \end{pmatrix}.$$

Conjecture a general result valid for $n \times n$ matrices and explain why it will be true. Can a similar thing be done with the columns?

13. Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix,

$$A = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & e^t \cos t - e^t \sin t & e^t \cos t + e^t \sin t \end{pmatrix}.$$

14. Let A be an $r \times r$ matrix and let B be an $m \times m$ matrix such that $r + m = n$. Consider the following $n \times n$ block matrix

$$C = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}.$$

where the D is an $m \times r$ matrix, and the 0 is a $r \times m$ matrix. Letting I_k denote the $k \times k$ identity matrix, tell why

$$C = \begin{pmatrix} A & 0 \\ D & I_m \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B \end{pmatrix}.$$

Now explain why $\det(C) = \det(A)\det(B)$. **Hint:** Part of this will require an explantion of why

$$\det \begin{pmatrix} A & 0 \\ D & I_m \end{pmatrix} = \det(A).$$

See Corollary 3.1.11.

15. Suppose Q is an orthogonal matrix. This means Q is a real $n \times n$ matrix which satisfies

$$QQ^T = I$$

Find the possible values for $\det(Q)$.

16. Suppose $Q(t)$ is an orthogonal matrix. This means $Q(t)$ is a real $n \times n$ matrix which satisfies
$$Q(t) Q(t)^T = I$$
Suppose $Q(t)$ is continuous for $t \in [a,b]$, some interval. Also suppose $\det(Q(t)) = 1$. Show that it follows $\det(Q(t)) = 1$ for all $t \in [a,b]$.

3.3 The Mathematical Theory Of Determinants

It is easiest to give a different definition of the determinant which is clearly well defined and then prove the earlier one in terms of Laplace expansion. Let $(i_1, \cdots, i_n)$ be an ordered list of numbers from $\{1, \cdots, n\}$. This means the order is important so $(1,2,3)$ and $(2,1,3)$ are different. There will be some repetition between this section and the earlier section on determinants. The main purpose is to give all the missing proofs. Two books which give a good introduction to determinants are Apostol [1] and Rudin [16]. A recent book which also has a good introduction is Baker [2]

3.3.1 The Function sgn

The following Lemma will be essential in the definition of the determinant.

Lemma 3.3.1 *There exists a unique function,* sgn_n *which maps each ordered list of numbers from* $\{1, \cdots, n\}$ *to one of the three numbers,* $0, 1,$ *or* -1 *which also has the following properties.*
$$\mathrm{sgn}_n(1, \cdots, n) = 1 \tag{3.4}$$
$$\mathrm{sgn}_n(i_1, \cdots, p, \cdots, q, \cdots, i_n) = -\mathrm{sgn}_n(i_1, \cdots, q, \cdots, p, \cdots, i_n) \tag{3.5}$$
In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1*. Also, in the case where* $n > 1$ *and* $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$ *so that every number from* $\{1, \cdots, n\}$ *appears in the ordered list,* $(i_1, \cdots, i_n)$*,*
$$\mathrm{sgn}_n(i_1, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_n) \equiv$$
$$(-1)^{n-\theta} \mathrm{sgn}_{n-1}(i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_n) \tag{3.6}$$
where $n = i_\theta$ *in the ordered list,* $(i_1, \cdots, i_n)$*.*

Proof: To begin with, it is necessary to show the existence of such a function. This is clearly true if $n = 1$. Define $\mathrm{sgn}_1(1) \equiv 1$ and observe that it works. No switching is possible. In the case where $n = 2$, it is also clearly true. Let $\mathrm{sgn}_2(1,2) = 1$ and $\mathrm{sgn}_2(2,1) = -1$ while $\mathrm{sgn}_2(2,2) = \mathrm{sgn}_2(1,1) = 0$ and verify it works. Assuming such a function exists for n, sgn_{n+1} will be defined in terms of sgn_n. If there are any repeated numbers in $(i_1, \cdots, i_{n+1})$, $\mathrm{sgn}_{n+1}(i_1, \cdots, i_{n+1}) \equiv 0$. If there are no repeats, then $n+1$ appears somewhere in the ordered list. Let θ be the position of the number $n+1$ in the list. Thus, the list is of the form $(i_1, \cdots, i_{\theta-1}, n+1, i_{\theta+1}, \cdots, i_{n+1})$. From 3.6 it must be that
$$\mathrm{sgn}_{n+1}(i_1, \cdots, i_{\theta-1}, n+1, i_{\theta+1}, \cdots, i_{n+1}) \equiv$$
$$(-1)^{n+1-\theta} \mathrm{sgn}_n(i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n+1}).$$
It is necessary to verify this satisfies 3.4 and 3.5 with n replaced with $n+1$. The first of these is obviously true because
$$\mathrm{sgn}_{n+1}(1, \cdots, n, n+1) \equiv (-1)^{n+1-(n+1)} \mathrm{sgn}_n(1, \cdots, n) = 1.$$

If there are repeated numbers in $(i_1, \cdots, i_{n+1})$, then it is obvious 3.5 holds because both sides would equal zero from the above definition. It remains to verify 3.5 in the case where there are no numbers repeated in $(i_1, \cdots, i_{n+1})$. Consider

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right),$$

where the r above the p indicates the number p is in the r^{th} position and the s above the q indicates that the number, q is in the s^{th} position. Suppose first that $r < \theta < s$. Then

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{\theta}{n+1}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right)$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{\theta}{n+1}, \cdots, \overset{s}{p}, \cdots, i_{n+1}\right) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s-1}{p}, \cdots, i_{n+1}\right)$$

and so, by induction, a switch of p and q introduces a minus sign in the result. Similarly, if $\theta > s$ or if $\theta < r$ it also follows that 3.5 holds. The interesting case is when $\theta = r$ or $\theta = s$. Consider the case where $\theta = r$ and note the other case is entirely similar.

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{n+1}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) \equiv$$

$$(-1)^{n+1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) \tag{3.7}$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s}{n+1}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-s} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right). \tag{3.8}$$

By making $s-1-r$ switches, move the q which is in the $s-1^{th}$ position in 3.7 to the r^{th} position in 3.8. By induction, each of these switches introduces a factor of -1 and so

$$\operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) = (-1)^{s-1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right).$$

Therefore,

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{n+1}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) = (-1)^{n+1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right)$$

$$= (-1)^{n+1-r} (-1)^{s-1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right)$$

$$= (-1)^{n+s} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right) = (-1)^{2s-1} (-1)^{n+1-s} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right)$$

$$= -\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s}{n+1}, \cdots, i_{n+1}\right).$$

This proves the existence of the desired function.

To see this function is unique, note that you can obtain any ordered list of distinct numbers from a sequence of switches. If there exist two functions, f and g both satisfying 3.4 and 3.5, you could start with $f(1, \cdots, n) = g(1, \cdots, n)$ and applying the same sequence of switches, eventually arrive at $f(i_1, \cdots, i_n) = g(i_1, \cdots, i_n)$. If any numbers are repeated, then 3.5 gives both functions are equal to zero for that ordered list. This proves the lemma.

In what follows sgn will often be used rather than sgn_n because the context supplies the appropriate n.

3.3.2 The Definition Of The Determinant

Definition 3.3.2 *Let f be a real valued function which has the set of ordered lists of numbers from $\{1,\cdots,n\}$ as its domain. Define*

$$\sum_{(k_1,\cdots,k_n)} f(k_1\cdots k_n)$$

to be the sum of all the $f(k_1\cdots k_n)$ for all possible choices of ordered lists $(k_1,\cdots,k_n)$ of numbers of $\{1,\cdots,n\}$. For example,

$$\sum_{(k_1,k_2)} f(k_1,k_2) = f(1,2)+f(2,1)+f(1,1)+f(2,2).$$

Definition 3.3.3 *Let $(a_{ij}) = A$ denote an $n\times n$ matrix. The determinant of A, denoted by $\det(A)$ is defined by*

$$\det(A) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots a_{nk_n}$$

where the sum is taken over all ordered lists of numbers from $\{1,\cdots,n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}(k_1,\cdots,k_n)=0$ and so that term contributes 0 to the sum.

Let A be an $n\times n$ matrix, $A=(a_{ij})$ and let $(r_1,\cdots,r_n)$ denote an ordered list of n numbers from $\{1,\cdots,n\}$. Let $A(r_1,\cdots,r_n)$ denote the matrix whose k^{th} row is the r_k row of the matrix, A. Thus

$$\det(A(r_1,\cdots,r_n)) = \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n} \tag{3.9}$$

and

$$A(1,\cdots,n) = A.$$

Proposition 3.3.4 *Let*

$$(r_1,\cdots,r_n)$$

be an ordered list of numbers from $\{1,\cdots,n\}$. Then

$$\operatorname{sgn}(r_1,\cdots,r_n)\det(A)$$

$$= \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n} \tag{3.10}$$

$$= \det(A(r_1,\cdots,r_n)). \tag{3.11}$$

Proof: Let $(1,\cdots,n)=(1,\cdots,r,\cdots s,\cdots,n)$ so $r<s$.

$$\det(A(1,\cdots,r,\cdots,s,\cdots,n)) = \tag{3.12}$$

$$\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_r,\cdots,k_s,\cdots,k_n)\, a_{1k_1}\cdots a_{rk_r}\cdots a_{sk_s}\cdots a_{nk_n},$$

and renaming the variables, calling k_s, k_r and k_r, k_s, this equals

$$= \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_s,\cdots,k_r,\cdots,k_n)\, a_{1k_1}\cdots a_{rk_s}\cdots a_{sk_r}\cdots a_{nk_n}$$

$$= \sum_{(k_1,\cdots,k_n)} -\operatorname{sgn}\left(k_1,\cdots,\overbrace{k_r,\cdots,k_s}^{\text{These got switched}},\cdots,k_n\right) a_{1k_1}\cdots a_{sk_r}\cdots a_{rk_s}\cdots a_{nk_n}$$

$$= -\det(A(1,\cdots,s,\cdots,r,\cdots,n)). \tag{3.13}$$

Consequently,

$$\det(A(1,\cdots,s,\cdots,r,\cdots,n)) =$$

$$-\det(A(1,\cdots,r,\cdots,s,\cdots,n)) = -\det(A)$$

Now letting $A(1,\cdots,s,\cdots,r,\cdots,n)$ play the role of A, and continuing in this way, switching pairs of numbers,

$$\det(A(r_1,\cdots,r_n)) = (-1)^p \det(A)$$

where it took p switches to obtain$(r_1,\cdots,r_n)$ from $(1,\cdots,n)$. By Lemma 3.3.1, this implies

$$\det(A(r_1,\cdots,r_n)) = (-1)^p \det(A) = \operatorname{sgn}(r_1,\cdots,r_n)\det(A)$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $(r_1,\cdots,r_n)$. However, if there is a repeat, say the r^{th} row equals the s^{th} row, then the reasoning of 3.12 -3.13 shows that $A(r_1,\cdots,r_n) = 0$ and also $\operatorname{sgn}(r_1,\cdots,r_n) = 0$ so the formula holds in this case also.

Observation 3.3.5 *There are $n!$ ordered lists of distinct numbers from $\{1,\cdots,n\}$.*

To see this, consider n slots placed in order. There are n choices for the first slot. For each of these choices, there are $n-1$ choices for the second. Thus there are $n(n-1)$ ways to fill the first two slots. Then for each of these ways there are $n-2$ choices left for the third slot. Continuing this way, there are $n!$ ordered lists of distinct numbers from $\{1,\cdots,n\}$ as stated in the observation.

3.3.3 A Symmetric Definition

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det\left(A^T\right)$.

Corollary 3.3.6 *The following formula for $\det(A)$ is valid.*

$$\det(A) = \frac{1}{n!}\cdot$$

$$\sum_{(r_1,\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n)\operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n}. \tag{3.14}$$

And also $\det\left(A^T\right) = \det(A)$ where A^T is the transpose of A. (Recall that for $A^T = \left(a_{ij}^T\right)$, $a_{ij}^T = a_{ji}$.)

Proof: From Proposition 3.3.4, if the r_i are distinct,

$$\det(A) = \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n)\operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n}.$$

Summing over all ordered lists, $(r_1,\cdots,r_n)$ where the r_i are distinct, (If the r_i are not distinct, $\operatorname{sgn}(r_1,\cdots,r_n) = 0$ and so there is no contribution to the sum.)

$$n!\det(A) =$$

$$\sum_{(r_1,\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n)\operatorname{sgn}(k_1,\cdots,k_n)\, a_{r_1k_1}\cdots a_{r_nk_n}.$$

This proves the corollary since the formula gives the same number for A as it does for A^T.

Corollary 3.3.7 *If two rows or two columns in an $n\times n$ matrix, A, are switched, the determinant of the resulting matrix equals (-1) times the determinant of the original matrix. If A is an $n\times n$ matrix in which two rows are equal or two columns are equal then* $\det(A) = 0$. *Suppose the i^{th} row of A equals $(xa_1+yb_1,\cdots,xa_n+yb_n)$. Then*

$$\det(A) = x\det(A_1) + y\det(A_2)$$

where the i^{th} row of A_1 is $(a_1,\cdots,a_n)$ and the i^{th} row of A_2 is $(b_1,\cdots,b_n)$, all other rows of A_1 and A_2 coinciding with those of A. In other words, det *is a linear function of each row A. The same is true with the word "row" replaced with the word "column".*

Proof: By Proposition 3.3.4 when two rows are switched, the determinant of the resulting matrix is (-1) times the determinant of the original matrix. By Corollary 3.3.6 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if A_1 is the matrix obtained from A by switching two columns,

$$\det(A) = \det\left(A^T\right) = -\det\left(A_1^T\right) = -\det(A_1).$$

If A has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det(A) = -\det(A)$ and so $\det(A) = 0$.

It remains to verify the last assertion.

$$\det(A) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots(xa_{k_i}+yb_{k_i})\cdots a_{nk_n}$$

$$= x\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots a_{k_i}\cdots a_{nk_n}$$

$$+y\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, a_{1k_1}\cdots b_{k_i}\cdots a_{nk_n}$$

$$\equiv x\det(A_1) + y\det(A_2).$$

The same is true of columns because $\det\left(A^T\right) = \det(A)$ and the rows of A^T are the columns of A.

3.3.4 Basic Properties Of The Determinant

Definition 3.3.8 *A vector,* $\mathbf{w}$*, is a linear combination of the vectors* $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ *if there exist scalars,* $c_1, \cdots c_r$ *such that* $\mathbf{w} = \sum_{k=1}^{r} c_k \mathbf{v}_k$*. This is the same as saying*

$$\mathbf{w} \in \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}.$$

The following corollary is also of great use.

Corollary 3.3.9 *Suppose* A *is an* $n \times n$ *matrix and some column (row) is a linear combination of* r *other columns (rows). Then* $\det(A) = 0$.

Proof: Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be the columns of A and suppose the condition that one column is a linear combination of r of the others is satisfied. Then by using Corollary 3.3.7 you may rearrange the columns to have the n^{th} column a linear combination of the first r columns. Thus $\mathbf{a}_n = \sum_{k=1}^{r} c_k \mathbf{a}_k$ and so

$$\det(A) = \det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \sum_{k=1}^{r} c_k \mathbf{a}_k \end{pmatrix}.$$

By Corollary 3.3.7

$$\det(A) = \sum_{k=1}^{r} c_k \det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_k \end{pmatrix} = 0.$$

The case for rows follows from the fact that $\det(A) = \det\left(A^T\right)$. This proves the corollary.

Recall the following definition of matrix multiplication.

Definition 3.3.10 *If* A *and* B *are* $n \times n$ *matrices,* $A = (a_{ij})$ *and* $B = (b_{ij})$*,* $AB = (c_{ij})$ *where*

$$c_{ij} \equiv \sum_{k=1}^{n} a_{ik} b_{kj}.$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 3.3.11 *Let* A *and* B *be* $n \times n$ *matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof: Let c_{ij} be the ij^{th} entry of AB. Then by Proposition 3.3.4,

$$\det(AB) =$$

$$\begin{aligned}
&\sum_{(k_1,\cdots,k_n)} \text{sgn}(k_1, \cdots, k_n)\, c_{1k_1} \cdots c_{nk_n} \\
= &\sum_{(k_1,\cdots,k_n)} \text{sgn}(k_1, \cdots, k_n) \left(\sum_{r_1} a_{1r_1} b_{r_1k_1}\right) \cdots \left(\sum_{r_n} a_{nr_n} b_{r_nk_n}\right) \\
= &\sum_{(r_1\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \text{sgn}(k_1, \cdots, k_n)\, b_{r_1k_1} \cdots b_{r_nk_n} \left(a_{1r_1} \cdots a_{nr_n}\right) \\
= &\sum_{(r_1\cdots,r_n)} \text{sgn}(r_1 \cdots r_n)\, a_{1r_1} \cdots a_{nr_n} \det(B) = \det(A)\det(B).
\end{aligned}$$

This proves the theorem.

3.3.5 Expansion Using Cofactors

Lemma 3.3.12 *Suppose a matrix is of the form*

$$M = \begin{pmatrix} A & * \\ \mathbf{0} & a \end{pmatrix} \tag{3.15}$$

or

$$M = \begin{pmatrix} A & \mathbf{0} \\ * & a \end{pmatrix} \tag{3.16}$$

where a is a number and A is an $(n-1)\times(n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $\mathbf{0}$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then

$$\det(M) = a\det(A).$$

Proof: Denote M by (m_{ij}). Thus in the first case, $m_{nn} = a$ and $m_{ni} = 0$ if $i \neq n$ while in the second case, $m_{nn} = a$ and $m_{in} = 0$ if $i \neq n$. From the definition of the determinant,

$$\det(M) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}_n(k_1,\cdots,k_n)\, m_{1k_1}\cdots m_{nk_n}$$

Letting θ denote the position of n in the ordered list, $(k_1,\cdots,k_n)$ then using the earlier conventions used to prove Lemma 3.3.1, $\det(M)$ equals

$$\sum_{(k_1,\cdots,k_n)} (-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(k_1,\cdots,k_{\theta-1},\overset{\theta}{k_{\theta+1}},\cdots,\overset{n-1}{k_n}\right) m_{1k_1}\cdots m_{nk_n}$$

Now suppose 3.16. Then if $k_n \neq n$, the term involving m_{nk_n} in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta = n$ or in other words, those for which $k_n = n$. Therefore, the above expression reduces to

$$a\sum_{(k_1,\cdots,k_{n-1})} \operatorname{sgn}_{n-1}(k_1,\cdots k_{n-1})\, m_{1k_1}\cdots m_{(n-1)k_{n-1}} = a\det(A).$$

To get the assertion in the situation of 3.15 use Corollary 3.3.6 and 3.16 to write

$$\det(M) = \det\left(M^T\right) = \det\left(\begin{pmatrix} A^T & \mathbf{0} \\ * & a \end{pmatrix}\right) = a\det\left(A^T\right) = a\det(A).$$

This proves the lemma.

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 3.3.13 *Let $A = (a_{ij})$ be an $n\times n$ matrix. Then a new matrix called the cofactor matrix,* $\operatorname{cof}(A)$ *is defined by* $\operatorname{cof}(A) = (c_{ij})$ *where to obtain c_{ij} delete the i^{th} row and the j^{th} column of A, take the determinant of the $(n-1)\times(n-1)$ matrix which results, (This is called the ij^{th} minor of A.) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember,* $\operatorname{cof}(A)_{ij}$ *will denote the ij^{th} entry of the cofactor matrix.*

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 3.3.14 *Let A be an $n \times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}. \tag{3.17}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Proof: Let $(a_{i1}, \cdots, a_{in})$ be the i^{th} row of A. Let B_j be the matrix obtained from A by leaving every row the same except the i^{th} row which in B_j equals $(0, \cdots, 0, a_{ij}, 0, \cdots, 0)$. Then by Corollary 3.3.7,

$$\det(A) = \sum_{j=1}^{n} \det(B_j)$$

For example if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$$

and $i = 2$, then

$$B_1 = \begin{pmatrix} a & b & c \\ d & 0 & 0 \\ h & i & j \end{pmatrix}, B_2 = \begin{pmatrix} a & b & c \\ 0 & e & 0 \\ h & i & j \end{pmatrix}, B_3 = \begin{pmatrix} a & b & c \\ 0 & 0 & f \\ h & i & j \end{pmatrix}$$

Denote by A^{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A. Thus $\operatorname{cof}(A)_{ij} \equiv (-1)^{i+j} \det\left(A^{ij}\right)$. At this point, recall that from Proposition 3.3.4, when two rows or two columns in a matrix, M, are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 3.3.12,

$$\begin{aligned} \det(B_j) &= (-1)^{n-j} (-1)^{n-i} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) \\ &= (-1)^{i+j} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) = a_{ij} \operatorname{cof}(A)_{ij}. \end{aligned}$$

Therefore,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}$$

which is the formula for expanding $\det(A)$ along the i^{th} row. Also,

$$\begin{aligned} \det(A) &= \det\left(A^T\right) = \sum_{j=1}^{n} a_{ij}^T \operatorname{cof}\left(A^T\right)_{ij} \\ &= \sum_{j=1}^{n} a_{ji} \operatorname{cof}(A)_{ji} \end{aligned}$$

which is the formula for expanding $\det(A)$ along the i^{th} column. This proves the theorem.

3.3.6 A Formula For The Inverse

Note that this gives an easy way to write a formula for the inverse of an $n \times n$ matrix. Recall the definition of the inverse of a matrix in Definition 2.1.21 on Page 45.

Theorem 3.3.15 *A^{-1} exists if and only if $\det(A) \neq 0$. If $\det(A) \neq 0$, then $A^{-1} = \left(a_{ij}^{-1}\right)$ where*

$$a_{ij}^{-1} = \det(A)^{-1} \operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ the ij^{th} cofactor of A.

Proof: By Theorem 3.3.14 and letting $(a_{ir}) = A$, if $\det(A) \neq 0$,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Corollary 3.3.7. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Using the other formula in Theorem 3.3.14, and similar reasoning,

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

This proves that if $\det(A) \neq 0$, then A^{-1} exists with $A^{-1} = \left(a_{ij}^{-1}\right)$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

Now suppose A^{-1} exists. Then by Theorem 3.3.11,

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A) \det\left(A^{-1}\right)$$

so $\det(A) \neq 0$. This proves the theorem.

The next corollary points out that if an $n \times n$ matrix, A has a right or a left inverse, then it has an inverse.

Corollary 3.3.16 *Let A be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix, B such that $BA = I$. Then A^{-1} exists and $A^{-1} = B$. Also, if there exists C an $n \times n$ matrix such that $AC = I$, then A^{-1} exists and $A^{-1} = C$.*

Proof: Since $BA = I$, Theorem 3.3.11 implies

$$\det B \det A = 1$$

and so $\det A \neq 0$. Therefore from Theorem 3.3.15, A^{-1} exists. Therefore,

$$A^{-1} = (BA) A^{-1} = B\left(AA^{-1}\right) = BI = B.$$

The case where $CA = I$ is handled similarly.

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 3.3.15 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A.

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det(A)} \operatorname{cof}(A)_{ji}\, y_j.$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

Definition 3.3.17 *A matrix M, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii} as shown.*

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 3.3.14.

Corollary 3.3.18 *Let M be an upper (lower) triangular matrix. Then $\det(M)$ is obtained by taking the product of the entries on the main diagonal.*

3.3.7 Rank Of A Matrix

Definition 3.3.19 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A. Let A be an $m \times n$ matrix. The* ***determinant rank*** *of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The* ***row rank*** *is defined to be the dimension of the span of the rows. The* ***column rank*** *is defined to be the dimension of the span of the columns.*

Theorem 3.3.20 *If A, an $m \times n$ matrix has determinant rank, r, then there exist r rows of the matrix such that every other row is a linear combination of these r rows.*

Proof: Suppose the determinant rank of $A = (a_{ij})$ equals r. Thus some $r \times r$ submatrix has non zero determinant and there is no larger square submatrix which has non zero determinant. Suppose such a submatrix is determined by the r columns whose indices are

$$j_1 < \cdots < j_r$$

and the r rows whose indices are

$$i_1 < \cdots < i_r$$

I want to show that every row is a linear combination of these rows. Consider the l^{th} row and let p be an index between 1 and n. Form the following $(r+1) \times (r+1)$ matrix

$$\begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_r} & a_{i_1 p} \\ \vdots & & \vdots & \vdots \\ a_{i_r j_1} & \cdots & a_{i_r j_r} & a_{i_r p} \\ a_{l j_1} & \cdots & a_{l j_r} & a_{lp} \end{pmatrix}$$

Of course you can assume $l \notin \{i_1, \cdots, i_r\}$ because there is nothing to prove if the l^{th} row is one of the chosen ones. The above matrix has determinant 0. This is because if $p \notin \{j_1, \cdots, j_r\}$ then the above would be a submatrix of A which is too large to have non zero determinant. On the other hand, if $p \in \{j_1, \cdots, j_r\}$ then the above matrix has two columns which are equal so its determinant is still 0.

Expand the determinant of the above matrix along the last column. Let C_k denote the cofactor associated with the entry $a_{i_k p}$. This is not dependent on the choice of p. Remember, you delete the column and the row the entry is in and take the determinant of what is left and multiply by -1 raised to an appropriate power. Let C denote the cofactor associated with a_{lp}. This is given to be nonzero, it being the determinant of the matrix

$$\begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_r} \\ \vdots & & \vdots \\ a_{i_r j_1} & \cdots & a_{i_r j_r} \end{pmatrix}$$

Thus

$$0 = a_{lp} C + \sum_{k=1}^{r} C_k a_{i_k p}$$

which implies

$$a_{lp} = \sum_{k=1}^{r} \frac{-C_k}{C} a_{i_k p} \equiv \sum_{k=1}^{r} m_k a_{i_k p}$$

Since this is true for every p and since m_k does not depend on p, this has shown the l^{th} row is a linear combination of the $i_1, i_2, \cdots, i_r$ rows. This proves the theorem.

Corollary 3.3.21 *The determinant rank equals the row rank.*

Proof: From Theorem 3.3.20, every row is in the span of r rows where r is the determinant rank. Therefore, the row rank (dimension of the span of the rows) is no larger than the determinant rank. Could the row rank be smaller than the determinant rank? If so, it follows from Theorem 3.3.20 there exist p rows for $p < r \equiv$ determinant rank, such that the span of these p rows equals the row space. But then you could consider the $r \times r$ sub matrix which determines the determinant rank and it would follow that each of these rows would be in the span of the p rows just mentioned. By Theorem 2.4.3, the exchange theorem, the rows of this sub matrix would not be linearly independent and so some row is a linear combination of the others. By Corollary 3.3.9 the determinant would be 0, a contradiction. This proves the corollary.

Corollary 3.3.22 *If A has determinant rank, r, then there exist r columns of the matrix such that every other column is a linear combination of these r columns. Also the column rank equals the determinant rank.*

Proof: This follows from the above by considering A^T. The rows of A^T are the columns of A and the determinant rank of A^T and A are the same. Therefore, from Corollary 3.3.21, column rank of A = row rank of A^T = determinant rank of A^T = determinant rank of A.

The following theorem is of fundamental importance and ties together many of the ideas presented above.

Theorem 3.3.23 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) = 0$.
2. A, A^T *are not one to one.*
3. A *is not onto.*

Proof: Suppose $\det(A) = 0$. Then the determinant rank of $A = r < n$. Therefore, there exist r columns such that every other column is a linear combination of these columns by Theorem 3.3.20. In particular, it follows that for some m, the m^{th} column is a linear combination of all the others. Thus letting $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m & \cdots & \mathbf{a}_n \end{pmatrix}$ where the columns are denoted by $\mathbf{a}_i$, there exists scalars, α_i such that

$$\mathbf{a}_m = \sum_{k \neq m} \alpha_k \mathbf{a}_k.$$

Now consider the column vector, $\mathbf{x} \equiv \begin{pmatrix} \alpha_1 & \cdots & -1 & \cdots & \alpha_n \end{pmatrix}^T$. Then

$$A\mathbf{x} = -\mathbf{a}_m + \sum_{k \neq m} \alpha_k \mathbf{a}_k = \mathbf{0}.$$

Since also $A\mathbf{0} = \mathbf{0}$, it follows A is not one to one. Similarly, A^T is not one to one by the same argument applied to A^T. This verifies that 1.) implies 2.).

Now suppose 2.). Then since A^T is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A^T\mathbf{x} = \mathbf{0}.$$

Taking the transpose of both sides yields

$$\mathbf{x}^T A = \mathbf{0}^T$$

where the $\mathbf{0}^T$ is a $1 \times n$ matrix or row vector. Now if $A\mathbf{y} = \mathbf{x}$, then

$$|\mathbf{x}|^2 = \mathbf{x}^T (A\mathbf{y}) = \left(\mathbf{x}^T A\right) \mathbf{y} = \mathbf{0}\mathbf{y} = 0$$

contrary to $\mathbf{x} \neq \mathbf{0}$. Consequently there can be no $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$ and so A is not onto. This shows that 2.) implies 3.).

Finally, suppose 3.). If 1.) does not hold, then $\det(A) \neq 0$ but then from Theorem 3.3.15 A^{-1} exists and so for every $\mathbf{y} \in \mathbb{F}^n$ there exists a unique $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{y}$. In fact $\mathbf{x} = A^{-1}\mathbf{y}$. Thus A would be onto contrary to 3.). This shows 3.) implies 1.) and proves the theorem.

Corollary 3.3.24 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *det$(A) \neq 0$.*
2. *A and A^T are one to one.*
3. *A is onto.*

Proof: This follows immediately from the above theorem.

3.3.8 Summary Of Determinants

In all the following A, B are $n \times n$ matrices

1. $\det(A)$ is a number.
2. $\det(A)$ is linear in each row and in each column.
3. If you switch two rows or two columns, the determinant of the resulting matrix is -1 times the determinant of the unswitched matrix. (This and the previous one say

   $$(\mathbf{a}_1 \cdots \mathbf{a}_n) \to \det(\mathbf{a}_1 \cdots \mathbf{a}_n)$$

   is an alternating multilinear function or alternating tensor.
4. $\det(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$.
5. $\det(AB) = \det(A)\det(B)$
6. $\det(A)$ can be expanded along any row or any column and the same result is obtained.
7. $\det(A) = \det\left(A^T\right)$
8. A^{-1} exists if and only if $\det(A) \neq 0$ and in this case

   $$\left(A^{-1}\right)_{ij} = \frac{1}{\det(A)} \operatorname{cof}(A)_{ji} \tag{3.18}$$

9. Determinant rank, row rank and column rank are all the same number for any $m \times n$ matrix.

3.4 The Cayley Hamilton Theorem

Definition 3.4.1 *Let A be an $n \times n$ matrix. The characteristic polynomial is defined as*

$$p_A(t) \equiv \det(tI - A)$$

and the solutions to $p_A(t) = 0$ are called eigenvalues. For A a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

The explanation for the last term is that A^0 is interpreted as I, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $p_A(t) = 0$. It is one of the most important theorems in linear algebra[1]. The following lemma will help with its proof.

Lemma 3.4.2 *Suppose for all $|\lambda|$ large enough,*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = 0,$$

where the A_i are $n \times n$ matrices. Then each $A_i = 0$.

Proof: Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \cdots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \to \infty$ to obtain that all the $A_i = 0$. This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 3.4.3 *Let A_i and B_i be $n \times n$ matrices and suppose*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = B_0 + B_1\lambda + \cdots + B_m\lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all i. Consequently if λ is replaced by any $n \times n$ matrix, the two sides will be equal. That is, for C any $n \times n$ matrix,

$$A_0 + A_1C + \cdots + A_mC^m = B_0 + B_1C + \cdots + B_mC^m.$$

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 3.4.4 *Let A be an $n \times n$ matrix and let $p(\lambda) \equiv \det(\lambda I - A)$ be the characteristic polynomial. Then $p(A) = 0$.*

[1] A special case was first proved by Hamilton in 1853. The general case was announced by Cayley some time later and a proof was given by Frobenius in 1878.

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then λ cannot be in the finite list of eigenvalues of A and so for such λ, $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem 3.3.15

$$C(\lambda) = p(\lambda)(\lambda I - A)^{-1}.$$

Note that each entry in $C(\lambda)$ is a polynomial in λ having degree no more than $n-1$. Therefore, collecting the terms,

$$C(\lambda) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

for C_j some $n \times n$ matrix. It follows that for all $|\lambda|$ large enough,

$$(\lambda I - A)\left(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}\right) = p(\lambda) I$$

and so Corollary 3.4.3 may be used. It follows the matrix coefficients corresponding to equal powers of λ are equal on both sides of this equation. Therefore, if λ is replaced with A, the two sides will be equal. Thus

$$0 = (A - A)\left(C_0 + C_1 A + \cdots + C_{n-1}A^{n-1}\right) = p(A) I = p(A).$$

This proves the Cayley Hamilton theorem.

3.5 Block Multiplication Of Matrices

Consider the following problem

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

You know how to do this. You get

$$\begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

Now what if instead of numbers, the entries, A, B, C, D, E, F, G are matrices of a size such that the multiplications and additions needed in the above formula all make sense. Would the formula be true in this case? I will show below that this is true.

Suppose A is a matrix of the form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rm} \end{pmatrix} \tag{3.19}$$

where A_{ij} is a $s_i \times p_j$ matrix where s_i is constant for $j = 1, \cdots, m$ for each $i = 1, \cdots, r$. Such a matrix is called a **block matrix,** also a **partitioned matrix**. How do you get the block A_{ij}? Here is how for A an $m \times n$ matrix:

$$\overbrace{\begin{pmatrix} \mathbf{0} & I_{s_i \times s_i} & \mathbf{0} \end{pmatrix}}^{s_i \times m} A \overbrace{\begin{pmatrix} \mathbf{0} \\ I_{p_j \times p_j} \\ \mathbf{0} \end{pmatrix}}^{n \times p_j}. \tag{3.20}$$

In the block column matrix on the right, you need to have $c_j - 1$ rows of zeros above the small $p_j \times p_j$ identity matrix where the columns of A involved in A_{ij} are $c_j, \cdots, c_j + p_j - 1$ and in the block row matrix on the left, you need to have $r_i - 1$ columns of zeros to the left of the $s_i \times s_i$ identity matrix where the rows of A involved in A_{ij} are $r_i, \cdots, r_i + s_i$. An important observation to make is that the matrix on the right specifies columns to use in the block and the one on the left specifies the rows used. Thus the block A_{ij} in this case is a matrix of size $s_i \times p_j$. There is no overlap between the blocks of A. Thus the identity $n \times n$ identity matrix corresponding to multiplication on the right of A is of the form

$$\begin{pmatrix} I_{p_1 \times p_1} & & 0 \\ & \ddots & \\ 0 & & I_{p_m \times p_m} \end{pmatrix}$$

where these little identity matrices don't overlap. A similar conclusion follows from consideration of the matrices $I_{s_i \times s_i}$. Note that in 3.20 the matrix on the right is a block column matrix for the above block diagonal matrix and the matrix on the left in 3.20 is a block row matrix taken from a similar block diagonal matrix consisting of the $I_{s_i \times s_i}$.

Next consider the question of multiplication of two block matrices. Let B be a block matrix of the form

$$\begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rp} \end{pmatrix} \tag{3.21}$$

and A is a block matrix of the form

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pm} \end{pmatrix} \tag{3.22}$$

and that for all i, j, it makes sense to multiply $B_{is}A_{sj}$ for all $s \in \{1, \cdots, p\}$. (That is the two matrices, B_{is} and A_{sj} are conformable.) and that for fixed ij, it follows $B_{is}A_{sj}$ is the same size for each s so that it makes sense to write $\sum_s B_{is}A_{sj}$.

The following theorem says essentially that when you take the product of two matrices, you can do it two ways. One way is to simply multiply them forming BA. The other way is to partition both matrices, formally multiply the blocks to get another block matrix and this one will be BA partitioned. Before presenting this theorem, here is a simple lemma which is really a special case of the theorem.

Lemma 3.5.1 *Consider the following product.*

$$\begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & I & \mathbf{0} \end{pmatrix}$$

where the first is $n \times r$ and the second is $r \times n$. The small identity matrix I is an $r \times r$ matrix and there are l zero rows above I and l zero columns to the left of I in the right matrix. Then the product of these matrices is a block matrix of the form

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Proof: From the definition of the way you multiply matrices, the product is

$$\left(\left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{0} \quad \cdots \left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{0} \quad \left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{e}_1 \quad \cdots \quad \left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{e}_r \quad \left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{0} \quad \cdots \quad \left(\begin{array}{c} \mathbf{0} \\ I \\ \mathbf{0} \end{array} \right) \mathbf{0} \right)$$

which yields the claimed result. In the formula $\mathbf{e}_j$ refers to the column vector of length r which has a 1 in the j^{th} position. This proves the lemma.

Theorem 3.5.2 *Let B be a $q \times p$ block matrix as in 3.21 and let A be a $p \times n$ block matrix as in 3.22 such that B_{is} is conformable with A_{sj} and each product, $B_{is}A_{sj}$ for $s = 1, \cdots, p$ is of the same size so they can be added. Then BA can be obtained as a block matrix such that the ij^{th} block is of the form*

$$\sum_s B_{is}A_{sj}. \tag{3.23}$$

Proof: From 3.20

$$B_{is}A_{sj} = \left(\begin{array}{ccc} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{array} \right) B \left(\begin{array}{c} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{array} \right) \left(\begin{array}{ccc} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{array} \right) A \left(\begin{array}{c} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{array} \right)$$

where here it is assumed B_{is} is $r_i \times p_s$ and A_{sj} is $p_s \times q_j$. The product involves the s^{th} block in the i^{th} row of blocks for B and the s^{th} block in the j^{th} column of A. Thus there are the same number of rows above the $I_{p_s \times p_s}$ as there are columns to the left of $I_{p_s \times p_s}$ in those two inside matrices. Then from Lemma 3.5.1

$$\left(\begin{array}{c} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{array} \right) \left(\begin{array}{ccc} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{array} \right) = \left(\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right)$$

Since the blocks of small identity matrices do not overlap,

$$\sum_s \left(\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) = \left(\begin{array}{ccc} I_{p_1 \times p_1} & & 0 \\ & \ddots & \\ 0 & & I_{p_p \times p_p} \end{array} \right) = I$$

and so

$$\sum_s B_{is}A_{sj} =$$

$$\sum_s \left(\begin{array}{ccc} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{array} \right) B \left(\begin{array}{c} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{array} \right) \left(\begin{array}{ccc} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{array} \right) A \left(\begin{array}{c} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{array} \right)$$

$$= \left(\begin{array}{ccc} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{array} \right) B \sum_s \left(\begin{array}{c} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{array} \right) \left(\begin{array}{ccc} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{array} \right) A \left(\begin{array}{c} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{array} \right)$$

$$= \left(\begin{array}{ccc} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{array} \right) BIA \left(\begin{array}{c} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{array} \right) = \left(\begin{array}{ccc} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{array} \right) BA \left(\begin{array}{c} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{array} \right)$$

which equals the ij^{th} block of BA. Hence the ij^{th} block of BA equals the formal multiplication according to matrix multiplication,

$$\sum_s B_{is}A_{sj}.$$

This proves the theorem.

Example 3.5.3 *Let an $n \times n$ matrix have the form*

$$A = \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & P \end{pmatrix}$$

where P is $n-1 \times n-1$. Multiply it by

$$B = \begin{pmatrix} p & \mathbf{q} \\ \mathbf{r} & Q \end{pmatrix}$$

where B is also an $n \times n$ matrix and Q is $n-1 \times n-1$.

You use block multiplication

$$\begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & P \end{pmatrix} \begin{pmatrix} p & \mathbf{q} \\ \mathbf{r} & Q \end{pmatrix} = \begin{pmatrix} ap + \mathbf{br} & a\mathbf{q} + \mathbf{b}Q \\ p\mathbf{c} + P\mathbf{r} & \mathbf{cq} + PQ \end{pmatrix}$$

Note that this all makes sense. For example, $\mathbf{b} = 1 \times n-1$ and $\mathbf{r} = n-1 \times 1$ so $\mathbf{br}$ is a 1×1. Similar considerations apply to the other blocks.

Here is an interesting and significant application of block multiplication. In this theorem, $p_M(t)$ denotes the characteristic polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, M.

Theorem 3.5.4 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m} p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n-m$ extra zero eigenvalues. Here $p_A(t)$ denotes the characteristic polynomial of the matrix A.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

Since the two matrices above are similar it follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials. Therefore, noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

3.6 Exercises

1. ♠Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one. **Hint:** Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n-m) \times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$.

2. Show that matrix multiplication is associative. That is, $(AB)\,C = A\,(BC)$.

3. Show the inverse of a matrix, if it exists, is unique. Thus if $AB = BA = I$, then $B = A^{-1}$.

4. In the proof of Theorem 3.3.15 it was claimed that $\det(I) = 1$. Here $I = (\delta_{ij})$. Prove this assertion. Also prove Corollary 3.3.18.

5. Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be vectors in $\mathbb{F}^n$ and let $M(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ denote the matrix whose i^{th} column equals $\mathbf{v}_i$. Define

$$d(\mathbf{v}_1, \cdots, \mathbf{v}_n) \equiv \det(M(\mathbf{v}_1, \cdots, \mathbf{v}_n)).$$

Prove that d is linear in each variable, (multilinear), that

$$d(\mathbf{v}_1, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots, \mathbf{v}_n) = -d(\mathbf{v}_1, \cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_n), \tag{3.24}$$

and

$$d(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1 \tag{3.25}$$

where here $\mathbf{e}_j$ is the vector in $\mathbb{F}^n$ which has a zero in every position except the j^{th} position in which it has a one.

6. ♠Suppose $f : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ satisfies 3.24 and 3.25 and is linear in each variable. Show that $f = d$.

7. Show that if you replace a row (column) of an $n \times n$ matrix A with itself added to some multiple of another row (column) then the new matrix has the same determinant as the original one.

8. If $A = (a_{ij})$, show $\det(A) = \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_n)\, a_{k_1 1} \cdots a_{k_n n}$.

9. ♠Use the result of Problem 7 to evaluate by hand the determinant

$$\det \begin{pmatrix} 1 & 2 & 3 & 2 \\ -6 & 3 & 2 & 3 \\ 5 & 2 & 2 & 3 \\ 3 & 4 & 6 & 4 \end{pmatrix}.$$

10. Find the inverse if it exists of the matrix,

$$\begin{pmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{pmatrix}.$$

11. ♠Let $Ly = y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y$ where the a_i are given continuous functions defined on a closed interval, (a, b) and y is some function which has n derivatives so it makes sense to write Ly. Suppose $Ly_k = 0$ for $k = 1, 2, \cdots, n$. The Wronskian of these functions, y_i is defined as

$$W(y_1, \cdots, y_n)(x) \equiv \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

Show that for $W(x) = W(y_1, \cdots, y_n)(x)$ to save space,

$$W'(x) = \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{pmatrix}.$$

Now use the differential equation, $Ly = 0$ which is satisfied by each of these functions, y_i and properties of determinants presented above to verify that $W' + a_{n-1}(x) W = 0$. Give an explicit solution of this linear differential equation, Abel's formula, and use your answer to verify that the Wronskian of these solutions to the equation, $Ly = 0$ either vanishes identically on (a, b) or never.

12. ♠Two $n \times n$ matrices, A and B, are similar if $B = S^{-1}AS$ for some invertible $n \times n$ matrix, S. Show that if two matrices are similar, they have the same characteristic polynomials. The characteristic polynomial of A is $\det(\lambda I - A)$.

13. ♠Suppose the characteristic polynomial of an $n \times n$ matrix, A is of the form

$$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

and that $a_0 \neq 0$. Find a formula A^{-1} in terms of powers of the matrix, A. Show that A^{-1} exists if and only if $a_0 \neq 0$.

14. In constitutive modeling of the stress and strain tensors, one sometimes considers sums of the form $\sum_{k=0}^{\infty} a_k A^k$ where A is a 3×3 matrix. Show using the Cayley Hamilton theorem that if such a thing makes any sense, you can always obtain it as a finite sum having no more than n terms.

15. ♠Recall you can find the determinant from expanding along the j^{th} column.

$$\det(A) = \sum_i A_{ij} (cof(A))_{ij}$$

Think of $\det(A)$ as a function of the entries, A_{ij}. Explain why the ij^{th} cofactor is really just

$$\frac{\partial \det(A)}{\partial A_{ij}}.$$

16. ♠Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{g}:U \to \mathbb{R}^n$ be such that all the first partial derivatives of all components of $\mathbf{g}$ exist and are continuous. Under these conditions form the matrix $D\mathbf{g}(\mathbf{x})$ given by

$$D\mathbf{g}(\mathbf{x})_{ij} \equiv \frac{\partial g_i(\mathbf{x})}{\partial x_j} \equiv g_{i,j}(\mathbf{x})$$

The best kept secret in calculus courses is that the linear transformation determined by this matrix $D\mathbf{g}(\mathbf{x})$ is called the derivative of $\mathbf{g}$ and is the correct generalization of the concept of derivative of a function of one variable. Suppose the second partial derivatives also exist and are continuous. Then show that

$$\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} = 0.$$

Hint: First explain why

$$\sum_i g_{i,k} \operatorname{cof}(D\mathbf{g})_{ij} = \delta_{jk} \det(D\mathbf{g})$$

Next differentiate with respect to x_j and sum on j using the equality of mixed partial derivatives. Assume $\det(D\mathbf{g}) \neq 0$ to prove the identity in this special case. Then explain why there exists a sequence $\varepsilon_k \to 0$ such that for $\mathbf{g}_{\varepsilon_k}(\mathbf{x}) \equiv \mathbf{g}(\mathbf{x}) + \varepsilon_k \mathbf{x}$, $\det(D\mathbf{g}_{\varepsilon_k}) \neq 0$ and so the identity holds for $\mathbf{g}_{\varepsilon_k}$. Then take a limit to get the desired result in general. This is an extremely important identity which has surprising implications.

17. ♠A determinant of the form

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix}$$

is called a Vandermonde determinant. Show this determinant equals

$$\prod_{0 \le i < j \le n} (a_j - a_i)$$

By this is meant to take the product of all terms of the form $(a_j - a_i)$ such that $j > i$. **Hint:** Show it works if $n = 1$ so you are looking at

$$\begin{vmatrix} 1 & 1 \\ a_0 & a_1 \end{vmatrix}$$

Then suppose it holds for $n-1$ and consider the case n. Consider the following polynomial.

$$p(t) \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & t \\ a_0^2 & a_1^2 & \cdots & t^2 \\ \vdots & \vdots & & \vdots \\ a_0^{n-1} & a_1^{n-1} & \cdots & t^{n-1} \\ a_0^n & a_1^n & \cdots & t^n \end{vmatrix}.$$

Explain why $p(a_j) = 0$ for $i = 0, \cdots, n-1$. Thus

$$p(t) = c \prod_{i=0}^{n-1} (t - a_i).$$

Of course c is the coefficient of t^n. Find this coefficient from the above description of $p(t)$ and the induction hypothesis. Then plug in $t = a_n$ and observe you have the formula valid for n.

Row Operations

4.1 Elementary Matrices

The elementary matrices result from doing a row operation to the identity matrix.

Definition 4.1.1 *The row operations consist of the following*

1. *Switch two rows.*

2. *Multiply a row by a nonzero number.*

3. *Replace a row by a multiple of another row added to it.*

The elementary matrices are given in the following definition.

Definition 4.1.2 *The elementary matrices consist of those matrices which result by applying a row operation to an identity matrix. Those which involve switching rows of the identity are called permutation matrices*[1]*.*

As an example of why these elementary matrices are interesting, consider the following.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ x & y & z & w \\ f & g & h & i \end{pmatrix} = \begin{pmatrix} x & y & z & w \\ a & b & c & d \\ f & g & h & i \end{pmatrix}$$

A 3×4 matrix was multiplied on the left by an elementary matrix which was obtained from row operation 1 applied to the identity matrix. This resulted in applying the operation 1 to the given matrix. This is what happens in general.

[1] More generally, a permutation matrix is a matrix which comes by permuting the rows of the identity matrix, not just switching two rows.

Now consider what these elementary matrices look like. First consider the one which involves switching row i and row j where $i < j$. This matrix is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & & & & & \vdots \\ \vdots & & 1 & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & 1 & 0 & \cdots & 0 & & & \vdots \\ \vdots & & & \vdots & & \ddots & & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & & & 1 & & \vdots \\ \vdots & & & & & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

The two exceptional rows are shown. The i^{th} row was the j^{th} and the j^{th} row was the i^{th} in the identity matrix. Now consider what this does to a column vector.

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & & & & & \vdots \\ \vdots & & 1 & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & 1 & 0 & \cdots & 0 & & & \vdots \\ \vdots & & & \vdots & & \ddots & & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & & & 1 & & \vdots \\ \vdots & & & & & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_i \\ \vdots \\ \vdots \\ \vdots \\ v_j \\ \vdots \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_j \\ \vdots \\ \vdots \\ \vdots \\ v_i \\ \vdots \\ \vdots \\ v_n \end{pmatrix}$$

Now denote by P^{ij} the elementary matrix which comes from the identity from switching rows i and j. From what was just explained consider multiplication on the left by this elementary matrix.

$$P^{ij} \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & \cdots & a_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{j1} & a_{j2} & \cdots & \cdots & \cdots & \cdots & a_{jp} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

From the way you multiply matrices this is a matrix which has the indicated columns.

$$\begin{pmatrix} P^{ij}\begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{j1} \\ \vdots \\ a_{n1} \end{pmatrix}, P^{ij}\begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{j2} \\ \vdots \\ a_{n2} \end{pmatrix}, \cdots, P^{ij}\begin{pmatrix} a_{1p} \\ \vdots \\ a_{ip} \\ \vdots \\ a_{jp} \\ \vdots \\ a_{np} \end{pmatrix} \end{pmatrix}$$

$$= \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{j1} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{j2} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{n2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1p} \\ \vdots \\ a_{jp} \\ \vdots \\ a_{ip} \\ \vdots \\ a_{np} \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ a_{j1} & a_{j2} & \cdots & \cdots & \cdots & \cdots & a_{jp} \\ \vdots & \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & \cdots & a_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

This has established the following lemma.

Lemma 4.1.3 *Let P^{ij} denote the elementary matrix which involves switching the i^{th} and the j^{th} rows. Then*

$$P^{ij}A = B$$

where B is obtained from A by switching the i^{th} and the j^{th} rows.

Next consider the row operation which involves multiplying the i^{th} row by a nonzero constant, c. The elementary matrix which results from applying this operation to the i^{th} row of the identity matrix is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & 1 & & & & \vdots \\ \vdots & & & c & & & \vdots \\ \vdots & & & & 1 & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Now consider what this does to a column vector.

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & 1 & & & & \vdots \\ \vdots & & & c & & & \vdots \\ \vdots & & & & 1 & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_{i-1} \\ cv_i \\ v_{i+1} \\ \vdots \\ v_n \end{pmatrix}$$

Denote by $E(c,i)$ this elementary matrix which multiplies the i^{th} row of the identity by the nonzero constant, c. Then from what was just discussed and the way matrices are multiplied,

$$E(c,i) \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & \cdots & a_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{j2} & a_{j2} & \cdots & \cdots & \cdots & \cdots & a_{jp} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

equals a matrix having the columns indicated below.

$$= \left(E(c,i) \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{j1} \\ \vdots \\ a_{n1} \end{pmatrix}, E(c,i) \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{j2} \\ \vdots \\ a_{n2} \end{pmatrix}, \cdots, E(c,i) \begin{pmatrix} a_{1p} \\ \vdots \\ a_{ip} \\ \vdots \\ a_{jp} \\ \vdots \\ a_{np} \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ ca_{i1} & ca_{i2} & \cdots & \cdots & \cdots & \cdots & ca_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{j2} & a_{j2} & \cdots & \cdots & \cdots & \cdots & a_{jp} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

This proves the following lemma.

Lemma 4.1.4 *Let $E(c,i)$ denote the elementary matrix corresponding to the row operation in which the i^{th} row is multiplied by the nonzero constant, c. Thus $E(c,i)$ involves multiplying the i^{th} row of the identity matrix by c. Then*

$$E(c,i)\,A = B$$

where B is obtained from A by multiplying the i^{th} row of A by c.

Finally consider the third of these row operations. Denote by $E(c \times i + j)$ the elementary matrix which replaces the j^{th} row with itself added to c times the i^{th} row added to it. In case $i < j$ this will be of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & 1 & & & & \vdots \\ \vdots & & \vdots & \ddots & & & \vdots \\ \vdots & & c & \cdots & 1 & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Now consider what this does to a column vector.

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & 1 & & & & \vdots \\ \vdots & & \vdots & \ddots & & & \vdots \\ \vdots & & c & \cdots & 1 & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ cv_i + v_j \\ \vdots \\ v_n \end{pmatrix}$$

Now from this and the way matrices are multiplied,

$$E(c \times i + j) \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & \cdots & a_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{j2} & a_{j2} & \cdots & \cdots & \cdots & \cdots & a_{jp} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

equals a matrix of the following form having the indicated columns.

$$\left(E(c \times i + j) \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{j2} \\ \vdots \\ a_{n1} \end{pmatrix}, E(c \times i + j) \begin{pmatrix} a_{12} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{j2} \\ \vdots \\ a_{n2} \end{pmatrix}, \cdots E(c \times i + j) \begin{pmatrix} a_{1p} \\ \vdots \\ a_{ip} \\ \vdots \\ a_{jp} \\ \vdots \\ a_{np} \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1p} \\ \vdots & \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & \cdots & a_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{j2}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & \cdots & \cdots & \cdots & a_{jp}+ca_{ip} \\ \vdots & \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & \cdots & \cdots & a_{np} \end{pmatrix}$$

The case where $i > j$ is handled similarly. This proves the following lemma.

Lemma 4.1.5 *Let $E(c \times i + j)$ denote the elementary matrix obtained from I by replacing the j^{th} row with c times the i^{th} row added to it. Then*

$$E(c \times i + j) A = B$$

where B is obtained from A by replacing the j^{th} row of A with itself added to c times the i^{th} row of A.

The next theorem is the main result.

Theorem 4.1.6 *To perform any of the three row operations on a matrix, A it suffices to do the row operation on the identity matrix obtaining an elementary matrix, E and then take the product, EA. Furthermore, each elementary matrix is invertible and its inverse is an elementary matrix.*

Proof: The first part of this theorem has been proved in Lemmas 4.1.3 - 4.1.5. It only remains to verify the claim about the inverses. Consider first the elementary matrices corresponding to row operation of type three.

$$E(-c \times i + j) E(c \times i + j) = I$$

This follows because the first matrix takes c times row i in the identity and adds it to row j. When multiplied on the left by $E(-c \times i + j)$ it follows from the first part of this theorem that you take the i^{th} row of $E(c \times i + j)$ which coincides with the i^{th} row of I since that row was not changed, multiply it by $-c$ and add to the j^{th} row of $E(c \times i + j)$ which was the j^{th} row of I added to c times the i^{th} row of I. Thus $E(-c \times i + j)$ multiplied on the left, undoes the row operation which resulted in $E(c \times i + j)$. The same argument applied to the product

$$E(c \times i + j) E(-c \times i + j)$$

replacing c with $-c$ in the argument yields that this product is also equal to I. Therefore, $E(c \times i + j)^{-1} = E(-c \times i + j)$.

Similar reasoning shows that for $E(c, i)$ the elementary matrix which comes from multiplying the i^{th} row by the nonzero constant, c,

$$E(c, i)^{-1} = E\left(c^{-1}, i\right).$$

Finally, consider P^{ij} which involves switching the i^{th} and the j^{th} rows.

$$P^{ij} P^{ij} = I$$

because by the first part of this theorem, multiplying on the left by P^{ij} switches the i^{th} and j^{th} rows of P^{ij} which was obtained from switching the i^{th} and j^{th} rows of the identity.

First you switch them to get P^{ij} and then you multiply on the left by P^{ij} which switches these rows again and restores the identity matrix. Thus $\left(P^{ij}\right)^{-1} = P^{ij}$.

In rough terms, the following lemma states that **linear relationships** between columns in a matrix are preserved by row operations. This simple lemma is the main result in understanding all the major questions related to the row reduced echelon form as well as many other topics.

4.2 The Rank Of A Matrix

Recall the following definition of rank of a matrix.

Definition 4.2.1 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A. Let A be an $m \times n$ matrix. The **determinant rank** of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The **row rank** is defined to be the dimension of the span of the rows. The **column rank** is defined to be the dimension of the span of the columns.*

The following theorem is proved in the section on the theory of the determinant and is restated here for convenience.

Theorem 4.2.2 *Let A be an $m \times n$ matrix. Then the row rank, column rank and determinant rank are all the same.*

So how do you find the rank? It turns out that row operations are the key to the practical computation of the rank of a matrix.

In rough terms, the following lemma states that linear relationships between columns in a matrix are preserved by row operations.

Lemma 4.2.3 *Let B and A be two $m \times n$ matrices and suppose B results from a row operation applied to A. Then the k^{th} column of B is a linear combination of the $i_1, \cdots, i_r$ columns of B if and only if the k^{th} column of A is a linear combination of the $i_1, \cdots, i_r$ columns of A. Furthermore, the scalars in the linear combination are the same. (The linear relationship between the k^{th} column of A and the $i_1, \cdots, i_r$ columns of A is the same as the linear relationship between the k^{th} column of B and the $i_1, \cdots, i_r$ columns of B.)*

Proof: Let A equal the following matrix in which the $\mathbf{a}_k$ are the columns

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$$

and let B equal the following matrix in which the columns are given by the $\mathbf{b}_k$

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$$

Then by Theorem 4.1.6 on Page 110 $\mathbf{b}_k = E\mathbf{a}_k$ where E is an elementary matrix. Suppose then that one of the columns of A is a linear combination of some other columns of A. Say

$$\mathbf{a}_k = \sum_{r \in S} c_r \mathbf{a}_r.$$

Then multiplying by E,

$$\mathbf{b}_k = E\mathbf{a}_k = \sum_{r \in S} c_r E\mathbf{a}_r = \sum_{r \in S} c_r \mathbf{b}_r.$$

This proves the lemma.

Corollary 4.2.4 *Let A and B be two $m \times n$ matrices such that B is obtained by applying a row operation to A. Then the two matrices have the same rank.*

Proof: Lemma 4.2.3 says the linear relationships are the same between the columns of A and those of B. Therefore, the column rank of the two matrices is the same.

This suggests that to find the rank of a matrix, one should do row operations until a matrix is obtained in which its rank is obvious.

Example 4.2.5 *Find the rank of the following matrix and identify columns whose linear combinations yield all the other columns.*

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{pmatrix} \tag{4.1}$$

Take (-1) times the first row and add to the second and then take (-3) times the first row and add to the third. This yields

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 1 & 5 & -3 & 0 \end{pmatrix}$$

By the above corollary, this matrix has the same rank as the first matrix. Now take (-1) times the second row and add to the third row yielding

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

At this point it is clear the rank is 2. This is because every column is in the span of the first two and these first two columns are linearly independent.

Example 4.2.6 *Find the rank of the following matrix and identify columns whose linear combinations yield all the other columns.*

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 2 & 6 & 0 & 2 \\ 3 & 6 & 8 & 6 & 6 \end{pmatrix} \tag{4.2}$$

Take (-1) times the first row and add to the second and then take (-3) times the first row and add to the last row. This yields

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & 5 & -3 & 0 \\ 0 & 0 & 5 & -3 & 0 \end{pmatrix}$$

Now multiply the second row by $1/5$ and add 5 times it to the last row.

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Add (-1) times the second row to the first.

$$\begin{pmatrix} 1 & 2 & 0 & \frac{18}{5} & 2 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.3}$$

It is now clear the rank of this matrix is 2 because the first and third columns form a basis for the column space.

The matrix, 4.3 is the row reduced echelon form for the matrix, 4.2.

4.3 The Row Reduced Echelon Form

The following definition is for the row reduced echelon form of a matrix.

Definition 4.3.1 *Let $\mathbf{e}_i$ denote the column vector which has all zero entries except for the i^{th} slot which is one. An $m \times n$ matrix is said to be in row reduced echelon form if, in viewing successive columns from left to right, the first nonzero column encountered is $\mathbf{e}_1$ and if you have encountered $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k$, the next column is either $\mathbf{e}_{k+1}$ or is a linear combination of the vectors, $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k$.*

Theorem 4.3.2 *Let A be an $m \times n$ matrix. Then A has a row reduced echelon form determined by a simple process.*

Proof: Viewing the columns of A from left to right take the first nonzero column. Pick a nonzero entry in this column and switch the row containing this entry with the top row of A. Now divide this new top row by the value of this nonzero entry to get a 1 in this position and then use row operations to make all entries below this entry equal to zero. Thus the first nonzero column is now $\mathbf{e}_1$. Denote the resulting matrix by A_1. Consider the submatrix of A_1 to the right of this column and below the first row. Do exactly the same thing for it that was done for A. This time the $\mathbf{e}_1$ will refer to $\mathbb{F}^{m-1}$. Use this 1 and row operations to zero out every entry above it in the rows of A_1. Call the resulting matrix, A_2. Thus A_2 satisfies the conditions of the above definition up to the column just encountered. Continue this way till every column has been dealt with and the result must be in row reduced echelon form.

The following diagram illustrates the above procedure. Say the matrix looked something like the following.

$$\begin{pmatrix} 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & * & * & * & * & * \end{pmatrix}$$

First step would yield something like

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & * & * & * & * \end{pmatrix}$$

For the second step you look at the lower right corner as described,

$$\begin{pmatrix} * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix}$$

and if the first column consists of all zeros but the next one is not all zeros, you would get something like this.

$$\begin{pmatrix} 0 & 1 & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & * & * \end{pmatrix}$$

Thus, after zeroing out the term in the top row above the 1, you get the following for the next step in the computation of the row reduced echelon form for the original matrix.

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & * & * & * \end{pmatrix}.$$

Next you look at the lower right matrix below the top two rows and to the right of the first four columns and repeat the process.

Definition 4.3.3 *The first pivot column of A is the first nonzero column of A. The next pivot column is the first column after this which is not a linear combination of the columns to its left. The third pivot column is the next column after this which is not a linear combination of those columns to its left, and so forth. Thus by Lemma 4.2.3 if a pivot column occurs as the j^{th} column from the left, it follows that in the row reduced echelon form there will be one of the $\mathbf{e}_k$ as the j^{th} column.*

There are three choices for row operations at each step in the above theorem. A natural question is whether the same row reduced echelon matrix always results in the end from following the above algorithm applied in any way. The next corollary says this is the case.

Definition 4.3.4 *Two matrices are said to be **row equivalent** if one can be obtained from the other by a sequence of row operations.*

It has been shown above that every matrix is row equivalent to one which is in row reduced echelon form. Note

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$$

so to say two column vectors are equal is to say they are the same linear combination of the special vectors $\mathbf{e}_j$.

Corollary 4.3.5 *The row reduced echelon form is unique. That is if B, C are two matrices in row reduced echelon form and both are row equivalent to A, then $B = C$.*

Proof: Suppose B and C are both row reduced echelon forms for the matrix, A. Then they clearly have the same zero columns since row operations leave zero columns unchanged. If B has the sequence $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_r$ occurring for the first time in the positions, $i_1, i_2, \cdots, i_r$, the description of the row reduced echelon form means that each of these columns is not a linear combination of the preceding columns. Therefore, by Lemma 4.2.3, the same is true of the columns in positions $i_1, i_2, \cdots, i_r$ for C. It follows from the description of the row reduced echelon form that $\mathbf{e}_1, \cdots, \mathbf{e}_r$ occur respectively for the first time in columns $i_1, i_2, \cdots, i_r$ for C. Therefore, both B and C have the sequence $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_r$ occurring for the first time in the positions, $i_1, i_2, \cdots, i_r$. By Lemma 4.2.3, the columns between the i_k and i_{k+1} position in the two matrices are linear combinations, involving the same scalars, of the columns in the $i_1, \cdots, i_k$ position. Also the columns after the i_r position are linear combinations of the columns in the $i_1, \cdots, i_r$ positions involving the same scalars in both matrices. This is equivalent to the assertion that each of these columns is identical and this proves the corollary.

The following corollary follows.

Corollary 4.3.6 *Let A be an $m \times n$ matrix and let R denote the row reduced echelon form obtained from A by row operations. Then there exists a sequence of elementary matrices, $E_1, \cdots, E_p$ such that*

$$(E_p E_{p-1} \cdots E_1) A = R.$$

Proof: This follows from the fact that row operations are equivalent to multiplication on the left by an elementary matrix.

Corollary 4.3.7 *Let A be an invertible $n \times n$ matrix. Then A equals a finite product of elementary matrices.*

Proof: Since A^{-1} is given to exist, it follows A must have rank n because by Theorem 3.3.15 $\det(A) \neq 0$ which says the determinant rank and hence the column rank of A is n and so the row reduced echelon form of A is I because the columns of A form a linearly independent set. Therefore, by Corollary 4.3.6 there is a sequence of elementary matrices, $E_1, \cdots, E_p$ such that

$$(E_p E_{p-1} \cdots E_1) A = I.$$

But now multiply on the left on both sides by E_p^{-1} then by E_{p-1}^{-1} and then by E_{p-2}^{-1} etc. until you get

$$A = E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$$

and by Theorem 4.1.6 each of these in this product is an elementary matrix.

Corollary 4.3.8 *The rank of a matrix equals the number of nonzero pivot columns. Furthermore, every column is contained in the span of the pivot columns.*

Proof: Write the row reduced echelon form for the matrix. From Corollary 4.2.4 this row reduced matrix has the same rank as the original matrix. Deleting all the zero rows and all the columns in the row reduced echelon form which do not correspond to a pivot column, yields an $r \times r$ identity submatrix in which r is the number of pivot columns. Thus the rank is at least r.

From Lemma 4.2.3 every column of A is a linear combination of the pivot columns since this is true by definition for the row reduced echelon form. Therefore, the rank is no more than r. This proves the corollary.

Here is a fundamental observation related to the above.

Corollary 4.3.9 *Suppose A is an $m \times n$ matrix and that $m < n$. That is, the number of rows is less than the number of columns. Then one of the columns of A is a linear combination of the preceding columns of A.*

Proof: Since $m < n$, not all the columns of A can be pivot columns. That is, in the row reduced echelon form say $\mathbf{e}_i$ occurs for the first time at r_i where $r_1 < r_2 < \cdots < r_p$ where $p \leq m$. It follows since $m < n$, there exists some column in the row reduced echelon form which is a linear combination of the preceding columns. By Lemma 4.2.3 the same is true of the columns of A. This proves the corollary.

Definition 4.3.10 *Let A be an $m \times n$ matrix having rank, r. Then the nullity of A is defined to be $n - r$. Also define* $\ker(A) \equiv \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{0}\}$.

Observation 4.3.11 *Note that* $\ker(A)$ *is a subspace because if a, b are scalars and $\mathbf{x}, \mathbf{y}$ are vectors in* $\ker(A)$*, then*

$$A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Recall that the dimension of the column space of a matrix equals its rank and since the column space is just $A(\mathbb{F}^n)$, the rank is just the dimension of $A(\mathbb{F}^n)$. The next theorem shows that the nullilty equals the dimension of $\ker(A)$.

Theorem 4.3.12 *Let A be an $m \times n$ matrix. Then* $\operatorname{rank}(A) + \dim(\ker(A)) = n$..

Proof: Since $\ker(A)$ is a subspace, there exists a basis for $\ker(A)$, $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$. Now by Corollary 2.4.12 this basis may be extended to a basis of $\mathbb{F}^n$, $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_{n-k}\}$. If $\mathbf{z} \in A(\mathbb{F}^n)$, then there exist scalars, $c_i, i = 1, \cdots, k$ and $d_i, i = 1, \cdots, n-k$ such that

$$\begin{aligned} \mathbf{z} &= A\left(\sum_{i=1}^{k} c_i \mathbf{x}_i + \sum_{i=1}^{n-k} d_i \mathbf{y}_i\right) \\ &= \sum_{i=1}^{k} c_i A\mathbf{x}_i + \sum_{i=1}^{n-k} d_i A\mathbf{y}_i = \sum_{i=1}^{n-k} d_i A\mathbf{y}_i \end{aligned}$$

and this shows $\operatorname{span}(A\mathbf{y}_1, \cdots, A\mathbf{y}_{n-k}) = A(\mathbb{F}^n)$. Are the vectors, $\{A\mathbf{y}_1, \cdots, A\mathbf{y}_{n-k}\}$ independent? Suppose

$$\sum_{i=1}^{n-k} c_i A\mathbf{y}_i = \mathbf{0}.$$

Then since A is linear, it follows

$$A\left(\sum_{i=1}^{n-k} c_i \mathbf{y}_i\right) = \mathbf{0}$$

showing that $\sum_{i=1}^{n-k} c_i \mathbf{y}_i \in \ker(A)$. Therefore, there exists constants, $d_i, i = 1, \cdots, k$ such that

$$\sum_{i=1}^{n-k} c_i \mathbf{y}_i = \sum_{j=1}^{k} d_j \mathbf{x}_j. \tag{4.4}$$

If any of these constants, d_i or c_i is not equal to zero then

$$\mathbf{0} = \sum_{j=1}^{k} d_j \mathbf{x}_j + \sum_{i=1}^{k} (-c_i)\, \mathbf{y}_i$$

and this would be a nontrivial linear combination of the vectors, $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_{n-k}\}$ which equals zero contrary to the fact that $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_{n-k}\}$ is a basis. Therefore, all the constants, d_i and c_i in 4.4 must equal zero. It follows the vectors, $\{A\mathbf{y}_1, \cdots, A\mathbf{y}_{n-k}\}$ are linearly independent and so they must be a basis $A(\mathbb{F}^n)$. Therefore,

$$\operatorname{rank}(A) + \dim(\ker(A)) = n - k + k = n.$$

This proves the theorem.

4.4 Rank And Existence Of Solutions To Linear Systems

Consider the linear system of equations,

$$A\mathbf{x} = \mathbf{b} \tag{4.5}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is a $n \times 1$ column vector, and $\mathbf{b}$ is an $m \times 1$ column vector. Suppose

$$A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$$

where the $\mathbf{a}_k$ denote the columns of A. Then $\mathbf{x} = (x_1, \cdots, x_n)^T$ is a solution of the system 4.5, if and only if

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which says that $\mathbf{b}$ is a vector in $\operatorname{span}(\mathbf{a}_1, \cdots, \mathbf{a}_n)$. This shows that there exists a solution to the system, 4.5 if and only if $\mathbf{b}$ is contained in $\operatorname{span}(\mathbf{a}_1, \cdots, \mathbf{a}_n)$. In words, there is a solution to 4.5 if and only if $\mathbf{b}$ is in the column space of A. In terms of rank, the following proposition describes the situation.

Proposition 4.4.1 *Let A be an $m \times n$ matrix and let $\mathbf{b}$ be an $m \times 1$ column vector. Then there exists a solution to 4.5 if and only if*

$$\operatorname{rank}\begin{pmatrix} A & | & \mathbf{b} \end{pmatrix} = \operatorname{rank}(A). \tag{4.6}$$

Proof: Place $\begin{pmatrix} A & | & \mathbf{b} \end{pmatrix}$ and A in row reduced echelon form, respectively B and C. If the above condition on rank is true, then both B and C have the same number of nonzero rows. In particular, you cannot have a row of the form

$$\begin{pmatrix} 0 & \cdots & 0 & \blacksquare \end{pmatrix}$$

where $\blacksquare \neq 0$ in B. Therefore, there will exist a solution to the system 4.5.

Conversely, suppose there exists a solution. This means there cannot be such a row in B described above. Therefore, B and C must have the same number of zero rows and so they have the same number of nonzero rows. Therefore, the rank of the two matrices in 4.6 is the same. This proves the proposition.

4.5 Fredholm Alternative

There is a very useful version of Proposition 4.4.1 known as the **Fredholm alternative**. I will only present this for the case of real matrices here. Later a much more elegant and general approach is presented which allows for the general case of complex matrices.

The following definition is used to state the Fredholm alternative.

Definition 4.5.1 *Let $S \subseteq \mathbb{R}^m$. Then $S^\perp \equiv \{\mathbf{z} \in \mathbb{R}^m : \mathbf{z} \cdot \mathbf{s} = 0$ for every $\mathbf{s} \in S\}$. The funny exponent, $\perp$ is called "perp".*

Now note

$$\ker\left(A^T\right) \equiv \left\{\mathbf{z} : A^T\mathbf{z} = \mathbf{0}\right\} = \left\{\mathbf{z} : \sum_{k=1}^{m} z_k\mathbf{a}_k = 0\right\}$$

Lemma 4.5.2 *Let A be a real $m \times n$ matrix, let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Then*

$$(A\mathbf{x} \cdot \mathbf{y}) = \left(\mathbf{x} \cdot A^T\mathbf{y}\right)$$

Proof: This follows right away from the definition of the dot product and matrix multiplication.

$$\begin{aligned} (A\mathbf{x} \cdot \mathbf{y}) &= \sum_{k,l} A_{kl}x_l y_k \\ &= \sum_{k,l} \left(A^T\right)_{lk} x_l y_k \\ &= \left(\mathbf{x} \cdot A^T\mathbf{y}\right). \end{aligned}$$

This proves the lemma.

Now it is time to state the Fredholm alternative. The first version of this is the following theorem.

Theorem 4.5.3 *Let A be a real $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. There exists a solution, $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b} \in \ker\left(A^T\right)^\perp$.*

Proof: First suppose $\mathbf{b} \in \ker\left(A^T\right)^\perp$. Then this says that if $A^T\mathbf{x} = \mathbf{0}$, it follows that $\mathbf{b} \cdot \mathbf{x} = \mathbf{0}$. In other words, taking the transpose, if

$$\mathbf{x}^T A = \mathbf{0}, \text{ then } \mathbf{b} \cdot \mathbf{x} = 0.$$

In other words, letting $\mathbf{x} = (x_1, \cdots, x_m)^T$, it follows that if

$$\sum_{i=1}^{m} x_i A_{ij} = 0 \text{ for each } j,$$

then it follows

$$\sum_i b_i x_i = 0.$$

In other words, if you get a row of zeros in row reduced echelon form for A then you the same row operations produce a zero in the $m \times 1$ matrix $\mathbf{b}$.

Consequently

$$\operatorname{rank}\left(\begin{array}{c|c} A & \mathbf{b} \end{array}\right) = \operatorname{rank}(A)$$

and so by Proposition 4.4.1, there exists a solution, $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$. It remains to go the other direction.

Let $\mathbf{z} \in \ker\left(A^T\right)$ and suppose $A\mathbf{x} = \mathbf{b}$. I need to verify $\mathbf{b} \cdot \mathbf{z} = 0$. By Lemma 4.5.2,

$$\mathbf{b} \cdot \mathbf{z} = A\mathbf{x} \cdot \mathbf{z} = \mathbf{x} \cdot A^T\mathbf{z} = \mathbf{x} \cdot \mathbf{0} = 0$$

This proves the theorem.

This implies the following corollary which is also called the Fredholm alternative. The "alternative" becomes more clear in this corollary.

Corollary 4.5.4 *Let A be an $m \times n$ matrix. Then A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the only solution to $A^T\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*

Proof: If the only solution to $A^T\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, then $\ker\left(A^T\right) = \{\mathbf{0}\}$ and so $\ker\left(A^T\right)^\perp = \mathbb{R}^m$ because every $\mathbf{b} \in \mathbb{R}^m$ has the property that $\mathbf{b} \cdot \mathbf{0} = 0$. Therefore, $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^m$ because the $\mathbf{b}$ for which there is a solution are those in $\ker\left(A^T\right)^\perp$ by Theorem 4.5.3. In other words, A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

Conversely if A is onto, then by Theorem 4.5.3 every $\mathbf{b} \in \mathbb{R}^m$ is in $\ker\left(A^T\right)^\perp$ and so if $A^T\mathbf{x} = \mathbf{0}$, then $\mathbf{b} \cdot \mathbf{x} = 0$ for every $\mathbf{b}$. In particular, this holds for $\mathbf{b} = \mathbf{x}$. Hence if $A^T\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. This proves the corollary.

Here is an amusing example.

Example 4.5.5 *Let A be an $m \times n$ matrix in which $m > n$. Then A cannot map onto $\mathbb{R}^m$.*

The reason for this is that A^T is an $n \times m$ where $m > n$ and so in the augmented matrix,

$$\left(A^T|\mathbf{0}\right)$$

there must be some free variables. Thus there exists a nonzero vector $\mathbf{x}$ such that $A^T\mathbf{x} = \mathbf{0}$.

4.6 Exercises

1. ♠Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be vectors in $\mathbb{R}^n$. The parallelepiped determined by these vectors $P(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ is defined as

$$P(\mathbf{u}_1, \cdots, \mathbf{u}_n) \equiv \left\{ \sum_{k=1}^{n} t_k \mathbf{u}_k : t_k \in [0,1] \text{ for all } k \right\}.$$

Now let A be an $n \times n$ matrix. Show that

$$\{A\mathbf{x} : \mathbf{x} \in P(\mathbf{u}_1, \cdots, \mathbf{u}_n)\}$$

is also a parallelepiped.

2. ♠In the context of Problem 1, draw $P(\mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors for $\mathbb{R}^2$. Thus $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$. Now suppose

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

where E is the elementary matrix which takes the third row and adds to the first. Draw

$$\{E\mathbf{x} : \mathbf{x} \in P(\mathbf{e}_1, \mathbf{e}_2)\}.$$

In other words, draw the result of doing E to the vectors in $P(\mathbf{e}_1, \mathbf{e}_2)$. Next draw the results of doing the other elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2)$.

3. In the context of Problem 1, either draw or describe the result of doing elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Describe geometrically the conclusion of Corollary 4.3.7.

4. Determine which matrices are in row reduced echelon form.

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$

5. Row reduce the following matrices to obtain the row reduced echelon form. List the pivot columns in the original matrix.

(a) $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$

6. Find the rank and nullity of the following matrices. If the rank is r, identify r columns **in the original matrix** which have the property that every other column may be written as a linear combination of these.

(a) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 3 & 2 & 12 & 1 & 6 & 8 \\ 0 & 1 & 1 & 5 & 0 & 2 & 3 \\ 0 & 2 & 1 & 7 & 0 & 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 6 & 0 & 5 & 4 \\ 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 0 & 2 & 1 & 4 & 0 & 3 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 3 & 2 & 6 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 4 & 0 & 3 & 1 \end{pmatrix}$

7. Find the rank of the following matrices. If the rank is r, identify r columns **in the original matrix** which have the property that every other column may be written as a linear combination of these. Also find a basis for the row and column spaces of the matrices.

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 3 & 2 & 12 & 1 & 6 & 8 \\ 0 & 1 & 1 & 5 & 0 & 2 & 3 \\ 0 & 2 & 1 & 7 & 0 & 3 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 6 & 0 & 5 & 4 \\ 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 0 & 2 & 1 & 4 & 0 & 3 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 3 & 2 & 6 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 4 & 0 & 3 & 1 \end{pmatrix}$

8. ♠Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

9. ♠Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that A maps $\mathbb{F}^n$ onto $\mathbb{F}^m$. **Hint:** The vectors $\mathbf{e}_1, \cdots, \mathbf{e}_m$ occur as columns in the row reduced echelon form for A.

10. ♠Suppose A is an $m \times n$ matrix and that $m > n$. Show there exists $\mathbf{b} \in \mathbb{F}^m$ such that there is no solution to the equation

$$A\mathbf{x} = \mathbf{b}.$$

11. Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. **Hint:** If not, there exists a vector, $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and this implies at least one column of A is a linear combination of the others. Show this would require the column rank to be less than n.

12. ♠Explain why an $n \times n$ matrix, A is both one to one and onto if and only if its rank is n.

13. Suppose A is an $m \times n$ matrix and $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{F}^n) \subseteq \mathbb{F}^m$. Now suppose $A(\mathbf{z}_i) = \mathbf{w}_i$. Show $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is also independent.

14. Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Hint: Consider the subspace, $B(\mathbb{F}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$. Now suppose $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is a basis for $\ker(B)$. Let $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ be such that $B\mathbf{z}_i = \mathbf{w}_i$ and argue that

$$\ker(AB) \subseteq \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_r, \mathbf{z}_1, \cdots, \mathbf{z}_k).$$

Here is how you do this. Suppose $AB\mathbf{x} = \mathbf{0}$. Then $B\mathbf{x} \in \ker(A) \cap B(\mathbb{F}^p)$ and so $B\mathbf{x} = \sum_{i=1}^k B\mathbf{z}_i$ showing that

$$\mathbf{x} - \sum_{i=1}^k \mathbf{z}_i \in \ker(B).$$

15. ♠Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one.

16. ♠Let A be an $m \times n$ real matrix and let $\mathbf{b} \in \mathbb{R}^m$. Show there exists a solution, $\mathbf{x}$ to the system

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Next show that if $\mathbf{x}, \mathbf{x}_1$ are two solutions, then $A\mathbf{x} = A\mathbf{x}_1$. **Hint:** First show that $\left(A^T A\right)^T = A^T A$. Next show if $\mathbf{x} \in \ker\left(A^T A\right)$, then $A\mathbf{x} = \mathbf{0}$. Finally apply the Fredholm alternative. Show $A^T\mathbf{b} \in \ker(A^T A)^\perp$. This will give existence of a solution.

17. ♠Show that in the context of Problem 16 that if $\mathbf{x}$ is the solution there, then $|\mathbf{b} - A\mathbf{x}| \leq |\mathbf{b} - A\mathbf{y}|$ for every $\mathbf{y}$. Thus $A\mathbf{x}$ is the point of $A(\mathbb{R}^n)$ which is closest to $\mathbf{b}$ of every point in $A(\mathbb{R}^n)$. This is a solution to the least squares problem.

18. Suppose A, B are two invertible $n \times n$ matrices. Show there exists a sequence of row operations which when done to A yield B. **Hint:** Recall that every invertible matrix is a product of elementary matrices.

19. ♠Here are two matrices in row reduced echelon form

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Does there exist a sequence of row operations which when done to A will yield B? Explain.

20. Show an upper triangular matrix has rank equal to the number of nonzero entries on the main diagonal.

21. ♠Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$ be vectors in $\mathbb{F}^n$. Describe a systematic way to obtain a vector $\mathbf{v}_n$ which is perpendicular to each of these vectors. **Hint:** You might consider something like this

$$\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & \vdots & & \vdots \\ v_{(n-1)1} & v_{(n-1)2} & \cdots & v_{(n-1)n} \end{pmatrix}$$

where v_{ij} is the j^{th} entry of the vector $\mathbf{v}_i$. This is a lot like the cross product.

22. Let A be an $m \times n$ matrix. Then $\ker(A)$ is a subspace of $\mathbb{F}^n$. Is it true that every subspace of $\mathbb{F}^n$ is the kernel or null space of some matrix? Prove or disprove.

Some Factorizations

5.1 LU Factorization

An LU factorization of a matrix involves writing the given matrix as the product of a lower triangular matrix which has the main diagonal consisting entirely of ones, L, and an upper triangular matrix, U in the indicated order. The L goes with "lower" and the U with "upper". It turns out many matrices can be written in this way and when this is possible, people get excited about slick ways of solving the system of equations, $A\mathbf{x} = \mathbf{y}$. The method lacks generality but is of interest just the same.

Example 5.1.1 *Can you write* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *in the form* LU *as just described?*

To do so you would need

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ xa & xb+c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $b = 1$ and $a = 0$. Also, from the bottom rows, $xa = 1$ which can't happen and have $a = 0$. Therefore, you can't write this matrix in the form LU. It has no LU factorization. This is what I mean above by saying the method lacks generality.

Which matrices have an LU factorization? It turns out it is those whose row reduced echelon form can be achieved without switching rows and which only involve row operations of type 3 in which row j is replaced with a multiple of row i added to row j for $i < j$.

5.2 Finding An LU Factorization

There is a convenient procedure for finding an LU factorization. It turns out that it is only necessary to keep track of the **multipliers** which are used to row reduce to upper triangular form. This procedure is described in the following examples and is called the multiplier method. It is due to Dolittle.

Example 5.2.1 *Find an* LU *factorization for* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 2 \end{pmatrix}$

Write the matrix next to the identity matrix as shown.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 2 \end{pmatrix}.$$

The process involves doing row operations to the matrix on the right while simultaneously updating successive columns of the matrix on the left. First take -2 times the first row and add to the second in the matrix on the right.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -10 \\ 1 & 5 & 2 \end{pmatrix}$$

Note the method for updating the matrix on the left. The 2 in the second entry of the first column is there because -2 times the first row of A added to the second row of A produced a 0. Now replace the third row in the matrix on the right by -1 times the first row added to the third. Thus the next step is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -10 \\ 0 & 3 & -1 \end{pmatrix}$$

Finally, add the second row to the bottom row and make the following changes

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -10 \\ 0 & 0 & -11 \end{pmatrix}.$$

At this point, stop because the matrix on the right is upper triangular. An LU factorization is the above.

The justification for this gimmick will be given later.

Example 5.2.2 *Find an LU factorization for* $A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 2 & 3 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$.

This time everything is done at once for a whole column. This saves trouble. First multiply the first row by (-1) and then add to the last row. Next take (-2) times the first and add to the second and then (-2) times the first and add to the third.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -4 & 0 & -3 & -1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & -2 & 0 & -1 & 1 \end{pmatrix}.$$

This finishes the first column of L and the first column of U. Now take $-(1/4)$ times the second row in the matrix on the right and add to the third followed by $-(1/2)$ times the second added to the last.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1/4 & 1 & 0 \\ 1 & 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -4 & 0 & -3 & -1 \\ 0 & 0 & -1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 & 3/2 \end{pmatrix}$$

This finishes the second column of L as well as the second column of U. Since the matrix on the right is upper triangular, stop. The LU factorization has now been obtained. This technique is called Dolittle's method.

This process is entirely typical of the general case. The matrix U is just the first upper triangular matrix you come to in your quest for the row reduced echelon form using only

the row operation which involves replacing a row by itself added to a multiple of another row. The matrix, L is what you get by updating the identity matrix as illustrated above.

You should note that for a square matrix, the number of row operations necessary to reduce to LU form is about half the number needed to place the matrix in row reduced echelon form. This is why an LU factorization is of interest in solving systems of equations.

5.3 Solving Linear Systems Using An LU Factorization

The reason people care about the LU factorization is it allows the quick solution of systems of equations. Here is an example.

Example 5.3.1 *Suppose you want to find the solutions to* $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Of course one way is to write the augmented matrix and grind away. However, this involves more row operations than the computation of an LU factorization and it turns out that an LU factorization can give the solution quickly. Here is how. The following is an LU factorization for the matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Let $U\mathbf{x} = \mathbf{y}$ and consider $L\mathbf{y} = \mathbf{b}$ where in this case, $\mathbf{b} = (1, 2, 3)^T$. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which yields very quickly that $\mathbf{y} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. Now you can find $\mathbf{x}$ by solving $U\mathbf{x} = \mathbf{y}$. Thus in this case,

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

which yields

$$\mathbf{x} = \begin{pmatrix} -\frac{3}{5} + \frac{7}{5}t \\ \frac{9}{5} - \frac{11}{5}t \\ t \\ -1 \end{pmatrix}, t \in \mathbb{R}.$$

Work this out by hand and you will see the advantage of working only with triangular matrices.

It may seem like a trivial thing but it is used because it cuts down on the number of operations involved in finding a solution to a system of equations enough that it makes a difference for large systems.

5.4 The PLU Factorization

As indicated above, some matrices don't have an LU factorization. Here is an example.

$$M = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 1 & 1 \end{pmatrix} \tag{5.1}$$

In this case, there is another factorization which is useful called a PLU factorization. Here P is a permutation matrix.

Example 5.4.1 *Find a PLU factorization for the above matrix in 5.1.*

Proceed as before trying to find the row echelon form of the matrix. First add -1 times the first row to the second row and then add -4 times the first to the third. This yields

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & -5 & -11 & -7 \end{pmatrix}$$

There is no way to do only row operations involving replacing a row with itself added to a multiple of another row to the second matrix in such a way as to obtain an upper triangular matrix. Therefore, consider M with the bottom two rows switched.

$$M' = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

Now try again with this matrix. First take -1 times the first row and add to the bottom row and then take -4 times the first row and add to the second row. This yields

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The second matrix is upper triangular and so the LU factorization of the matrix, M' is

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Thus $M' = PM = LU$ where L and U are given above. Therefore, $M = P^2M = PLU$ and so

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

This process can always be followed and so there always exists a PLU factorization of a given matrix even though there isn't always an LU factorization.

Example 5.4.2 *Use a PLU factorization of $M \equiv \begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 1 & 1 \end{pmatrix}$ to solve the system $M\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1, 2, 3)^T$.*

Let $U\mathbf{x} = \mathbf{y}$ and consider $PL\mathbf{y} = \mathbf{b}$. In other words, solve,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Then multiplying both sides by P gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

and so

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Now $U\mathbf{x} = \mathbf{y}$ and so it only remains to solve

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

which yields

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} + \frac{7}{5}t \\ \frac{9}{10} - \frac{11}{5}t \\ t \\ -\frac{1}{2} \end{pmatrix} : t \in \mathbb{R}.$$

5.5 Justification For The Multiplier Method

Why does the multiplier method work for finding an LU factorization? Suppose A is a matrix which has the property that the row reduced echelon form for A may be achieved using only the row operations which involve replacing a row with itself added to a multiple of another row. It is not ever necessary to switch rows. Thus every row which is replaced using this row operation in obtaining the echelon form may be modified by using a row which is above it. Furthermore, in the multiplier method for finding the LU factorization, we zero out the elements below the pivot entry in first column and then the next and so on when scanning from the left. In terms of elementary matrices, this means the row operations used to reduce A to upper triangular form correspond to multiplication on the left by lower triangular matrices having all ones down the main diagonal.and the sequence of elementary matrices which row reduces A has the property that in scanning the list of elementary matrices from the right to the left, this list consists of several matrices which involve only changes from the identity in the first column, then several which involve only changes from the identity in the second column and so forth. More precisely, $E_p \cdots E_1 A = U$ where U is upper triangular, each E_i is a lower triangular elementary matrix having all ones down the main diagonal, for some r_i, each of $E_{r_1} \cdots E_1$ differs from the identity only in the first column, each of $E_{r_2} \cdots E_{r_1+1}$ differs from the identity only in the second column and so forth. Therefore, $A = \overbrace{E_1^{-1} \cdots E_{p-1}^{-1} E_p^{-1}}^{\text{Will be } L} U$. You multiply the inverses in the reverse order. Now each of the E_i^{-1} is also lower triangular with 1 down the main diagonal. Therefore their product has this property. Recall also that if E_i equals the identity matrix except

for having an a in the j^{th} column somewhere below the main diagonal, E_i^{-1} is obtained by replacing the a in E_i with $-a$ thus explaining why we replace with -1 times the multiplier in computing L. In the case where A is a $3 \times m$ matrix, $E_1^{-1} \cdots E_{p-1}^{-1} E_p^{-1}$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Note that scanning from left to right, the first two in the product involve changes in the identity only in the first column while in the third matrix, the change is only in the second. If the entries in the first column had been zeroed out in a different order, the following would have resulted.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

However, it is important to be working from the left to the right, one column at a time.

A similar observation holds in any dimension. Multiplying the elementary matrices which involve a change only in the j^{th} column you obtain A equal to an upper triangular, $n \times m$ matrix, U which is multiplied by a sequence of lower triangular matrices on its left which is of the following form in which the a_{ij} are negatives of multipliers used in row reducing to an upper triangular matrix.

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{11} & 1 & & & & \vdots \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ a_{1,n-1} & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & a_{21} & \ddots & & & \vdots \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & a_{2,n-2} & 0 & \cdots & \cdots & 1 \end{pmatrix} \cdots$$

$$\cdots \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & 1 & 0 \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & 1 \end{pmatrix}$$

From the matrix multiplication, this product equals

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_{11} & 1 & & & & \vdots \\ a_{12} & a_{21} & \ddots & & & \vdots \\ \vdots & a_{22} & a_{31} & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & 1 & 0 \\ a_{1,n-1} & a_{2,n-2} & a_{3,n-3} & \cdots & a_{n,n-1} & 1 \end{pmatrix}$$

Notice how the end result of the matrix multiplication made no change in the a_{ij}. It just filled in the empty spaces with the a_{ij} which occured in one of the matrices in the product. This is why, in computing L, it is sufficient to begin with the left column and work column by column toward the right, replacing entries with the negative of the multiplier used in the row operation which produces a zero in that entry.

5.6 Existence For The PLU Factorization

Here I will consider an invertible $n \times n$ matrix and show that such a matrix always has a PLU factorization. More general matrices could also be considered but this is all I will present.

Let A be such an invertible matrix and consider the first column of A. If $A_{11} \neq 0$, use this to zero out everything below it. The entry A_{11} is called the pivot. Thus in this case there is a lower triangular matrix, L_1 which has all ones on the diagonal such that

$$L_1 P_1 A = \begin{pmatrix} * & * \\ \mathbf{0} & A_1 \end{pmatrix} \tag{5.2}$$

Here $P_1 = I$. In case $A_{11} = 0$, let r be such that $A_{r1} \neq 0$ and r is the first entry for which this happens. In this case, let P_1 be the permutation matrix which switches the first row and the r^{th} row. Then as before, there exists a lower triangular matrix, L_1 which has all ones on the diagonal such that 5.2 holds in this case also. Thus L_1 is of the form

$$\begin{pmatrix} 1 & & & & & & 0 \\ 0 & 1 & & & & & \\ \vdots & 0 & \ddots & & & & \\ 0 & \vdots & & 1 & & & \\ * & 0 & \cdots & 0 & 1 & & \\ \vdots & & & & & \ddots & \\ * & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

By this it is meant that the first column of L_1 is the only one which has possibly nonzero entries below the main diagonal and these possibly nonzero entries start with the r^{th} position. That is, $L_{p1} = 0$ whenever $1 < p < r$.

Go to A_1. Following the same procedure as above, there exists a lower triangular matrix and permuation matrix L_2', P_2' such that

$$L_2' P_2' A_1 = \begin{pmatrix} * & * \\ \mathbf{0} & A_2 \end{pmatrix}$$

Let

$$L_2 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & L_2' \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2' \end{pmatrix}$$

Then using block multiplication, Theorem 3.5.2,

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & L_2' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2' \end{pmatrix} \begin{pmatrix} * & * \\ \mathbf{0} & A_1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & L_2' \end{pmatrix} \begin{pmatrix} * & * \\ \mathbf{0} & P_2' A_1 \end{pmatrix} = \begin{pmatrix} * & * \\ \mathbf{0} & L_2' P_2' A_1 \end{pmatrix}$$

$$\begin{pmatrix} * & \cdots & * \\ 0 & * & * \\ \mathbf{0} & \mathbf{0} & A_2 \end{pmatrix} = L_2P_2L_1P_1A$$

and L_2 has all the subdiagonal entries equal to 0 except possibly some nonzero entries in the second column starting with position r_2 where P_2 switches rows r_2 and 2. Continuing this way, it follows there are lower triangular matrices L_j having all ones down the diagonal and permutation matrices P_i which switch only two rows such that

$$L_{n-1}P_{n-1}L_{n-2}P_{n-2}L_{n-3}\cdots L_2P_2L_1P_1A = U \tag{5.3}$$

where U is upper triangular. The matrix L_j has all zeros below the main diagonal except for the j^{th} column and even in this column it has zeros between position j and r_j where P_j switches rows j and r_j. Of course in the case where no switching is necessary, you could get all nonzero entries below the main diagonal in the j^{th} column for L_j.

The fact that L_j is the identity except for the j^{th} column means that each P_k for $k > j$ almost commutes with L_j. Say P_k switches the k^{th} and the q^{th} rows for $q \geq k > j$. When you place P_k on the right of L_j it just switches the k^{th} and the q^{th} columns and leaves the j^{th} column unchanged. Therefore, the same result as placing P_k on the left of L_j can be obtained by placing P_k on the right of L_j and modifying L_j by switching the k^{th} and the q^{th} entries in the j^{th} column. (Note this could possibly interchange a 0 for something nonzero.) It follows from 5.3 there exists P, the product of permutation matrices, $P = P_{n-1}\cdots P_1$ each of which switches two rows, and L a lower triangular matrix having all ones on the main diagonal, $L = L'_{n-1}\cdots L'_2L'_1$, where the L'_j are obtained as just described by moving a succession of P_k from the left to the right of L_j and modifying the j^{th} column as indicated, such that

$$LPA = U.$$

Then

$$A = P^TL^{-1}U$$

It is customary to write this more simply as

$$A = PLU$$

where L is an upper triangular matrix having all ones on the diagonal and P is a permutation matrix consisting of $P_1\cdots P_{n-1}$ as described above. This proves the following theorem.

Theorem 5.6.1 *Let A be any invertible $n \times n$ matrix. Then there exists a permuation matrix P and a lower triangular matrix L having all ones on the main diagonal and an upper triangular matrix U such that*

$$A = PLU$$

5.7 The QR Factorization

As pointed out above, the LU factorization is not a mathematically respectable thing because it does not always exist. There is another factorization which does always exist. Much more can be said about it than I will say here. I will only deal with real matrices and so the dot product will be the usual real dot product.

Definition 5.7.1 *An $n \times n$ real matrix Q is called an orthogonal matrix if*

$$QQ^T = Q^TQ = I.$$

Thus an orthogonal matrix is one whose inverse is equal to its transpose.

First note that if a matrix is orthogonal this says

$$\sum_j Q_{ij}^T Q_{jk} = \sum_j Q_{ji} Q_{jk} = \delta_{ik}$$

Thus

$$|Q\mathbf{x}|^2 = \sum_i \left(\sum_j Q_{ij} x_j \right)^2 = \sum_i \sum_r \sum_s Q_{is} x_s Q_{ir} x_r$$
$$= \sum_i \sum_r \sum_s Q_{is} Q_{ir} x_s x_r = \sum_r \sum_s \sum_i Q_{is} Q_{ir} x_s x_r$$
$$= \sum_r \sum_s \delta_{sr} x_s x_r = \sum_r x_r^2 = |\mathbf{x}|^2$$

This shows that orthogonal transformations preserve distances. You can show that if you have a matrix which does preserve distances, then it must be orthogonal also.

Example 5.7.2 *One of the most important examples of an orthogonal matrix is the so called Householder matrix. You have* $\mathbf{v}$ *a unit vector and you form the matrix,*

$$I - 2\mathbf{v}\mathbf{v}^T$$

This is an orthogonal matrix which is also symmetric. To see this, you use the rules of matrix operations.

$$\begin{aligned} \left(I - 2\mathbf{v}\mathbf{v}^T\right)^T &= I^T - \left(2\mathbf{v}\mathbf{v}^T\right)^T \\ &= I - 2\mathbf{v}\mathbf{v}^T \end{aligned}$$

so it is symmetric. Now to show it is orthogonal,

$$\begin{aligned} \left(I - 2\mathbf{v}\mathbf{v}^T\right)\left(I - 2\mathbf{v}\mathbf{v}^T\right) &= I - 2\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T + 4\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T \\ &= I - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}\mathbf{v}^T = I \end{aligned}$$

because $\mathbf{v}^T\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = 1$. *Therefore, this is an example of an orthogonal matrix.*

Consider the following problem.

Problem 5.7.3 *Given two vectors* $\mathbf{x}, \mathbf{y}$ *such that* $|\mathbf{x}| = |\mathbf{y}| \neq 0$ *but* $\mathbf{x} \neq \mathbf{y}$ *and you want an orthogonal matrix,* Q *such that* $Q\mathbf{x} = \mathbf{y}$ *and* $Q\mathbf{y} = \mathbf{x}$. *The thing which works is the Householder matrix*

$$Q \equiv I - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y})^T$$

Here is why this works.

$$\begin{aligned} Q(\mathbf{x}-\mathbf{y}) &= (\mathbf{x}-\mathbf{y}) - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y}) \\ &= (\mathbf{x}-\mathbf{y}) - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}|\mathbf{x}-\mathbf{y}|^2 = \mathbf{y}-\mathbf{x} \end{aligned}$$

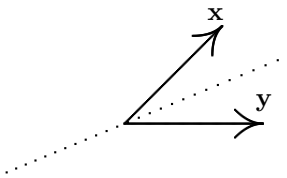
$$\begin{aligned} Q(\mathbf{x}+\mathbf{y}) &= (\mathbf{x}+\mathbf{y}) - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y})^T(\mathbf{x}+\mathbf{y}) \\ &= (\mathbf{x}+\mathbf{y}) - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}((\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})) \\ &= (\mathbf{x}+\mathbf{y}) - 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\left(|\mathbf{x}|^2 - |\mathbf{y}|^2\right) = \mathbf{x}+\mathbf{y} \end{aligned}$$

Hence

$$\begin{aligned} Q\mathbf{x}+Q\mathbf{y} &= \mathbf{x}+\mathbf{y} \\ Q\mathbf{x}-Q\mathbf{y} &= \mathbf{y}-\mathbf{x} \end{aligned}$$

Adding these equations, $2Q\mathbf{x}=2\mathbf{y}$ *and subtracting them yields* $2Q\mathbf{y}=2\mathbf{x}$.

A picture of the geometric significance follows.



The orthogonal matrix Q reflects across the dotted line taking $\mathbf{x}$ to $\mathbf{y}$ and $\mathbf{y}$ to $\mathbf{x}$.

Definition 5.7.4 *Let A be an $m\times n$ matrix. Then a QR factorization of A consists of two matrices, Q orthogonal and R upper triangular (right triangular) having all the entries on the main diagonal nonnegative such that $A=QR$.*

With the solution to this simple problem, here is how to obtain a QR factorization for any matrix A. Let

$$A=(\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n)$$

where the $\mathbf{a}_i$ are the columns. If $\mathbf{a}_1=\mathbf{0}$, let $Q_1=I$. If $\mathbf{a}_1\neq\mathbf{0}$, let

$$\mathbf{b}\equiv\begin{pmatrix} |\mathbf{a}_1| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and form the Householder matrix,

$$Q_1\equiv I-2\frac{(\mathbf{a}_1-\mathbf{b})}{|\mathbf{a}_1-\mathbf{b}|^2}(\mathbf{a}_1-\mathbf{b})^T$$

As in the above problem $Q_1\mathbf{a}_1=\mathbf{b}$ and so

$$Q_1A=\begin{pmatrix} |\mathbf{a}_1| & * \\ \mathbf{0} & A_2 \end{pmatrix}$$

where A_2 is a $m-1\times n-1$ matrix. Now find in the same way as was just done a $m-1\times m-1$ matrix $\widehat{Q}_2$ such that

$$\widehat{Q}_2A_2=\begin{pmatrix} * & * \\ \mathbf{0} & A_3 \end{pmatrix}$$

Let

$$Q_2\equiv\begin{pmatrix} 1 & 0 \\ \mathbf{0} & \widehat{Q}_2 \end{pmatrix}.$$

Then

$$Q_2Q_1A=\begin{pmatrix} 1 & 0 \\ \mathbf{0} & \widehat{Q}_2 \end{pmatrix}\begin{pmatrix} |\mathbf{a}_1| & * \\ \mathbf{0} & A_2 \end{pmatrix}$$

$$= \begin{pmatrix} |\mathbf{a}_1| & * & * \\ \vdots & * & * \\ 0 & \mathbf{0} & A_3 \end{pmatrix}$$

Continuing this way untill the result is upper triangular, you get a sequence of orthogonal matrices $Q_p Q_{p-1} \cdots Q_1$ such that

$$Q_p Q_{p-1} \cdots Q_1 A = R \tag{5.4}$$

where R is upper triangular.

Now if Q_1 and Q_2 are orthogonal, then from properties of matrix multiplication,

$$Q_1 Q_2 \left(Q_1 Q_2\right)^T = Q_1 Q_2 Q_2^T Q_1^T = Q_1 I Q_1^T = I$$

and similarly

$$\left(Q_1 Q_2\right)^T Q_1 Q_2 = I.$$

Thus the product of orthogonal matrices is orthogonal. Also the transpose of an orthogonal matrix is orthogonal directly from the definition. Therefore, from 5.4

$$A = \left(Q_p Q_{p-1} \cdots Q_1\right)^T R \equiv QR.$$

This proves the following theorem.

Theorem 5.7.5 *Let A be any real $m \times n$ matrix. Then there exists an orthogonal matrix, Q and an upper triangular matrix R having nonnegative entries on the main diagonal such that*

$$A = QR$$

and this factorization can be accomplished in a systematic manner.

5.8 Exercises

1. Find a LU factorization of $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

2. Find a LU factorization of $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{pmatrix}$.

3. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$.

4. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 1 \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}$.

5. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

6. Is there only one LU factorization for a given matrix? **Hint:** Consider the equation
$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

7. Here is a matrix and an LU factorization of it.
$$A = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 1 & 4 & 9 \\ 0 & 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & -1 & 9 \\ 0 & 0 & 1 & 14 \end{pmatrix}$$
Use this factorization to solve the system of equations
$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

8. ♠Find a QR factorization for the matrix
$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

9. ♠Find a QR factorization for the matrix
$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

10. ♠If you had a QR factorization, $A = QR$, describe how you could use it to solve the equation $A\mathbf{x} = \mathbf{b}$. Note, this is not how people usually solve systems of equations.

11. ♠If Q is an orthogonal matrix, show the columns are an orthonormal set. That is show that for
$$Q = \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{pmatrix}$$
it follows that $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$. Also show that any orthonormal set of vectors is linearly independent.

12. ♠Show you can't expect uniqueness for QR factorizations. Consider
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
and verify this equals
$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
and also
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
Using Definition 5.7.4, can it be concluded that if A is an invertible matrix it will follow there is only one QR factorization?

13. Suppose $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ are linearly independent vectors in $\mathbb{R}^p$ and let

$$A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$$

Form a QR factorization for A.

$$\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_p \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & r_{pn} \end{pmatrix}$$

Show that for each $k \leq n$,

$$\text{span}(\mathbf{a}_1, \cdots, \mathbf{a}_k) \subseteq \text{span}(\mathbf{q}_1, \cdots, \mathbf{q}_k)$$

and explain why, since $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ are linearly independent, these two subspaces must actually be equal. Prove that every subspace of $\mathbb{R}^n$ has an orthonormal basis. The procedure just described is similar to the Gram Schmidt procedure which will be presented later.

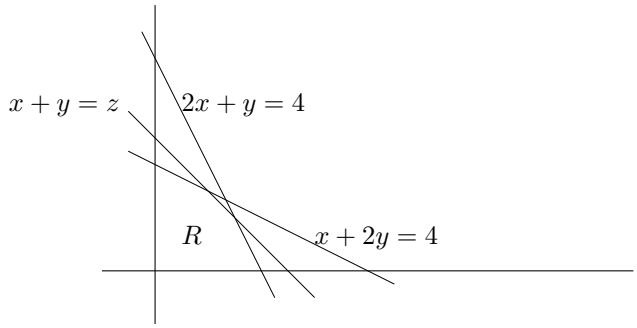
14. Suppose $Q_n R_n$ converges to an orthogonal matrix Q where Q_n is orthogonal and R_n is upper triangular having all positive entries on the diagonal. Show that then Q_n converges to Q and R_n converges to the identity.

Linear Programming

6.1 Simple Geometric Considerations

One of the most important uses of row operations is in solving linear program problems which involve maximizing a linear function subject to inequality constraints determined from linear equations. Here is an example. A certain hamburger store has 9000 hamburger pattys to use in one week and a limitless supply of special sauce, lettuce, tomatos, onions, and buns. They sell two types of hamburgers, the big stack and the basic burger. It has also been determined that the employees cannot prepare more than 9000 of either type in one week. The big stack, popular with the teen agers from the local high school, involves two pattys, lots of delicious sauce, condiments galore, and a divider between the two pattys. The basic burger, very popular with children, involves only one patty and some pickles and ketchup. Demand for the basic burger is twice what it is for the big stack. What is the maximum number of hamburgers which could be sold in one week given the above limitations?

Let x be the number of basic burgers and y the number of big stacks which could be sold in a week. Thus it is desired to maximize $z = x + y$ subject to the above constraints. The total number of pattys is 9000 and so the number of pattys used is $x+2y$. This number must satisfy $x+2y \leq 9000$ because there are only 9000 pattys available. Because of the limitation on the number the employees can prepare and the demand, it follows $2x + y \leq 9000$. You never sell a negative number of hamburgers and so $x, y \geq 0$. In simpler terms the problem reduces to maximizing $z = x + y$ subject to the two constraints, $x + 2y \leq 9000$ and $2x + y \leq 9000$. This problem is pretty easy to solve geometrically. Consider the following picture in which R labels the region described by the above inequalities and the line $z = x+y$ is shown for a particular value of z.



As you make z larger this line moves away from the origin, always having the same slope

and the desired solution would consist of a point in the region, R which makes z as large as possible or equivalently one for which the line is as far as possible from the origin. Clearly this point is the point of intersection of the two lines, $(3000, 3000)$ and so the maximum value of the given function is 6000. Of course this type of procedure is fine for a situation in which there are only two variables but what about a similar problem in which there are very many variables. In reality, this hamburger store makes many more types of burgers than those two and there are many considerations other than demand and available pattys. Each will likely give you a constraint which must be considered in order to solve a more realistic problem and the end result will likely be a problem in many dimensions, probably many more than three so your ability to draw a picture will get you nowhere for such a problem. Another method is needed. This method is the topic of this section. I will illustrate with this particular problem. Let $x_1 = x$ and $y = x_2$. Also let x_3 and x_4 be nonnegative variables such that

$$x_1 + 2x_2 + x_3 = 9000,\ 2x_1 + x_2 + x_4 = 9000.$$

To say that x_3 and x_4 are nonnegative is the same as saying $x_1 + 2x_2 \leq 9000$ and $2x_1 + x_2 \leq 9000$ and these variables are called slack variables at this point. They are called this because they "take up the slack". I will discuss these more later. First a general situation is considered.

6.2 The Simplex Tableau

Here is some notation.

Definition 6.2.1 *Let* $\mathbf{x}, \mathbf{y}$ *be vectors in* $\mathbb{R}^q$. *Then* $\mathbf{x} \leq \mathbf{y}$ *means for each* $i, x_i \leq y_i$.

The problem is as follows:

Let A be an $m \times (m+n)$ real matrix of rank m. It is desired to find $\mathbf{x} \in \mathbb{R}^{n+m}$ such that $\mathbf{x}$ satisfies the constraints,

$$\mathbf{x} \geq \mathbf{0},\ A\mathbf{x} = \mathbf{b} \tag{6.1}$$

and out of all such $\mathbf{x}$,

$$z \equiv \sum_{i=1}^{m+n} c_i x_i$$

is as large (or small) as possible. This is usually refered to as maximizing or minimizing z subject to the above constraints. First I will consider the constraints.

Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{n+m} \end{pmatrix}$. First you find a vector, $\mathbf{x}^0 \geq \mathbf{0}$, $A\mathbf{x}^0 = \mathbf{b}$ such that n of the components of this vector equal 0. Letting $i_1, \cdots, i_n$ be the positions of $\mathbf{x}^0$ for which $x_i^0 = 0$, suppose also that $\{\mathbf{a}_{j_1}, \cdots, \mathbf{a}_{j_m}\}$ is linearly independent for j_i the other positions of $\mathbf{x}^0$. Geometrically, this means that $\mathbf{x}^0$ is a corner of the feasible region, those $\mathbf{x}$ which satisfy the constraints. This is called a basic feasible solution. Also define

$$\begin{aligned} \mathbf{c}_B &\equiv (c_{j_1}, \cdots, c_{j_m}),\ \mathbf{c}_F \equiv (c_{i_1}, \cdots, c_{i_n}) \\ \mathbf{x}_B &\equiv (x_{j_1}, \cdots, x_{j_m}),\ \mathbf{x}_F \equiv (x_{i_1}, \cdots, x_{i_n}). \end{aligned}$$

and

$$z^0 \equiv z\left(\mathbf{x}^0\right) = \begin{pmatrix} \mathbf{c}_B & \mathbf{c}_F \end{pmatrix} \begin{pmatrix} \mathbf{x}_B^0 \\ \mathbf{x}_F^0 \end{pmatrix} = \mathbf{c}_B \mathbf{x}_B^0$$

since $\mathbf{x}_F^0 = \mathbf{0}$. The variables which are the components of the vector $\mathbf{x}_B$ are called the **basic variables** and the variables which are the entries of $\mathbf{x}_F$ are called the **free variables.** You

set $\mathbf{x}_F = \mathbf{0}$. Now $\left(\mathbf{x}^0, z^0\right)^T$ is a solution to

$$\begin{pmatrix} A & \mathbf{0} \\ -\mathbf{c} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$$

along with the constraints $\mathbf{x} \geq \mathbf{0}$. Writing the above in augmented matrix form yields

$$\begin{pmatrix} A & \mathbf{0} & \mathbf{b} \\ -\mathbf{c} & 1 & 0 \end{pmatrix} \tag{6.2}$$

Permute the columns and variables on the left if necessary to write the above in the form

$$\begin{pmatrix} B & F & \mathbf{0} \\ -\mathbf{c}_B & -\mathbf{c}_F & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_F \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} \tag{6.3}$$

or equivalently in the augmented matrix form keeping track of the variables on the bottom as

$$\begin{pmatrix} B & F & \mathbf{0} & \mathbf{b} \\ -\mathbf{c}_B & -\mathbf{c}_F & 1 & 0 \\ \mathbf{x}_B & \mathbf{x}_F & 0 & 0 \end{pmatrix}. \tag{6.4}$$

Here B pertains to the variables $x_{i_1}, \cdots, x_{j_m}$ and is an $m \times m$ matrix with linearly independent columns, $\{\mathbf{a}_{j_1}, \cdots, \mathbf{a}_{j_m}\}$, and F is an $m \times n$ matrix. Now it is assumed that

$$\begin{pmatrix} B & F \end{pmatrix} \begin{pmatrix} \mathbf{x}_B^0 \\ \mathbf{x}_F^0 \end{pmatrix} = \begin{pmatrix} B & F \end{pmatrix} \begin{pmatrix} \mathbf{x}_B^0 \\ \mathbf{0} \end{pmatrix} = B\mathbf{x}_B^0 = \mathbf{b}$$

and since B is assumed to have rank m, it follows

$$\mathbf{x}_B^0 = B^{-1}\mathbf{b} \geq \mathbf{0}. \tag{6.5}$$

This is very important to observe. $B^{-1}\mathbf{b} \geq \mathbf{0}$!

Do row operations on the top part of the matrix,

$$\begin{pmatrix} B & F & \mathbf{0} & \mathbf{b} \\ -\mathbf{c}_B & -\mathbf{c}_F & 1 & 0 \end{pmatrix} \tag{6.6}$$

and obtain its row reduced echelon form. Then after these row operations the above becomes

$$\begin{pmatrix} I & B^{-1}F & \mathbf{0} & B^{-1}\mathbf{b} \\ -\mathbf{c}_B & -\mathbf{c}_F & 1 & 0 \end{pmatrix}. \tag{6.7}$$

where $B^{-1}\mathbf{b} \geq 0$. Next do another row operation in order to get a $\mathbf{0}$ where you see a $-\mathbf{c}_B$. Thus

$$\begin{pmatrix} I & B^{-1}F & \mathbf{0} & B^{-1}\mathbf{b} \\ \mathbf{0} & \mathbf{c}_B B^{-1}F' - \mathbf{c}_F & 1 & \mathbf{c}_B B^{-1}\mathbf{b} \end{pmatrix} \tag{6.8}$$

$$= \begin{pmatrix} I & B^{-1}F & \mathbf{0} & B^{-1}\mathbf{b} \\ \mathbf{0} & \mathbf{c}_B B^{-1}F' - \mathbf{c}_F & 1 & \mathbf{c}_B \mathbf{x}_B^0 \end{pmatrix}$$

$$= \begin{pmatrix} I & B^{-1}F & \mathbf{0} & B^{-1}\mathbf{b} \\ \mathbf{0} & \mathbf{c}_B B^{-1}F - \mathbf{c}_F & 1 & z^0 \end{pmatrix} \tag{6.9}$$

The reason there is a z^0 on the bottom right corner is that $\mathbf{x}_F = 0$ and $\left(\mathbf{x}_B^0, \mathbf{x}_F^0, z^0\right)^T$ is a solution of the system of equations represented by the above augmented matrix because it is

a solution to the system of equations corresponding to the system of equations represented by 6.6 and row operations leave solution sets unchanged. Note how attractive this is. The z_0 is the value of z at the point $\mathbf{x}^0$. The augmented matrix of 6.9 is called the simplex tableau and it is the beginning point for the simplex algorithm to be described a little later. It is very convenient to express the simplex tableau in the above form in which the variables are possibly permuted in order to have $\begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}$ on the left side. However, as far as the simplex algorithm is concerned it is not necessary to be permuting the variables in this manner. Starting with 6.9 you could permute the variables and columns to obtain an augmented matrix in which the variables are in their original order. What is really required for the simplex tableau?

It is an augmented $m+1 \times m+n+2$ matrix which represents a system of equations which has the same set of solutions, $(\mathbf{x},z)^T$ as the system whose augmented matrix is

$$\begin{pmatrix} A & \mathbf{0} & \mathbf{b} \\ -\mathbf{c} & 1 & 0 \end{pmatrix}$$

(Possibly the variables for $\mathbf{x}$ are taken in another order.) There are m linearly independent columns in the first $m+n$ columns for which there is only one nonzero entry, a 1 in one of the first m rows, the "simple columns", the other first $m+n$ columns being the "nonsimple columns". As in the above, the variables corresponding to the simple columns are $\mathbf{x}_B$, the basic variables and those corresponding to the nonsimple columns are $\mathbf{x}_F$, the free variables. Also, the top m entries of the last column on the right are nonnegative. This is the description of a simplex tableau.

In a simplex tableau it is easy to spot a basic feasible solution. You can see one quickly by setting the variables, $\mathbf{x}_F$ corresponding to the nonsimple columns equal to zero. Then the other variables, corresponding to the simple columns are each equal to a nonnegative entry in the far right column. Lets call this an "obvious basic feasible solution". If a solution is obtained by setting the variables corresponding to the nonsimple columns equal to zero and the variables corresponding to the simple columns equal to zero this will be referred to as an "obvious" solution. Lets also call the first $m+n$ entries in the bottom row the "bottom left row". In a simplex tableau, the entry in the bottom right corner gives the value of the variable being maximized or minimized when the obvious basic feasible solution is chosen.

The following is a special case of the general theory presented above and shows how such a special case can be fit into the above framework. The following example is rather typical of the sorts of problems considered. It involves inequality constraints instead of $A\mathbf{x} = \mathbf{b}$. This is handled by adding in "slack variables" as explained below.

Example 6.2.2 *Consider $z = x_1 - x_2$ subject to the constraints, $x_1+2x_2 \leq 10, x_1+2x_2 \geq 2$, and $2x_1 + x_2 \leq 6, x_i \geq 0$. Find a simplex tableau for a problem of the form $\mathbf{x} \geq \mathbf{0}, A\mathbf{x} = \mathbf{b}$ which is equivalent to the above problem.*

You add in slack variables. These are positive variables, one for each of the first three constraints, which change the first three inequalities into equations. Thus the first three inequalities become $x_1+2x_2+x_3 = 10, x_1+2x_2-x_4 = 2$, and $2x_1+x_2+x_5 = 6, x_1, x_2, x_3, x_4, x_5 \geq 0$. Now it is necessary to find a basic feasible solution. You mainly need to find a positive solution to the equations,

$$\begin{array}{c} x_1 + 2x_2 + x_3 = 10 \\ x_1 + 2x_2 - x_4 = 2 \\ 2x_1 + x_2 + x_5 = 6 \end{array} .$$

the solution set for the above system is given by

$$x_2 = \frac{2}{3}x_4 - \frac{2}{3} + \frac{1}{3}x_5, x_1 = -\frac{1}{3}x_4 + \frac{10}{3} - \frac{2}{3}x_5, x_3 = -x_4 + 8.$$

An easy way to get a basic feasible solution is to let $x_4 = 8$ and $x_5 = 1$. Then a feasible solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 5, 0, 8, 1)\,.$$

It follows $z^0 = -5$ and the matrix 6.2, $\begin{pmatrix} A & \mathbf{0} & \mathbf{b} \\ -\mathbf{c} & 1 & 0 \end{pmatrix}$ with the variables kept track of on the bottom is

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 10 \\ 1 & 2 & 0 & -1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 0 & 6 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 \end{pmatrix}$$

and the first thing to do is to permute the columns so that the list of variables on the bottom will have x_1 and x_3 at the end.

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 0 & 10 \\ 2 & -1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 & 0 & 0 & 6 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ x_2 & x_4 & x_5 & x_1 & x_3 & 0 & 0 \end{pmatrix}$$

Next, as described above, take the row reduced echelon form of the top three lines of the above matrix. This yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Now do row operations to

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

to finally obtain

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 & -5 \end{pmatrix}$$

and this is a simplex tableau. The variables are $x_2, x_4, x_5, x_1, x_3, z$.

It isn't as hard as it may appear from the above. Lets not permute the variables and simply find an acceptable simplex tableau as described above.

Example 6.2.3 *Consider $z = x_1 - x_2$ subject to the constraints, $x_1 + 2x_2 \leq 10, x_1 + 2x_2 \geq 2$, and $2x_1 + x_2 \leq 6, x_i \geq 0$. Find a simplex tableau.*

Adding in slack variables, an augmented matrix which is descriptive of the constraints is

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 10 \\ 1 & 2 & 0 & -1 & 0 & 6 \\ 2 & 1 & 0 & 0 & 1 & 6 \end{pmatrix}$$

The obvious solution is not feasible because of that -1 in the fourth column. Consider the second column and select the 2 as a pivot to zero out that which is above and below the 2. This is because that 2 satisfies the criterion for being chosen as a pivot.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 4 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ \frac{3}{2} & 0 & 0 & \frac{1}{2} & 1 & 3 \end{pmatrix}$$

This one is good. The obvious solution is now feasible. You can now assemble the simplex tableau. The first step is to include a column and row for z. This yields

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 4 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & 0 & 3 \\ \frac{3}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 3 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Now you need to get zeros in the right places so the simple columns will be preserved as simple columns. This means you need to zero out the 1 in the third column on the bottom. A simplex tableau is now

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 4 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & 0 & 3 \\ \frac{3}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 3 \\ -1 & 0 & 0 & -1 & 0 & 1 & -4 \end{pmatrix}.$$

Note it is not the same one obtained earlier. There is no reason a simplex tableau should be unique. In fact, it follows from the above general description that you have one for each basic feasible point of the region determined by the constraints.

6.3 The Simplex Algorithm

6.3.1 Maximums

The simplex algorithm takes you from one basic feasible solution to another while maximizing or minimizing the function you are trying to maximize or minimize. Algebraically, it takes you from one simplex tableau to another in which the lower right corner either increases in the case of maximization or decreases in the case of minimization.

I will continue writing the simplex tableau in such a way that the simple columns having only one entry nonzero are on the left. As explained above, this amounts to permuting the variables. I will do this because it is possible to describe what is going on without onerous notation. However, in the examples, I won't worry so much about it. Thus, from a basic feasible solution, a simplex tableau of the following form has been obtained in which the columns for the basic variables, $\mathbf{x}_B$ are listed first and $\mathbf{b} \geq \mathbf{0}$.

$$\begin{pmatrix} I & F & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{c} & 1 & z^0 \end{pmatrix} \tag{6.10}$$

Let $x_i^0 = b_i$ for $i = 1, \cdots, m$ and $x_i^0 = 0$ for $i > m$. Then $\left(\mathbf{x}^0, z^0\right)$ is a solution to the above system and since $\mathbf{b} \geq \mathbf{0}$, it follows $\left(\mathbf{x}^0, z^0\right)$ is a basic feasible solution.

If $c_i < 0$ for some i, and if $F_{ji} \leq 0$ so that a whole column of $\begin{pmatrix} F \\ \mathbf{c} \end{pmatrix}$ is ≤ 0 with the bottom entry < 0, then letting x_i be the variable corresponding to that column, you could

leave all the other entries of $\mathbf{x}_F$ equal to zero but change x_i to be positive. Let the new vector be denoted by $\mathbf{x}'_F$ and letting $\mathbf{x}'_B = \mathbf{b} - F\mathbf{x}'_F$ it follows

$$\begin{aligned}(\mathbf{x}'_B)_k &= b_k - \sum_j F_{kj} (\mathbf{x}_F)_j \\ &= b_k - F_{ki} x_i \geq 0\end{aligned}$$

Now this shows $(\mathbf{x}'_B, \mathbf{x}'_F)$ is feasible whenever $x_i > 0$ and so you could let x_i become arbitrarily large and positive and conclude there is no maximum for z because

$$z = -\mathbf{c}\mathbf{x}'_F + z^0 = (-c_i) x_i + z^0 \tag{6.11}$$

If this happens in a simplex tableau, you can say there is no maximum and stop.

What if $\mathbf{c} \geq \mathbf{0}$? Then $z = z^0 - \mathbf{c}\mathbf{x}_F$ and to satisfy the constraints, $\mathbf{x}_F \geq 0$. Therefore, in this case, z^0 is the largest possible value of z and so the maximum has been found. You stop when this occurs. Next I explain what to do if neither of the above stopping conditions hold.

The only case which remains is that some $c_i < 0$ and some $F_{ji} > 0$. You pick a column in $\begin{pmatrix} F \\ \mathbf{c} \end{pmatrix}$ in which $c_i < 0$, usually the one for which c_i is the largest in absolute value. You pick $F_{ji} > 0$ as a pivot entry, divide the j^{th} row by F_{ji} and then use to obtain zeros above F_{ji} and below F_{ji}, thus obtaining a new simple column. This row operation also makes exactly one of the other simple columns into a nonsimple column. (In terms of variables, it is said that a free variable becomes a basic variable and a basic variable becomes a free variable.) Now permuting the columns and variables, yields

$$\begin{pmatrix} I & F' & \mathbf{0} & \mathbf{b}' \\ \mathbf{0} & \mathbf{c}' & 1 & z^{0\prime} \end{pmatrix}$$

where $z^{0\prime} \geq z^0$ because $z^{0\prime} = z^0 - c_i \left(\frac{b_j}{F_{ji}}\right)$ and $c_i < 0$. If $\mathbf{b}' \geq 0$, you are in the same position you were at the beginning but now z^0 is larger. Now here is the **important** thing. You don't pick just any F_{ji} when you do these row operations. You **pick the positive one for which the row operation results in $\mathbf{b}' \geq \mathbf{0}$**. Otherwise the obvious basic feasible solution obtained by letting $\mathbf{x}'_F = \mathbf{0}$ will fail to satisfy the constraint that $\mathbf{x} \geq \mathbf{0}$.

How is this done? You need

$$b'_p \equiv b_p - \frac{F_{pi} b_j}{F_{ji}} \geq 0 \tag{6.12}$$

for each $p = 1, \cdots, m$ or equivalently,

$$b_p \geq \frac{F_{pi} b_j}{F_{ji}}. \tag{6.13}$$

Now if $F_{pi} \leq 0$ the above holds. Therefore, you only need to check F_{pi} for $F_{pi} > 0$. The pivot, F_{ji} is the one which makes the quotients of the form

$$\frac{b_p}{F_{pi}}$$

for all positive F_{pi} the smallest. Having gotten a new simplex tableau, you do the same thing to it which was just done and continue. As long as $\mathbf{b} > \mathbf{0}$, so you don't encounter the degenerate case, the values for z associated with setting $\mathbf{x}_F = \mathbf{0}$ keep getting strictly larger every time the process is repeated. You keep going until you find $\mathbf{c} \geq \mathbf{0}$. Then you stop. You are at a maximum. Problems can occur in the process in the so called degenerate case when at some stage of the process some $b_j = 0$. In this case you can cycle through different values for $\mathbf{x}$ with no improvement in z. This case will not be discussed here.

6.3.2 Minimums

How does it differ if you are finding a minimum? From a basic feasible solution, a simplex tableau of the following form has been obtained in which the simple columns for the basic variables, $\mathbf{x}_B$ are listed first and $\mathbf{b} \geq \mathbf{0}$.

$$\begin{pmatrix} I & F & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{c} & 1 & z^0 \end{pmatrix} \tag{6.14}$$

Let $x_i^0 = b_i$ for $i = 1, \cdots, m$ and $x_i^0 = 0$ for $i > m$. Then $\left(\mathbf{x}^0, z^0\right)$ is a solution to the above system and since $\mathbf{b} \geq \mathbf{0}$, it follows $\left(\mathbf{x}^0, z^0\right)$ is a basic feasible solution. So far, there is no change.

Suppose first that some $c_i > 0$ and $F_{ji} \leq 0$ for each j. Then let $\mathbf{x}'_F$ consist of changing x_i by making it positive but leaving the other entries of $\mathbf{x}_F$ equal to 0. Then from the bottom row,

$$z = -\mathbf{c}\mathbf{x}'_F + z^0 = -c_i x_i + z^0$$

and you let $\mathbf{x}'_B = \mathbf{b} - F\mathbf{x}'_F \geq \mathbf{0}$. Thus the constraints continue to hold when x_i is made increasingly positive and it follows from the above equation that there is no minimum for z. You stop when this happens.

Next suppose $\mathbf{c} \leq \mathbf{0}$. Then in this case, $z = z^0 - \mathbf{c}\mathbf{x}_F$ and from the constraints, $\mathbf{x}_F \geq \mathbf{0}$ and so $-\mathbf{c}\mathbf{x}_F \geq 0$ and so z^0 is the minimum value and you stop since this is what you are looking for.

What do you do in the case where some $c_i > 0$ and some $F_{ji} > 0$? In this case, you use the simplex algorithm as in the case of maximums to obtain a new simplex tableau in which $z^{0\prime}$ is smaller. You choose F_{ji} the same way to be the positive entry of the i^{th} column such that $b_p/F_{pi} \geq b_j/F_{ji}$ for all positive entries, F_{pi} and do the same row operations. Now this time,

$$z^{0\prime} = z^0 - c_i \left(\frac{b_j}{F_{ji}}\right) < z^0$$

As in the case of maximums no problem can occur and the process will converge unless you have the degenerate case in which some $b_j = 0$. As in the earlier case, this is most unfortunate when it occurs. You see what happens of course. z^0 does not change and the algorithm just delivers different values of the variables forever with no improvement.

To summarize the geometrical significance of the simplex algorithm, it takes you from one corner of the feasible region to another. You go in one direction to find the maximum and in another to find the minimum. For the maximum you try to get rid of negative entries of $\mathbf{c}$ and for minimums you try to eliminate positive entries of $\mathbf{c}$ where the method of elimination involves the auspicious use of an appropriate pivot entry and row operations.

Now return to Example 6.2.2. It will be modified to be a maximization problem.

Example 6.3.1 *Maximize $z = x_1 - x_2$ subject to the constraints, $x_1 + 2x_2 \leq 10, x_1 + 2x_2 \geq 2$, and $2x_1 + x_2 \leq 6, x_i \geq 0$.*

Recall this is the same as maximizing $z = x_1 - x_2$ subject to

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix}, \mathbf{x} \geq \mathbf{0},$$

the variables, x_3, x_4, x_5 being slack variables. Recall the simplex tableau was

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 & -5 \end{pmatrix}$$

with the variables ordered as x_2, x_4, x_5, x_1, x_3 and so $\mathbf{x}_B = (x_2, x_4, x_5)$ and $\mathbf{x}_F = (x_1, x_3)$.

Apply the simplex algorithm to the fourth column because $-\frac{3}{2} < 0$ and this is the most negative entry in the bottom row. The pivot is $3/2$ because $1/(3/2) = 2/3 < 5/(1/2)$. Dividing this row by $3/2$ and then using this to zero out the other elements in that column, the new simplex tableau is

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{14}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & -1 & 1 & -4 \end{pmatrix}.$$

Now there is still a negative number in the bottom left row. Therefore, the process should be continued. This time the pivot is the $2/3$ in the top of the column. Dividing the top row by $2/3$ and then using this to zero out the entries below it,

$$\begin{pmatrix} \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 1 & 0 & 7 \\ -\frac{3}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & 3 \\ \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 & 1 & 3 \end{pmatrix}.$$

Now all the numbers on the bottom left row are nonnegative so the process stops. Now recall the variables and columns were ordered as x_2, x_4, x_5, x_1, x_3. The solution in terms of x_1 and x_2 is $x_2 = 0$ and $x_1 = 3$ and $z = 3$. Note that in the above, I did not worry about permuting the columns to keep those which go with the basic varibles on the left.

Here is a bucolic example.

Example 6.3.2 *Consider the following table.*

	F_1	F_2	F_3	F_4
iron	*1*	*2*	*1*	*3*
protein	*5*	*3*	*2*	*1*
folic acid	*1*	*2*	*2*	*1*
copper	*2*	*1*	*1*	*1*
calcium	*1*	*1*	*1*	*1*

This information is available to a pig farmer and F_i denotes a particular feed. The numbers in the table contain the number of units of a particular nutrient contained in one pound of the given feed. Thus F_2 has 2 units of iron in one pound. Now suppose the cost of each feed in cents per pound is given in the following table.

F_1	F_2	F_3	F_4
2	*3*	*2*	*3*

A typical pig needs 5 units of iron, 8 of protein, 6 of folic acid, 7 of copper and 4 of calcium. (The units may change from nutrient to nutrient.) How many pounds of each feed per pig should the pig farmer use in order to minimize his cost?

His problem is to minimize $C \equiv 2x_1 + 3x_2 + 2x_3 + 3x_4$ subject to the constraints

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 &\geq 5, \\ 5x_1 + 3x_2 + 2x_3 + x_4 &\geq 8, \\ x_1 + 2x_2 + 2x_3 + x_4 &\geq 6, \\ 2x_1 + x_2 + x_3 + x_4 &\geq 7, \\ x_1 + x_2 + x_3 + x_4 &\geq 4. \end{aligned}$$

where each $x_i \geq 0$. Add in the slack variables,

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 - x_5 &= 5 \\ 5x_1 + 3x_2 + 2x_3 + x_4 - x_6 &= 8 \\ x_1 + 2x_2 + 2x_3 + x_4 - x_7 &= 6 \\ 2x_1 + x_2 + x_3 + x_4 - x_8 &= 7 \\ x_1 + x_2 + x_3 + x_4 - x_9 &= 4 \end{aligned}$$

The augmented matrix for this system is

$$\begin{pmatrix} 1 & 2 & 1 & 3 & -1 & 0 & 0 & 0 & 0 & 5 \\ 5 & 3 & 2 & 1 & 0 & -1 & 0 & 0 & 0 & 8 \\ 1 & 2 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 6 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 7 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

How in the world can you find a basic feasible solution? Remember the simplex algorithm is designed to keep the entries in the right column nonnegative so you use this algorithm a few times till the obvious solution is a basic feasible solution.

Consider the first column. The pivot is the 5. Using the row operations described in the algorithm, you get

$$\begin{pmatrix} 0 & \frac{7}{5} & \frac{3}{5} & \frac{14}{5} & -1 & \frac{1}{5} & 0 & 0 & 0 & \frac{17}{5} \\ 1 & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5} & 0 & 0 & 0 & \frac{8}{5} \\ 0 & \frac{7}{5} & \frac{8}{5} & \frac{4}{5} & 0 & \frac{1}{5} & -1 & 0 & 0 & \frac{22}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & \frac{2}{5} & 0 & -1 & 0 & \frac{19}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 0 & \frac{1}{5} & 0 & 0 & -1 & \frac{12}{5} \end{pmatrix}$$

Now go to the second column. The pivot in this column is the 7/5. This is in a different row than the pivot in the first column so I will use it to zero out everything below it. This will get rid of the zeros in the fifth column and introduce zeros in the second. This yields

$$\begin{pmatrix} 0 & 1 & \frac{3}{7} & 2 & -\frac{5}{7} & \frac{1}{7} & 0 & 0 & 0 & \frac{17}{7} \\ 1 & 0 & \frac{1}{7} & -1 & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & \frac{2}{7} & 1 & -\frac{1}{7} & \frac{3}{7} & 0 & -1 & 0 & \frac{30}{7} \\ 0 & 0 & \frac{3}{7} & 0 & \frac{2}{7} & \frac{1}{7} & 0 & 0 & -1 & \frac{10}{7} \end{pmatrix}$$

Now consider another column, this time the fourth. I will pick this one because it has some negative numbers in it so there are fewer entries to check in looking for a pivot. Unfortunately, the pivot is the top 2 and I don't want to pivot on this because it would destroy the zeros in the second column. Consider the fifth column. It is also not a good choice because the pivot is the second entry from the top and this would destroy the zeros

in the first column. Consider the sixth column. I can use either of the two bottom entries as the pivot. The matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 3 & 0 & 2 & 1 & 0 & 0 & -7 & 10 \end{pmatrix}$$

Next consider the third column. The pivot is the 1 in the third row. This yields

$$\begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 6 & -1 & 1 & 3 & 0 & -7 & 7 \end{pmatrix}.$$

There are still 5 columns which consist entirely of zeros except for one entry. Four of them have that entry equal to 1 but one still has a -1 in it, the -1 being in the fourth column. I need to do the row operations on a nonsimple column which has the pivot in the fourth row. Such a column is the second to the last. The pivot is the 3. The new matrix is

$$\begin{pmatrix} 0 & 1 & 0 & \frac{7}{3} & -1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{8}{3} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{11}{3} & -1 & 1 & \frac{2}{3} & -\frac{7}{3} & 0 & \frac{28}{3} \end{pmatrix}. \tag{6.15}$$

Now the obvious basic solution is feasible. You let $x_4 = 0 = x_5 = x_7 = x_8$ and $x_1 = 8/3, x_2 = 2/3, x_3 = 1$, and $x_6 = 28/3$. You don't need to worry too much about this. It is the above matrix which is desired. Now you can assemble the simplex tableau and begin the algorithm. Remember $C \equiv 2x_1 + 3x_2 + 2x_3 + 3x_4$. First add the row and column which deal with C. This yields

$$\begin{pmatrix} 0 & 1 & 0 & \frac{7}{3} & -1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{8}{3} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{11}{3} & -1 & 1 & \frac{2}{3} & -\frac{7}{3} & 0 & 0 & \frac{28}{3} \\ -2 & -3 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.16}$$

Now you do row operations to keep the simple columns of 6.15 simple in 6.16. Of course you could permute the columns if you wanted but this is not necessary.

This yields the following for a simplex tableau. Now it is a matter of getting rid of the positive entries in the bottom row because you are trying to minimize.

$$\begin{pmatrix} 0 & 1 & 0 & \frac{7}{3} & -1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{8}{3} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{11}{3} & -1 & 1 & \frac{2}{3} & -\frac{7}{3} & 0 & 0 & \frac{28}{3} \\ 0 & 0 & 0 & \frac{2}{3} & -1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{28}{3} \end{pmatrix}$$

The most positive of them is the 2/3 and so I will apply the algorithm to this one first. The pivot is the 7/3. After doing the row operation the next tableau is

$$\begin{pmatrix} 0 & \frac{3}{7} & 0 & 1 & -\frac{3}{7} & 0 & \frac{1}{7} & \frac{1}{7} & 0 & 0 & \frac{2}{7} \\ 1 & -\frac{1}{7} & 0 & 0 & \frac{1}{7} & 0 & \frac{2}{7} & -\frac{5}{7} & 0 & 0 & \frac{18}{7} \\ 0 & \frac{6}{7} & 1 & 0 & \frac{1}{7} & 0 & -\frac{5}{7} & \frac{2}{7} & 0 & 0 & \frac{11}{7} \\ 0 & \frac{1}{7} & 0 & 0 & -\frac{1}{7} & 0 & -\frac{2}{7} & -\frac{2}{7} & 1 & 0 & \frac{3}{7} \\ 0 & -\frac{11}{7} & 0 & 0 & \frac{4}{7} & 1 & \frac{1}{7} & -\frac{20}{7} & 0 & 0 & \frac{58}{7} \\ 0 & -\frac{2}{7} & 0 & 0 & -\frac{5}{7} & 0 & -\frac{3}{7} & -\frac{3}{7} & 0 & 1 & \frac{64}{7} \end{pmatrix}$$

and you see that all the entries are negative and so the minimum is 64/7 and it occurs when $x_1 = 18/7, x_2 = 0, x_3 = 11/7, x_4 = 2/7$.

There is no maximum for the above problem. However, I will pretend I don't know this and attempt to use the simplex algorithm. You set up the simiplex tableau the same way. Recall it is

$$\begin{pmatrix} 0 & 1 & 0 & \frac{7}{3} & -1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{8}{3} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{11}{3} & -1 & 1 & \frac{2}{3} & -\frac{7}{3} & 0 & 0 & \frac{28}{3} \\ 0 & 0 & 0 & \frac{2}{3} & -1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{28}{3} \end{pmatrix}$$

Now to maximize, you try to get rid of the negative entries in the bottom left row. The most negative entry is the -1 in the fifth column. The pivot is the 1 in the third row of this column. The new tableau is

$$\begin{pmatrix} 0 & 1 & 1 & \frac{1}{3} & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & \frac{5}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{8}{3} \\ 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{3} & 0 & 1 & -\frac{1}{3} & -\frac{7}{3} & 0 & 0 & \frac{31}{3} \\ 0 & 0 & 1 & -\frac{4}{3} & 0 & 0 & -\frac{4}{3} & -\frac{1}{3} & 0 & 1 & \frac{31}{3} \end{pmatrix}.$$

Consider the fourth column. The pivot is the top 1/3. The new tableau is

$$\begin{pmatrix} 0 & 3 & 3 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 5 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 6 & 7 & 0 & 1 & 0 & -5 & 2 & 0 & 0 & 11 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & 0 & 0 & 1 & 3 & -4 & 0 & 0 & 2 \\ 0 & 4 & 5 & 0 & 0 & 0 & -4 & 1 & 0 & 1 & 17 \end{pmatrix}$$

There is still a negative in the bottom, the -4. The pivot in that column is the 3. The algorithm yields

$$\begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} & 1 & 0 & \frac{2}{3} & 0 & -\frac{5}{3} & 0 & 0 & \frac{19}{3} \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{1}{3} & 0 & 1 & \frac{5}{3} & 0 & -\frac{14}{3} & 0 & 0 & \frac{43}{3} \\ 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & -\frac{4}{3} & 1 & 0 & \frac{8}{3} \\ 0 & -\frac{5}{3} & -\frac{4}{3} & 0 & 0 & \frac{1}{3} & 1 & -\frac{4}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{8}{3} & -\frac{1}{3} & 0 & 0 & \frac{4}{3} & 0 & -\frac{13}{3} & 0 & 1 & \frac{59}{3} \end{pmatrix}$$

Note how z keeps getting larger. Consider the column having the $-13/3$ in it. The pivot is

the single positive entry, 1/3. The next tableau is

$$\begin{pmatrix} 5 & 3 & 2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 8 \\ 3 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 14 & 7 & 5 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 19 \\ 4 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 4 \\ 4 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 2 \\ 13 & 6 & 4 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 24 \end{pmatrix}.$$

There is a column consisting of all negative entries. There is therefore, no maximum. Note also how there is no way to pick the pivot in that column.

Example 6.3.3 *Minimize* $z = x_1 - 3x_2 + x_3$ *subject to the constraints* $x_1 + x_2 + x_3 \leq 10, x_1 + x_2 + x_3 \geq 2$, $x_1 + x_2 + 3x_3 \leq 8$ *and* $x_1 + 2x_2 + x_3 \leq 7$ *with all variables nonnegative.*

There exists an answer because the region defined by the contraints is closed and bounded. Adding in slack variables you get the following augmented matrix corresponding to the constraints.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 1 & 1 & 3 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Of course there is a problem with the obvious solution obtained by setting to zero all variables corresponding to a nonsimple column because of the simple column which has the -1 in it. Therefore, I will use the simplex algorithm to make this column non simple. The third column has the 1 in the second row as the pivot so I will use this column. This yields

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ -2 & -2 & 0 & 0 & 3 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 5 \end{pmatrix} \tag{6.17}$$

and the obvious solution is feasible. Now it is time to assemble the simplex tableau. First add in the bottom row and second to last column corresponding to the the equation for z. This yields

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 8 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 2 \\ -2 & -2 & 0 & 0 & 3 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Next you need to zero out the entries in the bottom row which are below one of the simple columns in 6.17. This yields the simplex tableau

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 8 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 2 \\ -2 & -2 & 0 & 0 & 3 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 4 & 0 & 0 & -1 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The desire is to minimize this so you need to get rid of the positive entries in the left bottom row. There is only one such entry, the 4. In that column the pivot is the 1 in the second

row of this column. Thus the next tableau is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 8 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 6 \\ -1 & 0 & -1 & 0 & 2 & 0 & 1 & 0 & 3 \\ -4 & 0 & -4 & 0 & 3 & 0 & 0 & 1 & -6 \end{pmatrix}$$

There is still a positive number there, the 3. The pivot in this column is the 2. Apply the algorithm again. This yields

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{13}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ -\frac{5}{2} & 0 & -\frac{5}{2} & 0 & 0 & 0 & -\frac{3}{2} & 1 & -\frac{21}{2} \end{pmatrix}.$$

Now all the entries in the left bottom row are nonpositive so the process has stopped. The minimum is $-21/2$. It occurs when $x_1 = 0$, $x_2 = 7/2, x_3 = 0$.

Now consider the same problem but change the word, minimize to the word, maximize.

Example 6.3.4 *Maximize $z = x_1 - 3x_2 + x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 10, x_1 + x_2 + x_3 \geq 2$, $x_1 + x_2 + 3x_3 \leq 8$ and $x_1 + 2x_2 + x_3 \leq 7$ with all variables nonnegative.*

The first part of it is the same. You wind up with the same simplex tableau,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 8 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 2 \\ -2 & -2 & 0 & 0 & 3 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 4 & 0 & 0 & -1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

but this time, you apply the algorithm to get rid of the negative entries in the left bottom row. There is a -1. Use this column. The pivot is the 3. The next tableau is

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & 0 & -\frac{1}{3} & 0 & 0 & \frac{22}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{8}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{5}{3} & 0 & 0 & 0 & -\frac{1}{3} & 1 & 0 & \frac{13}{3} \\ -\frac{2}{3} & \frac{10}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 1 & \frac{8}{3} \end{pmatrix}$$

There is still a negative entry, the $-2/3$. This will be the new pivot column. The pivot is the $2/3$ on the fourth row. This yields

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 3 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ 1 & \frac{5}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & \frac{13}{2} \\ 0 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 7 \end{pmatrix}$$

and the process stops. The maximum for z is 7 and it occurs when $x_1 = 13/2, x_2 = 0, x_3 = 1/2$.

6.4 Finding A Basic Feasible Solution

By now it should be fairly clear that finding a basic feasible solution can create considerable difficulty. Indeed, given a system of linear inequalities along with the requirement that each variable be nonnegative, do there even exist points satisfying all these inequalities? If you have many variables, you can't answer this by drawing a picture. Is there some other way to do this which is more systematic than what was presented above? The answer is yes. It is called the method of artificial variables. I will illustrate this method with an example.

Example 6.4.1 *Find a basic feasible solution to the system* $2x_1+x_2-x_3 \geq 3, x_1+x_2+x_3 \geq 2, x_1+x_2+x_3 \leq 7$ *and* $\mathbf{x} \geq \mathbf{0}$.

If you write the appropriate augmented matrix with the slack variables,

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & -1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 7 \end{pmatrix} \tag{6.18}$$

The obvious solution is not feasible. This is why it would be hard to get started with the simplex method. What is the problem? It is those -1 entries in the fourth and fifth columns. To get around this, you add in artificial variables to get an augmented matrix of the form

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 7 \end{pmatrix} \tag{6.19}$$

Thus the variables are $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$. Suppose you can find a feasible solution to the system of equations represented by the above augmented matrix. Thus all variables are nonnegative. Suppose also that it can be done in such a way that x_8 and x_7 happen to be 0. Then it will follow that $x_1, \cdots, x_6$ is a feasible solution for 6.18. Conversely, if you can find a feasible solution for 6.18, then letting x_7 and x_8 both equal zero, you have obtained a feasible solution to 6.19. Since all variables are nonnegative, x_7 and x_8 both equalling zero is equivalent to saying the minimum of $z = x_7 + x_8$ subject to the constraints represented by the above augmented matrix equals zero. This has proved the following simple observation.

Observation 6.4.2 *There exists a feasible solution to the constraints represented by the augmented matrix of 6.18 and* $\mathbf{x} \geq \mathbf{0}$ *if and only if the minimum of* $x_7 + x_8$ *subject to the constraints of 6.19 and* $\mathbf{x} \geq \mathbf{0}$ *exists and equals* 0.

Of course a similar observation would hold in other similar situations. Now the point of all this is that it is trivial to see a feasible solution to 6.19, namely $x_6 = 7, x_7 = 3, x_8 = 2$ and all the other variables may be set to equal zero. Therefore, it is easy to find an initial simplex tableau for the minimization problem just described. First add the column and row for z

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix}$$

Next it is necessary to make the last two columns on the bottom left row into simple columns. Performing the row operation, this yields an initial simplex tableau,

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 7 \\ 3 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}$$

Now the algorithm involves getting rid of the positive entries on the left bottom row. Begin with the first column. The pivot is the 2. An application of the simplex algorithm yields the new tableau

$$\begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & -1 & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{11}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & -1 & 0 & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

Now go to the third column. The pivot is the $3/2$ in the second row. An application of the simplex algorithm yields

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{5}{3} \\ 0 & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix} \tag{6.20}$$

and you see there are only nonpositive numbers on the bottom left column so the process stops and yields 0 for the minimum of $z = x_7 + x_8$. As for the other variables, $x_1 = 5/3, x_2 = 0, x_3 = 1/3, x_4 = 0, x_5 = 0, x_6 = 5$. Now as explained in the above observation, this is a basic feasible solution for the original system 6.18.

Now consider a maximization problem associated with the above constraints.

Example 6.4.3 *Maximize* $x_1 - x_2 + 2x_3$ *subject to the constraints,* $2x_1 + x_2 - x_3 \geq 3, x_1 + x_2 + x_3 \geq 2, x_1 + x_2 + x_3 \leq 7$ *and* $\mathbf{x} \geq \mathbf{0}$.

From 6.20 you can immediately assemble an initial simplex tableau. You begin with the first 6 columns and top 3 rows in 6.20. Then add in the column and row for z. This yields

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{5}{3} \\ 0 & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 5 \\ -1 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and you first do row operations to make the first and third columns simple columns. Thus the next simplex tableau is

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{5}{3} \\ 0 & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & \frac{7}{3} & 0 & \frac{1}{3} & -\frac{5}{3} & 0 & 1 & \frac{7}{3} \end{pmatrix}$$

You are trying to get rid of negative entries in the bottom left row. There is only one, the $-5/3$. The pivot is the 1. The next simplex tableau is then

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{10}{3} \\ 0 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{11}{3} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & \frac{7}{3} & 0 & \frac{1}{3} & 0 & \frac{5}{3} & 1 & \frac{32}{3} \end{pmatrix}$$

and so the maximum value of z is $32/3$ and it occurs when $x_1 = 10/3, x_2 = 0$ and $x_3 = 11/3$.

6.5 Duality

You can solve minimization problems by solving maximization problems. You can also go the other direction and solve maximization problems by minimization problems. Sometimes this makes things much easier. To be more specific, the two problems to be considered are

$A.$) Minimize $z = \mathbf{c}\mathbf{x}$ subject to $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \geq \mathbf{b}$ and
$B.$) Maximize $w = \mathbf{y}\mathbf{b}$ such that $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y}A \leq \mathbf{c}$,

$$\left(\text{equivalently } A^T\mathbf{y}^T \geq \mathbf{c}^T \text{ and } w = \mathbf{b}^T\mathbf{y}^T\right).$$

In these problems it is assumed A is an $m \times p$ matrix.

I will show how a solution of the first yields a solution of the second and then show how a solution of the second yields a solution of the first. The problems, $A.$) and $B.$) are called dual problems.

Lemma 6.5.1 *Let* $\mathbf{x}$ *be a solution of the inequalities of* $A.$) *and let* $\mathbf{y}$ *be a solution of the inequalities of* $B.$). *Then*

$$\mathbf{c}\mathbf{x} \geq \mathbf{y}\mathbf{b}.$$

and if equality holds in the above, then $\mathbf{x}$ *is the solution to* $A.$) *and* $\mathbf{y}$ *is a solution to* $B.$).

Proof: This follows immediately. Since $\mathbf{c} \geq \mathbf{y}A$,

$$\mathbf{c}\mathbf{x} \geq \mathbf{y}A\mathbf{x} \geq \mathbf{y}\mathbf{b}.$$

It follows from this lemma that if $\mathbf{y}$ satisfies the inequalitites of $B.$) and $\mathbf{x}$ satisfies the inequalities of $A.$) then if equality holds in the above lemma, it must be that $\mathbf{x}$ is a solution of $A.$) and $\mathbf{y}$ is a solution of $B.$). This proves the lemma.

Now recall that to solve either of these problems using the simplex method, you first add in slack variables. Denote by $\mathbf{x}'$ and $\mathbf{y}'$ the enlarged list of variables. Thus $\mathbf{x}'$ has at least m entries and so does $\mathbf{y}'$ and the inequalities involving A were replaced by equalities whose augmented matrices were of the form

$$\begin{pmatrix} A & -I & \mathbf{b} \end{pmatrix}, \text{ and } \begin{pmatrix} A^T & I & \mathbf{c}^T \end{pmatrix}$$

Then you included the row and column for z and w to obtain

$$\begin{pmatrix} A & -I & \mathbf{0} & \mathbf{b} \\ -\mathbf{c} & \mathbf{0} & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} A^T & I & \mathbf{0} & \mathbf{c}^T \\ -\mathbf{b}^T & \mathbf{0} & 1 & 0 \end{pmatrix}. \tag{6.21}$$

Then the problems have basic feasible solutions if it is possible to permute the first $p+m$ columns in the above two matrices and obtain matrices of the form

$$\begin{pmatrix} B & F & \mathbf{0} & \mathbf{b} \\ -\mathbf{c}_B & -\mathbf{c}_F & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} B_1 & F_1 & \mathbf{0} & \mathbf{c}^T \\ -\mathbf{b}_{B_1}^T & -\mathbf{b}_{F_1}^T & 1 & 0 \end{pmatrix} \tag{6.22}$$

where B, B_1 are invertible $m \times m$ and $p \times p$ matrices and denoting the variables associated with these columns by $\mathbf{x}_B, \mathbf{y}_B$ and those variables associated with F or F_1 by $\mathbf{x}_F$ and $\mathbf{y}_F$, it follows that letting $B\mathbf{x}_B = \mathbf{b}$ and $\mathbf{x}_F = \mathbf{0}$, the resulting vector, $\mathbf{x}'$ is a solution to $\mathbf{x}' \geq \mathbf{0}$ and $\begin{pmatrix} A & -I \end{pmatrix}\mathbf{x}' = \mathbf{b}$ with similar constraints holding for $\mathbf{y}'$. In other words, it is possible to obtain simplex tableaus,

$$\begin{pmatrix} I & B^{-1}F & \mathbf{0} & B^{-1}\mathbf{b} \\ \mathbf{0} & \mathbf{c}_B B^{-1}F - \mathbf{c}_F & 1 & \mathbf{c}_B B^{-1}\mathbf{b} \end{pmatrix}, \begin{pmatrix} I & B_1^{-1}F_1 & \mathbf{0} & B_1^{-1}\mathbf{c}^T \\ \mathbf{0} & \mathbf{b}_{B_1}^T B_1^{-1}F - \mathbf{b}_{F_1}^T & 1 & \mathbf{b}_{B_1}^T B_1^{-1}\mathbf{c}^T \end{pmatrix} \tag{6.23}$$

Similar considerations apply to the second problem. Thus as just described, a basic feasible solution is one which determines a simplex tableau like the above in which you get a feasible solution by setting all but the first m variables equal to zero. The simplex algorithm takes you from one basic feasible solution to another till eventually, if there is no degeneracy, you obtain a basic feasible solution which yields the solution of the problem of interest.

Theorem 6.5.2 *Suppose there exists a solution,* $\mathbf{x}$ *to* $A.$*) where* $\mathbf{x}$ *is a basic feasible solution of the inequalities of* $\mathbf{A}.$*). Then there exists a solution,* $\mathbf{y}$ *to* $B.$*) and* $\mathbf{cx} = \mathbf{by}$*. It is also possible to find* $\mathbf{y}$ *from* $\mathbf{x}$ *using a simple formula.*

Proof: Since the solution to $A.$) is basic and feasible, there exists a simplex tableau like 6.23 such that $\mathbf{x}'$ can be split into $\mathbf{x}_B$ and $\mathbf{x}_F$ such that $\mathbf{x}_F = 0$ and $\mathbf{x}_B = B^{-1}\mathbf{b}$. Now since it is a minimizer, it follows $\mathbf{c}_B B^{-1}F - \mathbf{c}_F \leq \mathbf{0}$ and the minimum value for $\mathbf{cx}$ is $\mathbf{c}_B B^{-1}\mathbf{b}$. Stating this again, $\mathbf{cx} = \mathbf{c}_B B^{-1}\mathbf{b}$. Is it posible you can take $\mathbf{y} = \mathbf{c}_B B^{-1}$? From Lemma 6.5.1 this will be so if $\mathbf{c}_B B^{-1}$ solves the constraints of problem $B.$). Is $\mathbf{c}_B B^{-1} \geq 0$? Is $\mathbf{c}_B B^{-1} A \leq \mathbf{c}$? These two conditions are satisfied if and only if $\mathbf{c}_B B^{-1} \begin{pmatrix} A & -I \end{pmatrix} \leq \begin{pmatrix} \mathbf{c} & \mathbf{0} \end{pmatrix}$. Referring to the process of permuting the columns of the first augmented matrix of 6.21 to get 6.22 and doing the same permutations on the columns of $\begin{pmatrix} A & -I \end{pmatrix}$ and $\begin{pmatrix} \mathbf{c} & \mathbf{0} \end{pmatrix}$, the desired inequality holds if and only if $\mathbf{c}_B B^{-1} \begin{pmatrix} B & F \end{pmatrix} \leq \begin{pmatrix} \mathbf{c}_B & \mathbf{c}_F \end{pmatrix}$ which is equivalent to saying $\begin{pmatrix} \mathbf{c}_B & \mathbf{c}_B B^{-1}F \end{pmatrix} \leq \begin{pmatrix} \mathbf{c}_B & \mathbf{c}_F \end{pmatrix}$ and this is true because $\mathbf{c}_B B^{-1}F - \mathbf{c}_F \leq \mathbf{0}$ due to the assumption that $\mathbf{x}$ is a minimizer. The simple formula is just

$$\mathbf{y} = \mathbf{c}_B B^{-1}.$$

This proves the theorem.

The proof of the following corollary is similar.

Corollary 6.5.3 *Suppose there exists a solution,* $\mathbf{y}$ *to* $B.$*) where* $\mathbf{y}$ *is a basic feasible solution of the inequalities of* $B.$*). Then there exists a solution,* $\mathbf{x}$ *to* $A.$*) and* $\mathbf{cx} = \mathbf{by}$*. It is also possible to find* $\mathbf{x}$ *from* $\mathbf{y}$ *using a simple formula. In this case, and referring to 6.23, the simple formula is* $\mathbf{x} = B_1^{-T}\mathbf{b}_{B_1}$.

As an example, consider the pig farmers problem. The main difficulty in this problem was finding an initial simplex tableau. Now consider the following example and marvel at how all the difficulties disappear.

Example 6.5.4 *minimize* $C \equiv 2x_1 + 3x_2 + 2x_3 + 3x_4$ *subject to the constraints*

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 &\geq 5, \\ 5x_1 + 3x_2 + 2x_3 + x_4 &\geq 8, \\ x_1 + 2x_2 + 2x_3 + x_4 &\geq 6, \\ 2x_1 + x_2 + x_3 + x_4 &\geq 7, \\ x_1 + x_2 + x_3 + x_4 &\geq 4. \end{aligned}$$

where each $x_i \geq 0$.

Here the dual problem is to maximize $w = 5y_1 + 8y_2 + 6y_3 + 7y_4 + 4y_5$ subject to the constraints

$$\begin{pmatrix} 1 & 5 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \end{pmatrix}.$$

Adding in slack variables, these inequalities are equivalent to the system of equations whose augmented matrix is

$$\begin{pmatrix} 1 & 5 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 3 \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 2 \\ 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Now the obvious solution is feasible so there is no hunting for an initial obvious feasible solution required. Now add in the row and column for w. This yields

$$\begin{pmatrix} 1 & 5 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ -5 & -8 & -6 & -7 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is a maximization problem so you want to eliminate the negatives in the bottom left row. Pick the column having the one which is most negative, the -8. The pivot is the top 5. Then apply the simplex algorithm to obtain

$$\begin{pmatrix} \frac{1}{5} & 1 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{2}{5} \\ \frac{7}{5} & 0 & \frac{7}{5} & -\frac{1}{5} & \frac{2}{5} & -\frac{3}{5} & 1 & 0 & 0 & 0 & \frac{9}{5} \\ \frac{3}{5} & 0 & \frac{8}{5} & \frac{1}{5} & \frac{3}{5} & -\frac{2}{5} & 0 & 1 & 0 & 0 & \frac{8}{5} \\ \frac{14}{5} & 0 & \frac{4}{5} & \frac{3}{5} & \frac{4}{5} & -\frac{1}{5} & 0 & 0 & 1 & 0 & \frac{13}{5} \\ -\frac{17}{5} & 0 & -\frac{22}{5} & -\frac{19}{5} & -\frac{12}{5} & \frac{8}{5} & 0 & 0 & 0 & 1 & \frac{16}{5} \end{pmatrix}.$$

There are still negative entries in the bottom left row. Do the simplex algorithm to the column which has the $-\frac{22}{5}$. The pivot is the $\frac{8}{5}$. This yields

$$\begin{pmatrix} \frac{1}{8} & 1 & 0 & \frac{3}{8} & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{8} & 0 & 0 & \frac{1}{4} \\ \frac{7}{8} & 0 & 0 & -\frac{3}{8} & -\frac{1}{8} & -\frac{1}{4} & 1 & -\frac{7}{8} & 0 & 0 & \frac{3}{4} \\ \frac{3}{8} & 0 & 1 & \frac{1}{8} & \frac{3}{8} & -\frac{1}{4} & 0 & \frac{5}{8} & 0 & 0 & \frac{3}{4} \\ \frac{5}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 2 \\ -\frac{7}{4} & 0 & 0 & -\frac{13}{4} & -\frac{3}{4} & \frac{1}{2} & 0 & \frac{11}{4} & 0 & 1 & \frac{13}{2} \end{pmatrix}$$

and there are still negative numbers. Pick the column which has the $-13/4$. The pivot is the $3/8$ in the top. This yields

$$\begin{pmatrix} \frac{1}{3} & \frac{8}{3} & 0 & 1 & \frac{1}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} \\ \frac{7}{3} & -\frac{4}{3} & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 1 & 0 & \frac{5}{3} \\ -\frac{2}{3} & \frac{26}{3} & 0 & 0 & \frac{1}{3} & \frac{8}{3} & 0 & \frac{5}{3} & 0 & 1 & \frac{26}{3} \end{pmatrix}$$

which has only one negative entry on the bottom left. The pivot for this first column is the $\frac{7}{3}$. The next tableau is

$$\begin{pmatrix} 0 & \frac{20}{7} & 0 & 1 & \frac{2}{7} & \frac{5}{7} & 0 & -\frac{2}{7} & -\frac{1}{7} & 0 & \frac{3}{7} \\ 0 & \frac{11}{7} & 0 & 0 & -\frac{1}{7} & \frac{1}{7} & 1 & -\frac{6}{7} & -\frac{3}{7} & 0 & \frac{2}{7} \\ 0 & -\frac{1}{7} & 1 & 0 & \frac{2}{7} & -\frac{2}{7} & 0 & \frac{5}{7} & -\frac{1}{7} & 0 & \frac{3}{7} \\ 1 & -\frac{4}{7} & 0 & 0 & \frac{1}{7} & -\frac{1}{7} & 0 & -\frac{1}{7} & \frac{3}{7} & 0 & \frac{5}{7} \\ 0 & \frac{58}{7} & 0 & 0 & \frac{3}{7} & \frac{18}{7} & 0 & \frac{11}{7} & \frac{2}{7} & 1 & \frac{64}{7} \end{pmatrix}$$

and all the entries in the left bottom row are nonnegative so the answer is $64/7$. This is the same as obtained before. So what values for $\mathbf{x}$ are needed? Here the basic variables are y_1, y_3, y_4, y_7. Consider the original augmented matrix, one step before the simplex tableau.

$$\begin{pmatrix} 1 & 5 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ -5 & -8 & -6 & -7 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Permute the columns to put the columns associated with these basic variables first. Thus

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 5 & 1 & 1 & 0 & 0 & 0 & 2 \\ 2 & 2 & 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 2 \\ 3 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\ -5 & -6 & -7 & 0 & -8 & -4 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix, B is

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix}$$

and so B^{-T} equals

$$\begin{pmatrix} -\frac{1}{7} & -\frac{2}{7} & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 1 \\ -\frac{1}{7} & \frac{5}{7} & -\frac{2}{7} & -\frac{6}{7} \\ \frac{3}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{3}{7} \end{pmatrix}$$

Also $\mathbf{b}_B^T = \begin{pmatrix} 5 & 6 & 7 & 0 \end{pmatrix}$ and so from Corollary 6.5.3,

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{7} & -\frac{2}{7} & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 1 \\ -\frac{1}{7} & \frac{5}{7} & -\frac{2}{7} & -\frac{6}{7} \\ \frac{3}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{18}{7} \\ 0 \\ \frac{11}{7} \\ \frac{2}{7} \end{pmatrix}$$

which agrees with the original way of doing the problem.

Two good books which give more discussion of linear programming are Strang [19] and Nobel and Daniels [14]. Also listed in these books are other references which may prove useful if you are interested in seeing more on these topics. There is a great deal more which can be said about linear programming.

6.6 Exercises

1. Maximize and minimize $z = x_1 - 2x_2 + x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 2$, and $x_1 + 2x_2 + x_3 \leq 7$ if possible. All variables are nonnegative.

2. Maximize and minimize the following is possible. All variables are nonnegative.

   (a) $z = x_1 - 2x_2$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$

   (b) $z = x_1 - 2x_2 - 3x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 8$, $x_1 + x_2 + 3x_3 \geq 1$, and $x_1 + x_2 + x_3 \leq 7$

   (c) $z = 2x_1 + x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$

   (d) $z = x_1 + 2x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$

3. Consider contradictory constraints, $x_1 + x_2 \geq 12$ and $x_1 + 2x_2 \leq 5$. You know these two contradict but show they contradict using the simplex algorithm.

4. Find a solution to the following inequalities $x, y \geq 0$ and if it is possible to do so. If it is not possible, prove it is not possible.

   (a)
   $$\begin{aligned} 6x + 3y &\geq 4 \\ 8x + 4y &\leq 5 \end{aligned}$$

   (b)
   $$\begin{aligned} 6x_1 + 4x_3 &\leq 11 \\ 5x_1 + 4x_2 + 4x_3 &\geq 8 \\ 6x_1 + 6x_2 + 5x_3 &\leq 11 \end{aligned}$$

   (c)
   $$\begin{aligned} 6x_1 + 4x_3 &\leq 11 \\ 5x_1 + 4x_2 + 4x_3 &\geq 9 \\ 6x_1 + 6x_2 + 5x_3 &\leq 9 \end{aligned}$$

   (d)
   $$\begin{aligned} x_1 - x_2 + x_3 &\leq 2 \\ x_1 + 2x_2 &\geq 4 \\ 3x_1 + 2x_3 &\leq 7 \end{aligned}$$

   (e)
   $$\begin{aligned} 5x_1 - 2x_2 + 4x_3 &\leq 1 \\ 6x_1 - 3x_2 + 5x_3 &\geq 2 \\ 5x_1 - 2x_2 + 4x_3 &\leq 5 \end{aligned}$$

5. Minimize $z = x_1 + x_2$ subject to $x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 20,\ x_1 + x_2 \leq 18$. Change to a maximization problem and solve as follows: Let $y_i = M - x_i$. Formulate in terms of y_1, y_2.

Spectral Theory

Spectral Theory refers to the study of eigenvalues and eigenvectors of a matrix. It is of fundamental importance in many areas. Row operations will no longer be such a useful tool in this subject.

7.1 Eigenvalues And Eigenvectors Of A Matrix

The field of scalars in spectral theory is best taken to equal $\mathbb{C}$ although I will sometimes refer to it as $\mathbb{F}$.

Definition 7.1.1 *Let M be an $n \times n$ matrix and let $\mathbf{x} \in \mathbb{C}^n$ be a nonzero vector for which*

$$M\mathbf{x} = \lambda\mathbf{x} \tag{7.1}$$

for some scalar, λ. Then $\mathbf{x}$ is called an eigenvector and λ is called an eigenvalue (characteristic value) of the matrix, M.

Eigenvectors are never equal to zero!

The set of all eigenvalues of an $n \times n$ matrix, M, is denoted by $\sigma(M)$ and is referred to as the spectrum of M.

Eigenvectors are vectors which are shrunk, stretched or reflected upon multiplication by a matrix. How can they be identified? Suppose $\mathbf{x}$ satisfies 7.1. Then

$$(\lambda I - M)\mathbf{x} = \mathbf{0}$$

for some $\mathbf{x} \neq \mathbf{0}$. Therefore, the matrix $M - \lambda I$ cannot have an inverse and so by Theorem 3.3.15

$$\det(\lambda I - M) = 0. \tag{7.2}$$

In other words, λ must be a zero of the characteristic polynomial. Since M is an $n \times n$ matrix, it follows from the theorem on expanding a matrix by its cofactor that this is a polynomial equation of degree n. As such, it has a solution, $\lambda \in \mathbb{C}$. Is it actually an eigenvalue? The answer is yes and this follows from Theorem 3.3.23 on Page 93. Since $\det(\lambda I - M) = 0$ the matrix, $\lambda I - M$ cannot be one to one and so there exists a nonzero vector, $\mathbf{x}$ such that $(\lambda I - M)\mathbf{x} = \mathbf{0}$. This proves the following corollary.

Corollary 7.1.2 *Let M be an $n \times n$ matrix and $\det(M - \lambda I) = 0$. Then there exists $\mathbf{x} \in \mathbb{C}^n$ such that $(M - \lambda I)\mathbf{x} = \mathbf{0}$.*

As an example, consider the following.

Example 7.1.3 *Find the eigenvalues and eigenvectors for the matrix,*

$$A = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}.$$

You first need to identify the eigenvalues. Recall this requires the solution of the equation

$$\det\left(\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right) = 0$$

When you expand this determinant, you find the equation is

$$(\lambda - 5)\left(\lambda^2 - 20\lambda + 100\right) = 0$$

and so the eigenvalues are

$$5, 10, 10.$$

I have listed 10 twice because it is a zero of multiplicity two due to

$$\lambda^2 - 20\lambda + 100 = (\lambda - 10)^2.$$

Having found the eigenvalues, it only remains to find the eigenvectors. First find the eigenvectors for $\lambda = 5$. As explained above, this requires you to solve the equation,

$$\left(5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is you need to find the solution to

$$\begin{pmatrix} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By now this is an old problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\begin{pmatrix} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{pmatrix}. \tag{7.3}$$

The reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the solution is any vector of the form

$$\begin{pmatrix} \frac{5}{4}z \\ \frac{-1}{2}z \\ z \end{pmatrix} = z\begin{pmatrix} \frac{5}{4} \\ \frac{-1}{2} \\ 1 \end{pmatrix}$$

where $z \in \mathbb{F}$. You would obtain the same collection of vectors if you replaced z with $4z$. Thus a simpler description for the solutions to this system of equations whose augmented matrix is in 7.3 is

$$z \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \tag{7.4}$$

where $z \in \mathbb{F}$. Now you need to remember that you can't take $z = 0$ because this would result in the zero vector and

Eigenvectors are never equal to zero!

Other than this value, every other choice of z in 7.4 results in an eigenvector. It is a good idea to check your work! To do so, I will take the original matrix and multiply by this vector and see if I get 5 times this vector.

$$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 25 \\ -10 \\ 20 \end{pmatrix} = 5 \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

so it appears this is correct. Always check your work on these problems if you care about getting the answer right.

The variable, z is called a free variable or sometimes a parameter. The set of vectors in 7.4 is called the eigenspace and it equals $\ker(\lambda I - A)$. You should observe that in this case the eigenspace has dimension 1 because there is one vector which spans the eigenspace. In general, you obtain the solution from the row echelon form and the number of different free variables gives you the dimension of the eigenspace. Just remember that not every vector in the eigenspace is an eigenvector. The vector, $\mathbf{0}$ is not an eigenvector although it is in the eigenspace because

Eigenvectors are never equal to zero!

Next consider the eigenvectors for $\lambda = 10$. These vectors are solutions to the equation,

$$\left(10 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

That is you must find the solutions to

$$\begin{pmatrix} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which reduces to consideration of the augmented matrix,

$$\begin{pmatrix} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{pmatrix}$$

The row reduced echelon form for this matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the eigenvectors are of the form

$$\begin{pmatrix} -2y-z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

You can't pick z and y both equal to zero because this would result in the zero vector and

Eigenvectors are never equal to zero!

However, every other choice of z and y does result in an eigenvector for the eigenvalue $\lambda = 10$. As in the case for $\lambda = 5$ you should check your work if you care about getting it right.

$$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ 10 \end{pmatrix} = 10\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

so it worked. The other vector will also work. Check it.

The above example shows how to find eigenvectors and eigenvalues algebraically. You may have noticed it is a bit long. Sometimes students try to first row reduce the matrix before looking for eigenvalues. This is a terrible idea because row operations destroy the value of the eigenvalues. The eigenvalue problem is really not about row operations. A general rule to remember about the eigenvalue problem is this.

If it is not long and hard it is usually wrong!

The eigenvalue problem is the hardest problem in algebra and people still do research on ways to find eigenvalues. Now if you are so fortunate as to find the eigenvalues as in the above example, then finding the eigenvectors does reduce to row operations and this part of the problem is easy. However, finding the eigenvalues is anything but easy because for an $n \times n$ matrix, it involves solving a polynomial equation of degree n and none of us are very good at doing this. If you only find a good approximation to the eigenvalue, it won't work. It either is or is not an eigenvalue and if it is not, the only solution to the equation, $(\lambda I - M)\,\mathbf{x} = \mathbf{0}$ will be the zero solution as explained above and

Eigenvectors are never equal to zero!

Here is another example.

Example 7.1.4 *Let*

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

First find the eigenvalues.

$$\det\left(\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}\right) = 0$$

This is $\lambda^3 - 6\lambda^2 + 8\lambda = 0$ and the solutions are 0, 2, and 4.

0 Can be an Eigenvalue!

Now find the eigenvectors. For $\lambda = 0$ the augmented matrix for finding the solutions is

$$\begin{pmatrix} 2 & 2 & -2 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

and the row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the eigenvectors are of the form

$$z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

where $z \neq 0$.

Next find the eigenvectors for $\lambda = 2$. The augmented matrix for the system of equations needed to find these eigenvectors is

$$\begin{pmatrix} 0 & -2 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

and the row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the eigenvectors are of the form

$$z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $z \neq 0$.

Finally find the eigenvectors for $\lambda = 4$. The augmented matrix for the system of equations needed to find these eigenvectors is

$$\begin{pmatrix} 2 & -2 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 \end{pmatrix}$$

and the row reduced echelon form is

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenvectors are of the form

$$y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

where $y \neq 0$.

Example 7.1.5 *Let*

$$A = \begin{pmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ 14 & 25 & 14 \end{pmatrix}.$$

Find the eigenvectors and eigenvalues.

In this case the eigenvalues are $3, 6, 6$ where I have listed 6 twice because it is a zero of algebraic multiplicity two, the characteristic equation being

$$(\lambda - 3)(\lambda - 6)^2 = 0.$$

It remains to find the eigenvectors for these eigenvalues. First consider the eigenvectors for $\lambda = 3$. You must solve

$$\left(3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ 14 & 25 & 14 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ -14 & -25 & -11 & 0 \end{pmatrix}$$

and the row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the eigenvectors are nonzero vectors of the form

$$\begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Next consider the eigenvectors for $\lambda = 6$. This requires you to solve

$$\left(6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ 14 & 25 & 14 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and the augmented matrix for this system of equations is

$$\begin{pmatrix} 4 & 2 & 1 & 0 \\ 2 & 7 & 2 & 0 \\ -14 & -25 & -8 & 0 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & \frac{1}{8} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the eigenvectors for $\lambda = 6$ are of the form

$$z\begin{pmatrix} -\frac{1}{8} \\ -\frac{1}{4} \\ 1 \end{pmatrix}$$

or written more simply,

$$z\begin{pmatrix} -1 \\ -2 \\ 8 \end{pmatrix}$$

where $z \in \mathbb{F}$.

Note that in this example the eigenspace for the eigenvalue, $\lambda = 6$ is of dimension 1 because there is only one parameter which can be chosen. However, this eigenvalue is of multiplicity two as a root to the characteristic equation.

Definition 7.1.6 *If A is an $n \times n$ matrix with the property that some eigenvalue has algebraic multiplicity as a root of the characteristic equation which is greater than the dimension of the eigenspace associated with this eigenvalue, then the matrix is called defective.*

There may be repeated roots to the characteristic equation, 7.2 and it is not known whether the dimension of the eigenspace equals the multiplicity of the eigenvalue. However, the following theorem is available.

Theorem 7.1.7 *Suppose $M\mathbf{v}_i = \lambda_i \mathbf{v}_i, i = 1, \cdots, r$, $\mathbf{v}_i \neq 0$, and that if $i \neq j$, then $\lambda_i \neq \lambda_j$. Then the set of eigenvectors, $\{\mathbf{v}_i\}_{i=1}^r$ is linearly independent.*

Proof: If the conclusion of this theorem is not true, then there exist non zero scalars, c_{k_j} such that

$$\sum_{j=1}^{m} c_{k_j} \mathbf{v}_{k_j} = \mathbf{0}. \tag{7.5}$$

Since any nonempty set of non negative integers has a smallest integer in the set, take m is as small as possible for this to take place. Then solving for $\mathbf{v}_{k_1}$

$$\mathbf{v}_{k_1} = \sum_{k_j \neq k_1} d_{k_j} \mathbf{v}_{k_j} \tag{7.6}$$

where $d_{k_j} = c_{k_j}/c_{k_1} \neq 0$. Multiplying both sides by M,

$$\lambda_{k_1} \mathbf{v}_{k_1} = \sum_{k_j \neq k_1} d_{k_j} \lambda_{k_j} \mathbf{v}_{k_j},$$

which from 7.6 yields

$$\sum_{k_j \neq k_1} d_{k_j} \lambda_{k_1} \mathbf{v}_{k_j} = \sum_{k_j \neq k_1} d_{k_j} \lambda_{k_j} \mathbf{v}_{k_j}$$

and therefore,

$$\mathbf{0} = \sum_{k_j \neq k_1} d_{k_j} \left(\lambda_{k_1} - \lambda_{k_j}\right) \mathbf{v}_{k_j},$$

a sum having fewer than m terms. However, from the assumption that m is as small as possible for 7.5 to hold with all the scalars, c_{k_j} non zero, it follows that for some $j \neq 1$,

$$d_{k_j} \left(\lambda_{k_1} - \lambda_{k_j}\right) = 0$$

which implies $\lambda_{k_1} = \lambda_{k_j}$, a contradiction.

In words, this theorem says that eigenvectors associated with distinct eigenvalues are linearly independent.

Sometimes you have to consider eigenvalues which are complex numbers. This occurs in differential equations for example. You do these problems exactly the same way as you do the ones in which the eigenvalues are real. Here is an example.

Example 7.1.8 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

You need to find the eigenvalues. Solve

$$\det\left(\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\right) = 0.$$

This reduces to $(\lambda - 1)\left(\lambda^2 - 4\lambda + 5\right) = 0$. The solutions are $\lambda = 1, \lambda = 2 + i, \lambda = 2 - i$.

There is nothing new about finding the eigenvectors for $\lambda = 1$ so consider the eigenvalue $\lambda = 2 + i$. You need to solve

$$\left((2+i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In other words, you must consider the augmented matrix,

$$\begin{pmatrix} 1+i & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{pmatrix}$$

for the solution. Divide the top row by $(1 + i)$ and then take $-i$ times the second row and add to the bottom. This yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now multiply the second row by $-i$ to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the eigenvectors are of the form

$$z\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}.$$

You should find the eigenvectors for $\lambda = 2 - i$. These are

$$z\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}.$$

As usual, if you want to get it right you had better check it.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1-2i \\ 2-i \end{pmatrix} = (2-i)\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

so it worked.

7.2 Some Applications Of Eigenvalues And Eigenvectors

Recall that $n \times n$ matrices can be considered as linear transformations. If F is a 3×3 real matrix having positive determinant, it can be shown that $F = RU$ where R is a rotation matrix and U is a symmetric real matrix having positive eigenvalues. An application of this wonderful result, known to mathematicians as the right polar decomposition, is to continuum mechanics where a chunk of material is identified with a set of points in three dimensional space.

The linear transformation, F in this context is called the deformation gradient and it describes the local deformation of the material. Thus it is possible to consider this deformation in terms of two processes, one which distorts the material and the other which just rotates it. It is the matrix, U which is responsible for stretching and compressing. This is why in continuum mechanics, the stress is often taken to depend on U which is known in this context as the right Cauchy Green strain tensor. This process of writing a matrix as a product of two such matrices, one of which preserves distance and the other which distorts is also important in applications to geometric measure theory an interesting field of study in mathematics and to the study of quadratic forms which occur in many applications such as statistics. Here I am emphasizing the application to mechanics in which the eigenvectors of U determine the principle directions, those directions in which the material is stretched or compressed to the maximum extent.

Example 7.2.1 *Find the principle directions determined by the matrix,*

$$\begin{pmatrix} \frac{29}{11} & \frac{6}{11} & \frac{6}{11} \\ \frac{6}{11} & \frac{41}{44} & \frac{19}{44} \\ \frac{6}{11} & \frac{19}{44} & \frac{41}{44} \end{pmatrix}$$

The eigenvalues are $3, 1,$ *and* $\frac{1}{2}$.

It is nice to be given the eigenvalues. The largest eigenvalue is 3 which means that in the direction determined by the eigenvector associated with 3 the stretch is three times as large. The smallest eigenvalue is $1/2$ and so in the direction determined by the eigenvector for $1/2$ the material is compressed, becoming locally half as long. It remains to find these directions. First consider the eigenvector for 3. It is necessary to solve

$$\left(3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{29}{11} & \frac{6}{11} & \frac{6}{11} \\ \frac{6}{11} & \frac{41}{44} & \frac{19}{44} \\ \frac{6}{11} & \frac{19}{44} & \frac{41}{44} \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus the augmented matrix for this system of equations is

$$\begin{pmatrix} \frac{4}{11} & -\frac{6}{11} & -\frac{6}{11} & 0 \\ -\frac{6}{11} & \frac{91}{44} & -\frac{19}{44} & 0 \\ -\frac{6}{11} & -\frac{19}{44} & \frac{91}{44} & 0 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the principle direction for the eigenvalue, 3 in which the material is stretched to the maximum extent is

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

A direction vector in this direction is

$$\begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}.$$

You should show that the direction in which the material is compressed the most is in the direction

$$\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Note this is meaningful information which you would have a hard time finding without the theory of eigenvectors and eigenvalues.

Another application is to the problem of finding solutions to systems of differential equations. It turns out that vibrating systems involving masses and springs can be studied in the form

$$\mathbf{x}'' = A\mathbf{x} \tag{7.7}$$

where A is a real symmetric $n \times n$ matrix which has nonpositive eigenvalues. This is analogous to the case of the scalar equation for undamped oscillation, $x'' + \omega^2 x = 0$. The main difference is that here the scalar ω^2 is replaced with the matrix, $-A$. Consider the problem of finding solutions to 7.7. You look for a solution which is in the form

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} \tag{7.8}$$

and substitute this into 7.7. Thus

$$\mathbf{x}'' = \mathbf{v}\lambda^2 e^{\lambda t} = e^{\lambda t} A\mathbf{v}$$

and so

$$\lambda^2 \mathbf{v} = A\mathbf{v}.$$

Therefore, λ^2 needs to be an eigenvalue of A and $\mathbf{v}$ needs to be an eigenvector. Since A has nonpositive eigenvalues, $\lambda^2 = -a^2$ and so $\lambda = \pm ia$ where $-a^2$ is an eigenvalue of A. Corresponding to this you obtain solutions of the form

$$\mathbf{x}(t) = \mathbf{v}\cos(at), \mathbf{v}\sin(at).$$

Note these solutions oscillate because of the $\cos(at)$ and $\sin(at)$ in the solutions. Here is an example.

Example 7.2.2 *Find oscillatory solutions to the system of differential equations,* $\mathbf{x}'' = A\mathbf{x}$ *where*

$$A = \begin{pmatrix} -\frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{13}{6} & \frac{5}{6} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{13}{6} \end{pmatrix}.$$

The eigenvalues are $-1, -2,$ *and* -3.

According to the above, you can find solutions by looking for the eigenvectors. Consider the eigenvectors for -3. The augmented matrix for finding the eigenvectors is

$$\begin{pmatrix} -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{6} & -\frac{5}{6} & 0 \\ \frac{1}{3} & -\frac{5}{6} & -\frac{5}{6} & 0 \end{pmatrix}$$

and its row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenvectors are of the form

$$\mathbf{v} = z \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

It follows

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cos\left(\sqrt{3}t\right), \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \sin\left(\sqrt{3}t\right)$$

are both solutions to the system of differential equations. You can find other oscillatory solutions in the same way by considering the other eigenvalues. You might try checking these answers to verify they work.

This is just a special case of a procedure used in differential equations to obtain closed form solutions to systems of differential equations using linear algebra. The overall philosophy is to take one of the easiest problems in analysis and change it into the eigenvalue problem which is the most difficult problem in algebra. However, when it works, it gives precise solutions in terms of known functions.

7.3 Exercises

1. ♠If A is the matrix of a linear transformation which rotates all vectors in $\mathbb{R}^2$ through $30°$, explain why A cannot have any real eigenvalues.

2. ♠If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA.

3. ♠If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

4. ♠Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose $\mathbf{x}$ is an eigenvector of B. Show that then $A\mathbf{x}$ must also be an eigenvector for B.

5. ♠Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

6. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$, then whenever a, b are scalars,

$$A(a\mathbf{x} + b\mathbf{y}) = \lambda(a\mathbf{x} + b\mathbf{y}).$$

   Does this imply that $a\mathbf{x} + b\mathbf{y}$ is an eigenvector? Explain.

7. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -19 & -14 & -1 \\ 8 & 4 & 8 \\ 15 & 30 & -3 \end{pmatrix}.$$

   Determine whether the matrix is defective.

8. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} -3 & -30 & 15 \\ 0 & 12 & 0 \\ 15 & 30 & -3 \end{pmatrix}.$$
Determine whether the matrix is defective.

9. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} 8 & 4 & 5 \\ 0 & 12 & 9 \\ -2 & 2 & 10 \end{pmatrix}.$$
Determine whether the matrix is defective.

10. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} 12 & -12 & 6 \\ 0 & 18 & 0 \\ 6 & 12 & 12 \end{pmatrix}$$

11. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} -5 & -1 & 10 \\ -15 & 9 & -6 \\ 8 & -8 & 2 \end{pmatrix}.$$
Determine whether the matrix is defective.

12. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} -10 & -8 & 8 \\ -4 & -14 & -4 \\ 0 & 0 & -18 \end{pmatrix}.$$
Determine whether the matrix is defective.

13. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} 1 & 26 & -17 \\ 4 & -4 & 4 \\ -9 & -18 & 9 \end{pmatrix}.$$
Determine whether the matrix is defective.

14. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} 8 & 4 & 5 \\ 0 & 12 & 9 \\ -2 & 2 & 10 \end{pmatrix}.$$
Determine whether the matrix is defective.

15. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} 9 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & -6 & 9 \end{pmatrix}.$$
Determine whether the matrix is defective.

16. Find the eigenvalues and eigenvectors of the matrix
$$\begin{pmatrix} -10 & -2 & 11 \\ -18 & 6 & -9 \\ 10 & -10 & -2 \end{pmatrix}.$$
Determine whether the matrix is defective.

17. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 2 & 0 & 2 \end{pmatrix}$. Determine whether the matrix is defective.

18. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ -2 & 2 & -2 \end{pmatrix}$. Determine whether the matrix is defective.

19. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 & -6 \\ 7 & -5 & -6 \\ -1 & 7 & 2 \end{pmatrix}$. Determine whether the matrix is defective.

20. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ -2 & 2 & 6 \end{pmatrix}$. Determine whether the matrix is defective.

21. Here is a matrix.
$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 2 & c \\ 0 & 0 & 0 & 2 \end{pmatrix}$$
Find values of a, b, c for which the matrix is defective and values of a, b, c for which it is nondefective.

22. ♠Here is a matrix.
$$\begin{pmatrix} a & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & c \end{pmatrix}$$
where a, b, c are numbers. Show this is sometimes defective depending on the choice of a, b, c. What is an easy case which will ensure it is not defective?

23. ♠Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a+ib$ is a complex eigenvalue having the eigenvector, $\mathbf{x} + i\mathbf{y}$. Here $\mathbf{x}$ and $\mathbf{y}$ are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $\mathbf{x} - i\mathbf{y}$. **Hint:** You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a + ib$ is a complex number whose conjugate equals $a - ib$.

24. ♠Recall an $n \times n$ matrix is said to be symmetric if it has all real entries and if $A = A^T$. Show the eigenvalues of a real symmetric matrix are real and for each eigenvalue, it has a real eigenvector.

25. ♠Recall an $n \times n$ matrix is said to be skew symmetric if it has all real entries and if $A = -A^T$. Show that any nonzero eigenvalues must be of the form ib where $i^2 = -1$. In words, the eigenvalues are either 0 or pure imaginary.

26. ♠Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

27. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$, then whenever a, b are scalars,

$$A(a\mathbf{x} + b\mathbf{y}) = \lambda(a\mathbf{x} + b\mathbf{y}).$$

28. Show that the eigenvalues and eigenvectors of a real matrix occur in conjugate pairs.

29. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix}.$$

Can you find three independent eigenvectors?

30. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Can you find three independent eigenvectors in this case?

31. ♠Let M be an $n \times n$ matrix and suppose $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n eigenvectors which form a linearly independent set. Form the matrix S by making the columns these vectors. Show that S^{-1} exists and that $S^{-1}MS$ is a diagonal matrix (one having zeros everywhere except on the main diagonal) having the eigenvalues of M on the main diagonal. When this can be done the matrix is diagonalizable.

32. ♠Show that a matrix, M is diagonalizable if and only if it has a basis of eigenvectors. **Hint:** The first part is done in Problem 42. It only remains to show that if the matrix can be diagonalized by some matrix, S giving $D = S^{-1}MS$ for D a diagonal matrix, then it has a basis of eigenvectors. Try using the columns of the matrix S.

33. ♠Suppose A is an $n \times n$ matrix having all real eigenvalues which are distinct. Show there exists S such that $S^{-1}AS = D$, a diagonal matrix. If

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

define e^D by

$$e^D \equiv \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

and define

$$e^A \equiv Se^DS^{-1}.$$

Next show that if A is as just described, so is tA where t is a real number and the eigenvalues of At are $t\lambda_k$. If you differentiate a matrix of functions entry by entry so that for the ij^{th} entry of $A'(t)$ you get $a'_{ij}(t)$ where $a_{ij}(t)$ is the ij^{th} entry of $A(t)$, show

$$\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$$

Next show $\det\left(e^{At}\right) \neq 0$. This is called the matrix exponential. Note I have only defined it for the case where the eigenvalues of A are real, but the same procedure will work even for complex eigenvalues. All you have to do is to define what is meant by e^{a+ib}.

34. Find the principle directions determined by the matrix, $\begin{pmatrix} \frac{7}{12} & -\frac{1}{4} & \frac{1}{6} \\ -\frac{1}{4} & \frac{7}{12} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}$. The eigenvalues are $\frac{1}{3}, 1$, and $\frac{1}{2}$ listed according to multiplicity.

35. Find the principle directions determined by the matrix, $\begin{pmatrix} \frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{7}{6} \end{pmatrix}$ The eigenvalues are $1, 2$, and 1. What is the physical interpretation of the repeated eigenvalue?

36. Find the principle directions determined by the matrix, $\begin{pmatrix} \frac{19}{54} & \frac{1}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{11}{27} & \frac{2}{27} \\ \frac{1}{27} & \frac{2}{27} & \frac{11}{27} \end{pmatrix}$ The eigenvalues are $\frac{1}{2}$, $\frac{1}{3}$, and$\frac{1}{3}$.

37. Find the principle directions determined by the matrix, $\begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$ The eigenvalues are 2, $\frac{1}{2}$, and 2. What is the physical interpretation of the repeated eigenvalue?

38. Find oscillatory solutions to the system of differential equations, $\mathbf{x}'' = A\mathbf{x}$ where $A = \begin{pmatrix} -3 & -1 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$ The eigenvalues are $-1, -4$, and -2.

39. Find oscillatory solutions to the system of differential equations, $\mathbf{x}'' = A\mathbf{x}$ where $A = \begin{pmatrix} -\frac{7}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{11}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{11}{6} \end{pmatrix}$ The eigenvalues are $-1, -3$, and -2.

40. Let A and B be $n \times n$ matrices and let the columns of B be

$$\mathbf{b}_1, \cdots, \mathbf{b}_n$$

and the rows of A are

$$\mathbf{a}_1^T, \cdots, \mathbf{a}_n^T.$$

Show the columns of AB are

$$A\mathbf{b}_1 \cdots A\mathbf{b}_n$$

and the rows of AB are

$$\mathbf{a}_1^T B \cdots \mathbf{a}_n^T B.$$

41. ♠Let M be an $n \times n$ matrix. Then define the adjoint of M, denoted by M^* to be the transpose of the conjugate of M. For example,

$$\begin{pmatrix} 2 & i \\ 1+i & 3 \end{pmatrix}^* = \begin{pmatrix} 2 & 1-i \\ -i & 3 \end{pmatrix}.$$

A matrix, M, is self adjoint if $M^* = M$. Show the eigenvalues of a self adjoint matrix are all real.

42. Let M be an $n \times n$ matrix and suppose $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n eigenvectors which form a linearly independent set. Form the matrix S by making the columns these vectors. Show that S^{-1} exists and that $S^{-1}MS$ is a diagonal matrix (one having zeros everywhere except on the main diagonal) having the eigenvalues of M on the main diagonal. When this can be done the matrix is diagonalizable.

43. Show that a $n \times n$ matrix, M is diagonalizable if and only if $\mathbb{F}^n$ has a basis of eigenvectors. **Hint:** The first part is done in Problem 42. It only remains to show that if the matrix can be diagonalized by some matrix, S giving $D = S^{-1}MS$ for D a diagonal matrix, then it has a basis of eigenvectors. Try using the columns of the matrix S.

44. Let

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 1 \end{matrix}} & \boxed{3} \end{pmatrix}$$

and let

$$B = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}} \\ \boxed{\begin{matrix} 2 & 1 \end{matrix}} \end{pmatrix}$$

Multiply AB verifying the block multiplication formula. Here $A_{11} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $A_{21} = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $A_{22} = (3)$.

45. ♠Suppose A, B are $n \times n$ matrices and λ is a nonzero eigenvalue of AB. Show that then it is also an eigenvalue of BA. **Hint:** Use the definition of what it means for λ to be an eigenvalue. That is,

$$AB\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{x} \neq \mathbf{0}$. Maybe you should multiply both sides by B.

46. ♠Using the above problem show that if A, B are $n \times n$ matrices, it is not possible that $AB - BA = aI$ for any $a \neq 0$. **Hint:** First show that if A is a matrix, then the eigenvalues of $A - aI$ are $\lambda - a$ where λ is an eigenvalue of A.

47. ♠Consider the following matrix.

$$C = \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & -a_1 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$$

Show $\det(\lambda I - C) = a_0 + \lambda a_1 + \cdots a_{n-1}\lambda^{n-1} + \lambda^n$. This matrix is called a companion matrix for the given polynomial.

48. A discreet dynamical system is of the form

$$\mathbf{x}(k+1) = A\mathbf{x}(k),\ \mathbf{x}(0) = \mathbf{x}_0$$

where A is an $n \times n$ matrix and $\mathbf{x}(k)$ is a vector in $\mathbb{R}^n$. Show first that

$$\mathbf{x}(k) = A^k \mathbf{x}_0$$

for all $k \geq 1$. If A is nondefective so that it has a basis of eigenvectors, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ where

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

you can write the initial condition $\mathbf{x}_0$ in a unique way as a linear combination of these eigenvectors. Thus

$$\mathbf{x}_0 = \sum_{j=1}^{n} a_j \mathbf{v}_j$$

Now explain why

$$\mathbf{x}(k) = \sum_{j=1}^{n} a_j A^k \mathbf{v}_j = \sum_{j=1}^{n} a_j \lambda_j^k \mathbf{v}_j$$

which gives a formula for $\mathbf{x}(k)$, the solution of the dynamical system.

49. Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector such that $A\mathbf{v} = \lambda\mathbf{v}$. Also suppose the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Explain why

$$\left(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I\right)\mathbf{v} = \mathbf{0}$$

If A is nondefective, give a very easy proof of the Cayley Hamilton theorem based on this. Recall this theorem says A satisfies its characteristic equation,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

50. Suppose an $n \times n$ nondefective matrix A has only 1 and -1 as eigenvalues. Find A^{12}.

51. Suppose the characteristic polynomial of an $n \times n$ matrix A is $1 - \lambda^n$. Find A^{mn} where m is an integer. **Hint:** Note first that A is nondefective. Why?

52. Sometimes sequences come in terms of a recursion formula. An example is the Fibonacci sequence.

$$x_0 = 1 = x_1,\ x_{n+1} = x_n + x_{n-1}$$

Show this can be considered as a discreet dynamical system as follows.

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix},\ \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now use the technique of Problem 48 to find a formula for x_n.

53. Let A be an $n \times n$ matrix having characteristic polynomial

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Show that $a_0 = (-1)^n \det(A)$.

7.4 Shur's Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur's theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 7.4.1 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Then there exists an orthonormal basis for $\mathbb{F}^n$, $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ which has the property that for each $k \leq n$, span$(\mathbf{x}_1, \cdots, \mathbf{x}_k) =$ span $(\mathbf{u}_1, \cdots, \mathbf{u}_k)$.*

Proof: Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Let $\mathbf{u}_1 \equiv \mathbf{x}_1 / |\mathbf{x}_1|$. Thus for $k = 1$, span $(\mathbf{u}_1) =$ span $(\mathbf{x}_1)$ and $\{\mathbf{u}_1\}$ is an orthonormal set. Now suppose for some $k < n$, $\mathbf{u}_1, \cdots, \mathbf{u}_k$ have been chosen such that $(\mathbf{u}_j \cdot \mathbf{u}_l) = \delta_{jl}$ and span $(\mathbf{x}_1, \cdots, \mathbf{x}_k) =$ span $(\mathbf{u}_1, \cdots, \mathbf{u}_k)$. Then define

$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \mathbf{u}_j}{\left| \mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \mathbf{u}_j \right|}, \tag{7.9}$$

where the denominator is not equal to zero because the $\mathbf{x}_j$ form a basis and so

$$\mathbf{x}_{k+1} \notin \text{span}\,(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}\,(\mathbf{u}_1, \cdots, \mathbf{u}_k)$$

Thus by induction,

$$\mathbf{u}_{k+1} \in \text{span}\,(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{x}_{k+1}) = \text{span}\,(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}).$$

Also, $\mathbf{x}_{k+1} \in \text{span}\,(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1})$ which is seen easily by solving 7.9 for $\mathbf{x}_{k+1}$ and it follows

$$\text{span}\,(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}) = \text{span}\,(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
(\mathbf{u}_{k+1} \cdot \mathbf{u}_l) &= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, (\mathbf{u}_j \cdot \mathbf{u}_l) \right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \delta_{lj} \right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - (\mathbf{x}_{k+1} \cdot \mathbf{u}_l) \right) = 0.
\end{aligned}$$

The vectors, $\{\mathbf{u}_j\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.

Definition 7.4.2 *An $n \times n$ matrix, U, is unitary if $UU^* = I = U^*U$ where U^* is defined to be the transpose of the conjugate of U.*

Theorem 7.4.3 *Let A be an $n \times n$ matrix. Then there exists a unitary matrix, U such that*

$$U^* A U = T, \tag{7.10}$$

where T is an upper triangular matrix having the eigenvalues of A on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $\mathbf{v}_1$ be a unit eigenvector for A . Then there exists λ_1 such that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,\ |\mathbf{v}_1| = 1.$$

Extend $\{\mathbf{v}_1\}$ to a basis and then use Lemma 7.4.1 to obtain $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, an orthonormal basis in $\mathbb{F}^n$. Let U_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then from the above, it follows U_0 is unitary. Then $U_0^* A U_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $n-1 \times n-1$ matrix. Repeat the process for the matrix, A_1 above. There exists a unitary matrix $\widetilde{U}_1$ such that $\widetilde{U}_1^* A_1 \widetilde{U}_1$ is of the form

$$\begin{pmatrix} \lambda_2 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{pmatrix}.$$

Now let U_1 be the $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}.$$

This is also a unitary matrix because by block multiplication,

$$\begin{aligned} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}^* \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \widetilde{U}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \end{aligned}$$

Then using block multiplication, $U_1^* U_0^* A U_0 U_1$ is of the form

$$\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{pmatrix}$$

where A_2 is an $n-2 \times n-2$ matrix. Continuing in this way, there exists a unitary matrix, U given as the product of the U_i in the above construction such that

$$U^* A U = T$$

where T is some upper triangular matrix similar to A which consequently has the same eigenvalues with the same algebraic multiplicities as A. Since the matrix is upper triangular, the characteristic equation for both A and T is $\prod_{i=1}^n (\lambda - \lambda_i)$ where the λ_i are the diagonal entries of T. Therefore, the λ_i are the eigenvalues.

Now here is a streamlined proof.

Theorem 7.4.4 *Let A be an $n \times n$ matrix. Then there exists a unitary matrix, U such that*

$$U^* AU = T, \tag{7.11}$$

where T is an upper triangular matrix having the eigenvalues of A on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: The theorem is clearly true if A is a 1×1 matrix. Just let $U = 1$ the 1×1 matrix which has 1 down the main diagonal and zeros elsewhere. Suppose it is true for $(n-1) \times (n-1)$ matrices and let A be an $n \times n$ matrix. Then let $\mathbf{v}_1$ be a unit eigenvector for A . Then there exists λ_1 such that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ |\mathbf{v}_1| = 1.$$

Extend $\{\mathbf{v}_1\}$ to a basis and then use Lemma 7.4.1 to obtain $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, an orthonormal basis in $\mathbb{F}^n$. Let U_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then from the above, it follows U_0 is unitary. Then $U_0^* A U_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $n-1 \times n-1$ matrix. Now by induction there exists an $(n-1) \times (n-1)$ unitary matrix $\widetilde{U}_1$ such that

$$\widetilde{U}_1^* A_1 \widetilde{U}_1 = T_{n-1},$$

an upper triangular matrix. Consider

$$U_1 \equiv \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}$$

This is a unitary matrix and

$$\begin{aligned} U_1^* U_0^* A U_0 U_1 &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \end{pmatrix} \begin{pmatrix} \lambda_1 & * \\ \mathbf{0} & A_1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & * \\ \mathbf{0} & T_{n-1} \end{pmatrix} \equiv T \end{aligned}$$

where T is upper triangular. Then let $U = U_0 U_1$. Since $(U_0 U_1)^* = U_1^* U_0^*$, it follows A is similar to T and that $U_0 U_1$ is unitary. Hence A and T have the same characteristic polynomials and since the eigenvalues of T are the diagonal entries listed according to algebraic multiplicity, this proves the theorem.

As a simple consequence of the above theorem, here is an interesting lemma.

Lemma 7.4.5 *Let A be of the form*

$$A = \begin{pmatrix} P_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_s \end{pmatrix}$$

where P_k is an $m_k \times m_k$ matrix. Then

$$\det(A) = \prod_k \det(P_k).$$

Proof: Let U_k be an $m_k \times m_k$ unitary matrix such that

$$U_k^* P_k U_k = T_k$$

where T_k is upper triangular. Then letting

$$U = \begin{pmatrix} U_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & U_s \end{pmatrix},$$

it follows

$$U^* = \begin{pmatrix} U_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & U_s^* \end{pmatrix}$$

and

$$\begin{pmatrix} U_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & U_s^* \end{pmatrix} \begin{pmatrix} P_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_s \end{pmatrix} \begin{pmatrix} U_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & U_s \end{pmatrix}$$

$$= \begin{pmatrix} T_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_s \end{pmatrix}$$

and so

$$\det(A) = \prod_k \det(T_k) = \prod_k \det(P_k).$$

This proves the lemma.

What if A is a real matrix and you only want to consider real unitary matrices?

Theorem 7.4.6 *Let A be a real $n \times n$ matrix. Then there exists a real unitary matrix, Q and a matrix T of the form*

$$T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix} \tag{7.12}$$

where P_i equals either a real 1×1 matrix or P_i equals a real 2×2 matrix having as its eigenvalues a conjugate pair of eigenvalues of A such that $Q^T A Q = T$. The matrix, T is called the real Schur form of the matrix A. Recall that a real unitary matrix is also called an orthogonal matrix.

Proof: Suppose

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ |\mathbf{v}_1| = 1$$

where λ_1 is real. Then let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be an orthonormal basis of vectors in $\mathbb{R}^n$. Let Q_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is a real $n-1\times n-1$ matrix. This is just like the proof of Theorem 7.4.3 up to this point.

Now consider the case where $\lambda_1=\alpha+i\beta$ where $\beta\neq 0$. It follows since A is real that $\mathbf{v}_1=\mathbf{z}_1+i\mathbf{w}_1$ and that $\overline{\mathbf{v}}_1=\mathbf{z}_1-i\mathbf{w}_1$ is an eigenvector for the eigenvalue, $\alpha-i\beta$. Here $\mathbf{z}_1$ and $\mathbf{w}_1$ are real vectors. Since $\overline{\mathbf{v}}_1$ and $\mathbf{v}_1$ are eigenvectors corresponding to distinct eigenvalues, they form a linearly independent set. From this it follows that $\{\mathbf{z}_1,\mathbf{w}_1\}$ is an independent set of vectors in $\mathbb{C}^n$, hence in $\mathbb{R}^n$. Indeed,$\{\mathbf{v}_1,\overline{\mathbf{v}}_1\}$ is an independent set and also $\operatorname{span}(\mathbf{v}_1,\overline{\mathbf{v}}_1)=\operatorname{span}(\mathbf{z}_1,\mathbf{w}_1)$. Now using the Gram Schmidt theorem in $\mathbb{R}^n$, there exists $\{\mathbf{u}_1,\mathbf{u}_2\}$, an orthonormal set of real vectors such that $\operatorname{span}(\mathbf{u}_1,\mathbf{u}_2)=\operatorname{span}(\mathbf{v}_1,\overline{\mathbf{v}}_1)$. For example,

$$\mathbf{u}_1=\mathbf{z}_1/|\mathbf{z}_1|,\ \mathbf{u}_2=\frac{|\mathbf{z}_1|^2\mathbf{w}_1-(\mathbf{w}_1\cdot\mathbf{z}_1)\mathbf{z}_1}{\left||\mathbf{z}_1|^2\mathbf{w}_1-(\mathbf{w}_1\cdot\mathbf{z}_1)\mathbf{z}_1\right|}$$

Let $\{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_n\}$ be an orthonormal basis in $\mathbb{R}^n$ and let Q_0 be a unitary matrix whose i^{th} column is $\mathbf{u}_i$ so Q_0 is a real orthogonal matrix. Then $A\mathbf{u}_j$ are both in $\operatorname{span}(\mathbf{u}_1,\mathbf{u}_2)$ for $j=1,2$ and so $\mathbf{u}_k^TA\mathbf{u}_j=0$ whenever $k\geq 3$. It follows that $Q_0^*AQ_0$ is of the form

$$\begin{aligned}Q_0^*AQ_0 &= \begin{pmatrix}* & * & \cdots & *\\ * & * & & \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}\\ &= \begin{pmatrix}P_1 & *\\ 0 & A_1\end{pmatrix}\end{aligned}$$

where A_1 is now an $n-2\times n-2$ matrix and P_1 is a 2×2 matrix. Now this is similar to A and so two of its eigenvalues are $\alpha+i\beta$ and $\alpha-i\beta$.

Now find $\widetilde{Q}_1$ an $n-2\times n-2$ matrix to put A_1 in an appropriate form as above and come up with A_2 either an $n-4\times n-4$ matrix or an $n-3\times n-3$ matrix. Then the only other difference is to let

$$Q_1=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & \widetilde{Q}_1 & \\ 0 & 0 & & & \end{pmatrix}$$

thus putting a 2×2 identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to 7.12 where Q is the product of real unitary matrices Q_i above. When the block P_i is 2×2, its eigenvalues are a conjugate pair of eigenvalues of A and if it is 1×1 it is a real eigenvalue of A.

Here is why this last claim is true

$$\lambda I-T=\begin{pmatrix}\lambda I_1-P_1 & \cdots & *\\ & \ddots & \vdots\\ 0 & & \lambda I_r-P_r\end{pmatrix}$$

where I_k is the 2×2 identity matrix in the case that P_k is 2×2 and is the number 1 in the case where P_k is a 1×1 matrix. Now by Lemma 7.4.5,

$$\det(\lambda I-T)=\prod_{k=1}^{r}\det(\lambda I_k-P_k).$$

Therefore, λ is an eigenvalue of T if and only if it is an eigenvalue of some P_k. This proves the theorem since the eigenvalues of T are the same as those of A including multiplicity because they have the same characteristic polynomial due to the similarity of A and T. This proves the theorem.

Corollary 7.4.7 *Let A be a real $n \times n$ matrix having only real eigenvalues. Then there exists a real orthogonal matrix Q and an upper triangular matrix T such that*

$$Q^T A Q = T$$

and furthermore, if the eigenvalues of A are listed in decreasing order,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Q can be chosen such that T is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Proof: Most of this follows right away from Theorem 7.4.6. It remains to verify the claim that the diagonal entries can be arranged in the desired order. However, this follows from a simple modification of the above argument. When you find $\mathbf{v}_1$ the eigenvalue of λ_1, just be sure λ_1 is chosen to be the largest eigenvalue. Then observe that from Lemma 7.4.5 applied to the characteristic equation, the eigenvalues of the $(n-1) \times (n-1)$ matrix A_1 are $\{\lambda_1, \cdots, \lambda_n\}$. Then pick λ_2 to continue the process of construction with A_1. This proves the corollary.

Of course there is a similar conclusion which can be proved exactly the same way in the case where A has complex eigenvalues.

Corollary 7.4.8 *Let A be a real $n \times n$ matrix. Then there exists a real orthogonal matrix Q and an upper triangular matrix T such that*

$$Q^T A Q = T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix}$$

where P_i equals either a real 1×1 matrix or P_i equals a real 2×2 matrix having as its eigenvalues a conjugate pair of eigenvalues of A. If P_k corresponds to the two eigenvalues $\alpha_k \pm i\beta_k \equiv \sigma(P_k)$, Q can be chosen such that

$$|\sigma(P_1)| \geq |\sigma(P_2)| \geq \cdots$$

where

$$|\sigma(P_k)| \equiv \sqrt{\alpha_k^2 + \beta_k^2}$$

The blocks, P_k can be arranged in any other order also.

Definition 7.4.9 *When a linear transformation, A, mapping a linear space, V to V has a basis of eigenvectors, the linear transformation is called non defective. Otherwise it is called defective. An $n \times n$ matrix, A, is called normal if $AA^* = A^*A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, A is self adjoint or Hermitian if $A = A^*$.*

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

Lemma 7.4.10 *If T is upper triangular and normal, then T is a diagonal matrix.*

Proof: This is obviously true if T is 1×1. In fact, it can't help being diagonal in this case. Suppose then that the lemma is true for $(n-1) \times (n-1)$ matrices and let T be an upper triangular normal $n \times n$ matrix. Thus T is of the form

$$T = \begin{pmatrix} t_{11} & \mathbf{a}^* \\ \mathbf{0} & T_1 \end{pmatrix}, \; T^* = \begin{pmatrix} \overline{t_{11}} & \mathbf{0}^T \\ \mathbf{a} & T_1^* \end{pmatrix}$$

Then

$$\begin{aligned} TT^* &= \begin{pmatrix} t_{11} & \mathbf{a}^* \\ \mathbf{0} & T_1 \end{pmatrix} \begin{pmatrix} \overline{t_{11}} & \mathbf{0}^T \\ \mathbf{a} & T_1^* \end{pmatrix} = \begin{pmatrix} |t_{11}|^2 + \mathbf{a}^*\mathbf{a} & \mathbf{a}^* T_1^* \\ T_1\mathbf{a} & T_1 T_1^* \end{pmatrix} \\ T^*T &= \begin{pmatrix} \overline{t_{11}} & \mathbf{0}^T \\ \mathbf{a} & T_1^* \end{pmatrix} \begin{pmatrix} t_{11} & \mathbf{a}^* \\ \mathbf{0} & T_1 \end{pmatrix} = \begin{pmatrix} |t_{11}|^2 & \overline{t_{11}}\mathbf{a}^* \\ \mathbf{a}t_{11} & \mathbf{a}\mathbf{a}^* + T_1^* T_1 \end{pmatrix} \end{aligned}$$

Since these two matrices are equal, it follows $\mathbf{a} = \mathbf{0}$. But now it follows that $T_1^* T_1 = T_1 T_1^*$ and so by induction T_1 is a diagonal matrix D_1. Therefore,

$$T = \begin{pmatrix} t_{11} & \mathbf{0}^T \\ \mathbf{0} & D_1 \end{pmatrix}$$

a diagonal matrix.

Now here is a proof which doesn't involve block multiplication. Since T is normal, $T^*T = TT^*$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the ik^{th} entry of $T^*T = TT^*$.

$$\overbrace{\sum_j t_{ij} t_{jk}^* = \sum_j t_{ij} \overline{t_{kj}}}^{TT^*} = \overbrace{\sum_j t_{ij}^* t_{jk} = \sum_j \overline{t_{ji}} t_{jk}}^{T^*T}.$$

Now use the fact that T is upper triangular and let $i = k = 1$ to obtain the following from the above.

$$\sum_j |t_{1j}|^2 = \sum_j |t_{j1}|^2 = |t_{11}|^2$$

You see, $t_{j1} = 0$ unless $j = 1$ due to the assumption that T is upper triangular. This shows T is of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

Now do the same thing only this time take $i = k = 2$ and use the result just established. Thus, from the above,

$$\sum_j |t_{2j}|^2 = \sum_j |t_{j2}|^2 = |t_{22}|^2,$$

showing that $t_{2j} = 0$ if $j > 2$ which means T has the form

$$\begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Next let $i = k = 3$ and obtain that T looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows T is a diagonal matrix. This proves the lemma.

Theorem 7.4.11 *Let A be a normal matrix. Then there exists a unitary matrix, U such that U^*AU is a diagonal matrix.*

Proof: From Theorem 7.4.3 there exists a unitary matrix, U such that U^*AU equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if A is normal and if $B = U^*AU$, then B is also normal. But this is easy.

$$\begin{aligned} B^*B &= U^*A^*UU^*AU = U^*A^*AU \\ &= U^*AA^*U = U^*AUU^*A^*U = BB^*. \end{aligned}$$

Therefore, U^*AU is a normal and upper triangular matrix and by Lemma 7.4.10 it must be a diagonal matrix. This proves the theorem.

Corollary 7.4.12 *If A is Hermitian, then all the eigenvalues of A are real and there exists an orthonormal basis of eigenvectors.*

Proof: Since A is normal, there exists unitary, U such that $U^*AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, $D^* = U^*A^*U = U^*AU = D$ showing D is real.

Finally, let

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$$

where the $\mathbf{u}_i$ denote the columns of U and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The equation, $U^*AU = D$ implies

$$\begin{aligned} AU &= \begin{pmatrix} A\mathbf{u}_1 & A\mathbf{u}_2 & \cdots & A\mathbf{u}_n \end{pmatrix} \\ &= UD = \begin{pmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{pmatrix} \end{aligned}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and since the matrix is unitary, the ij^{th} entry of U^*U equals δ_{ij} and so

$$\delta_{ij} = \mathbf{u}_i^*\mathbf{u}_j \equiv \mathbf{u}_j \cdot \mathbf{u}_i.$$

This proves the corollary because it shows the vectors $\{\mathbf{u}_i\}$ are orthonormal. Therefore, they form a basis because every orthonormal set of vectors is linearly independent. This proves the corollary.

Corollary 7.4.13 *If A is a real symmetric matrix, then A is Hermitian and there exists a real unitary matrix, U such that $U^TAU = D$ where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. By arranging the columns of U the diagonal entries of D can be made to appear in any order.*

Proof: This follows from Theorem 7.4.6 and Corollary 7.4.12. Let

$$U = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$$

Then $AU = UD$ so

$$\begin{aligned} AU &= \begin{pmatrix} A\mathbf{u}_1 & \cdots & A\mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} D \\ &= \begin{pmatrix} \lambda_1\mathbf{u}_1 & \cdots & \lambda_n\mathbf{u}_n \end{pmatrix} \end{aligned}$$

Hence each column of U is an eigenvector of A. It follows that by rearranging these columns, the entries of D on the main diagonal can be made to appear in any order. To see this, consider such a rearrangement resulting in an orthogonal matrix U' given by

$$U' = \begin{pmatrix} \mathbf{u}_{i_1} & \cdots & \mathbf{u}_{i_n} \end{pmatrix}$$

Then

$$\begin{aligned} U'^T AU' &= U'^T \begin{pmatrix} A\mathbf{u}_{i_1} & \cdots & A\mathbf{u}_{i_n} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}_{i_1}^T \\ \vdots \\ \mathbf{u}_{i_n}^T \end{pmatrix} \begin{pmatrix} \lambda_{i_1}\mathbf{u}_{i_1} & \cdots & \lambda_{i_n}\mathbf{u}_{i_n} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{i_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{i_n} \end{pmatrix} \end{aligned}$$

This proves the corollary.

7.5 Trace And Determinant

The determinant has already been discussed. It is also clear that if $A = S^{-1}BS$, then

$$\begin{aligned} \det(A) &= \det\left(S^{-1}\right)\det(S)\det(B) \\ &= \det\left(S^{-1}S\right)\det(B) \\ &= \det(I)\det(B) = \det(B) \end{aligned}$$

The **trace** is defined in the following definition.

Definition 7.5.1 *Let A be an $n \times n$ matrix whose ij^{th} entry is denoted as a_{ij}. Then*

$$\operatorname{trace}(A) \equiv \sum_i a_{ii}$$

In other words it is the sum of the entries down the main diagonal.

With this definition, it is easy to see that if $A = S^{-1}BS$, then

$$\operatorname{trace}(A) = \operatorname{trace}(B).$$

Here is why.

$$\begin{aligned} \operatorname{trace}(A) &\equiv \sum_i A_{ii} = \sum_{i,j,k} \left(S^{-1}\right)_{ij} B_{jk} S_{ki} \\ &= \sum_{j,k} B_{jk} \sum_i S_{ki} \left(S^{-1}\right)_{ij} \\ &= \sum_{j,k} B_{jk}\delta_{kj} = \sum_k B_{kk} = \operatorname{trace}(B). \end{aligned}$$

Alternatively,

$$\text{trace}(AB) \equiv \sum_{ij} A_{ij}B_{ji} = \text{trace}(BA).$$

Therefore,

$$\text{trace}\left(S^{-1}AS\right) = \text{trace}\left(ASS^{-1}\right) = \text{trace}(A).$$

Theorem 7.5.2 *Let A be an $n \times n$ matrix. Then* $\text{trace}(A)$ *equals the sum of the eigenvalues of A and* $\det(A)$ *equals the product of the eigenvalues of A.*

This is proved using Schur's theorem and is in Problem 14 below. Another important property of the trace is in the following theorem.

Theorem 7.5.3 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Then*

$$\text{trace}(AB) = \text{trace}(BA).$$

Proof:

$$\text{trace}(AB) \equiv \sum_i \left(\sum_k A_{ik}B_{ki} \right) = \sum_k \sum_i B_{ki}A_{ik} = \text{trace}(BA)$$

7.6 Quadratic Forms

Definition 7.6.1 *A quadratic form in three dimensions is an expression of the form*

$$\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.13}$$

where A is a 3×3 symmetric matrix. In higher dimensions the idea is the same except you use a larger symmetric matrix in place of A. In two dimensions A is a 2×2 matrix.

For example, consider

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & -4 & 1 \\ -4 & 0 & -4 \\ 1 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.14}$$

which equals $3x^2 - 8xy + 2xz - 8yz + 3z^2$. This is very awkward because of the mixed terms such as $-8xy$. The idea is to pick different axes such that if x, y, z are taken with respect to these axes, the quadratic form is much simpler. In other words, look for new variables, x', y', and z' and a unitary matrix, U such that

$$U \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.15}$$

and if you write the quadratic form in terms of the primed variables, there will be no mixed terms. Any symmetric real matrix is Hermitian and is therefore normal. From Corollary

7.4.13, it follows there exists a real unitary matrix, U, (an orthogonal matrix) such that $U^T AU = D$ a diagonal matrix. Thus in the quadratic form, 7.13

$$\begin{aligned}\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x' & y' & z' \end{pmatrix} U^T AU \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\ &= \begin{pmatrix} x' & y' & z' \end{pmatrix} D \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\end{aligned}$$

and in terms of these new variables, the quadratic form becomes

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \lambda_3 (z')^2$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. Similar considerations apply equally well in any other dimension. For the given example,

$$\begin{pmatrix} -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & -4 & 1 \\ -4 & 0 & -4 \\ 1 & -4 & 3 \end{pmatrix} \cdot$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

and so if the new variables are given by

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

it follows that in terms of the new variables the quadratic form is $2(x')^2 - 4(y')^2 + 8(z')^2$. You can work other examples the same way.

7.7 Second Derivative Test

Under certain conditions the **mixed partial derivatives** will always be equal. This astonishing fact is due to Euler in 1734.

Theorem 7.7.1 *Suppose $f : U \subseteq \mathbb{F}^2 \to \mathbb{R}$ where U is an open set on which f_x, f_y, f_{xy} and f_{yx} exist. Then if f_{xy} and f_{yx} are continuous at the point $(x, y) \in U$, it follows*

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Proof: Since U is open, there exists $r > 0$ such that $B((x, y), r) \subseteq U$. Now let $|t|, |s| < r/2, t, s$ real numbers and consider

$$\Delta(s, t) \equiv \frac{1}{st}\{\overbrace{f(x + t, y + s) - f(x + t, y)}^{h(t)} - \overbrace{(f(x, y + s) - f(x, y))}^{h(0)}\}. \tag{7.16}$$

Note that $(x+t, y+s) \in U$ because

$$\begin{aligned} |(x+t, y+s) - (x,y)| &= |(t,s)| = \left(t^2+s^2\right)^{1/2} \\ &\leq \left(\frac{r^2}{4}+\frac{r^2}{4}\right)^{1/2} = \frac{r}{\sqrt{2}} < r. \end{aligned}$$

As implied above, $h(t) \equiv f(x+t, y+s) - f(x+t, y)$. Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

$$\begin{aligned} \Delta(s,t) &= \frac{1}{st}(h(t) - h(0)) = \frac{1}{st} h'(\alpha t)\, t \\ &= \frac{1}{s}\left(f_x(x+\alpha t, y+s) - f_x(x+\alpha t, y)\right) \end{aligned}$$

for some $\alpha \in (0,1)$. Applying the mean value theorem again,

$$\Delta(s,t) = f_{xy}(x+\alpha t, y+\beta s)$$

where $\alpha, \beta \in (0,1)$.

If the terms $f(x+t, y)$ and $f(x, y+s)$ are interchanged in 7.16, $\Delta(s,t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in (0,1)$ such that

$$\Delta(s,t) = f_{yx}(x+\gamma t, y+\delta s).$$

Letting $(s,t) \to (0,0)$ and using the continuity of f_{xy} and f_{yx} at (x,y),

$$\lim_{(s,t)\to(0,0)} \Delta(s,t) = f_{xy}(x,y) = f_{yx}(x,y).$$

This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 7.7.2 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{R}$ has the property that for two indices, k, l, f_{x_k}, $f_{x_l}, f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

Thus the theorem asserts that the mixed partial derivatives are equal at $\mathbf{x}$ if they are defined near $\mathbf{x}$ and continuous at $\mathbf{x}$.

Now recall the Taylor formula with the Lagrange form of the remainder. What follows is a proof of this important result based on the mean value theorem or Rolle's theorem.

Theorem 7.7.3 *Suppose f has $n+1$ derivatives on an interval, (a,b) and let $c \in (a,b)$. Then if $x \in (a,b)$, there exists ξ between c and x such that*

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

(In this formula, the symbol $\sum_{k=1}^{0} a_k$ will denote the number 0.)

Proof: If $n = 0$ then the theorem is true because it is just the mean value theorem. Suppose the theorem is true for $n-1, n \geq 1$. It can be assumed $x \neq c$ because if $x = c$ there is nothing to show. Then there exists K such that

$$f(x) - \left(f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + K(x-c)^{n+1}\right) = 0 \tag{7.17}$$

In fact,

$$K = \frac{-f(x) + \left(f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k\right)}{(x-c)^{n+1}}.$$

Now define $F(t)$ for t in the closed interval determined by x and c by

$$F(t) \equiv f(x) - \left(f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!} (x-t)^k + K(x-t)^{n+1}\right).$$

The c in 7.17 got replaced by t.

Therefore, $F(c) = 0$ by the way K was chosen and also $F(x) = 0$. By the mean value theorem or Rolle's theorem, there exists t_1 between x and c such that $F'(t_1) = 0$. Therefore,

$$\begin{aligned} 0 &= f'(t_1) - \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!} k (x-t_1)^{k-1} - K(n+1)(x-t_1)^n \\ &= f'(t_1) - \left(f'(c) + \sum_{k=1}^{n-1} \frac{f^{(k+1)}(c)}{k!} (x-t_1)^k\right) - K(n+1)(x-t_1)^n \\ &= f'(t_1) - \left(f'(c) + \sum_{k=1}^{n-1} \frac{f'^{(k)}(c)}{k!} (x-t_1)^k\right) - K(n+1)(x-t_1)^n \end{aligned}$$

By induction applied to f', there exists ξ between x and t_1 such that the above simplifies to

$$\begin{aligned} 0 &= \frac{f'^{(n)}(\xi)(x-t_1)^n}{n!} - K(n+1)(x-t_1)^n \\ &= \frac{f^{(n+1)}(\xi)(x-t_1)^n}{n!} - K(n+1)(x-t_1)^n \end{aligned}$$

therefore,

$$K = \frac{f^{(n+1)}(\xi)}{(n+1)\,n!} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

and the formula is true for n. This proves the theorem.

The following is a special case and is what will be used.

Theorem 7.7.4 *Let $h : (-\delta, 1+\delta) \to \mathbb{R}$ have $m+1$ derivatives. Then there exists $t \in [0,1]$ such that*

$$h(1) = h(0) + \sum_{k=1}^{m} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m+1)}(t)}{(m+1)!}.$$

Now let $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$ and suppose $f \in C^m(U)$. Let $\mathbf{x} \in U$ and let $r > 0$ be such that

$$B(\mathbf{x}, r) \subseteq U.$$

Then for $||\mathbf{v}|| < r$, consider

$$f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x}) \equiv h(t)$$

for $t \in [0,1]$. Then by the chain rule,

$$h'(t) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} (\mathbf{x} + t\mathbf{v}) v_k$$

$$h''(t) = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{x}+t\mathbf{v})\, v_k v_j$$

Then from the Taylor formula stopping at the second derivative, the following theorem can be obtained.

Theorem 7.7.5 *Let $f : U \to \mathbb{R}$ and let $f \in C^2(U)$. Then if*

$$B(\mathbf{x},r) \subseteq U,$$

and $||\mathbf{v}|| < r$, there exists $t \in (0,1)$ such that.

$$\begin{aligned} f(\mathbf{x}+\mathbf{v}) &= f(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\mathbf{x})\, v_k \\ &\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{x}+t\mathbf{v})\, v_k v_j \end{aligned} \tag{7.18}$$

Definition 7.7.6 *Define the following matrix.*

$$H_{ij}(\mathbf{x}+t\mathbf{v}) \equiv \frac{\partial^2 f(\mathbf{x}+t\mathbf{v})}{\partial x_j \partial x_i}.$$

It is called the Hessian matrix. From Corollary 7.7.2, this is a symmetric matrix. Then in terms of this matrix, 7.18 can be written as

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x})\, v_k + \frac{1}{2}\mathbf{v}^T H(\mathbf{x}+t\mathbf{v})\,\mathbf{v}$$

Then this implies

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x})\, v_k + \frac{1}{2}\mathbf{v}^T H(\mathbf{x})\,\mathbf{v} +$$

$$\frac{1}{2}\left(\mathbf{v}^T\left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right). \tag{7.19}$$

Using the above formula, here is the second derivative test.

Theorem 7.7.7 *In the above situation, suppose $f_{x_j}(\mathbf{x}) = 0$ for each x_j. Then if $H(\mathbf{x})$ has all positive eigenvalues, $\mathbf{x}$ is a local minimum for f. If $H(\mathbf{x})$ has all negative eigenvalues, then $\mathbf{x}$ is a local maximum. If $H(\mathbf{x})$ has a positive eigenvalue, then there exists a direction in which f has a local minimum at $\mathbf{x}$, while if $H(\mathbf{x})$ has a negative eigenvalue, there exists a direction in which $H(\mathbf{x})$ has a local maximum at $\mathbf{x}$.*

Proof: Since $f_{x_j}(\mathbf{x}) = 0$ for each x_j, formula 7.19 implies

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \frac{1}{2}\mathbf{v}^T H(\mathbf{x})\,\mathbf{v} + \frac{1}{2}\left(\mathbf{v}^T\left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right)$$

where $H(\mathbf{x})$ is a symmetric matrix. Thus, by Corollary 7.4.12 $H(\mathbf{x})$ has all real eigenvalues. Suppose first that $H(\mathbf{x})$ has all positive eigenvalues and that all are larger than $\delta^2 > 0$.

Then $H(\mathbf{x})$ has an orthonormal basis of eigenvectors, $\{\mathbf{v}_i\}_{i=1}^n$ and if $\mathbf{u}$ is an arbitrary vector, $\mathbf{u} = \sum_{j=1}^n u_j \mathbf{v}_j$ where $u_j = \mathbf{u} \cdot \mathbf{v}_j$. Thus

$$\mathbf{u}^T H(\mathbf{x})\,\mathbf{u} = \left(\sum_{k=1}^n u_k \mathbf{v}_k^T\right) H(\mathbf{x}) \left(\sum_{j=1}^n u_j \mathbf{v}_j\right)$$

$$= \sum_{j=1}^n u_j^2 \lambda_j \geq \delta^2 \sum_{j=1}^n u_j^2 = \delta^2 |\mathbf{u}|^2.$$

From 7.19 and the continuity of H, if $\mathbf{v}$ is small enough,

$$f(\mathbf{x}+\mathbf{v}) \geq f(\mathbf{x}) + \frac{1}{2}\delta^2 |\mathbf{v}|^2 - \frac{1}{4}\delta^2 |\mathbf{v}|^2 = f(\mathbf{x}) + \frac{\delta^2}{4} |\mathbf{v}|^2.$$

This shows the first claim of the theorem. The second claim follows from similar reasoning. Suppose $H(\mathbf{x})$ has a positive eigenvalue λ^2. Then let $\mathbf{v}$ be an eigenvector for this eigenvalue. From 7.19,

$$f(\mathbf{x}+t\mathbf{v}) = f(\mathbf{x}) + \frac{1}{2}t^2 \mathbf{v}^T H(\mathbf{x})\,\mathbf{v} +$$

$$\frac{1}{2}t^2 \left(\mathbf{v}^T \left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right)$$

which implies

$$\begin{aligned} f(\mathbf{x}+t\mathbf{v}) &= f(\mathbf{x}) + \frac{1}{2}t^2\lambda^2 |\mathbf{v}|^2 + \frac{1}{2}t^2 \left(\mathbf{v}^T \left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right) \\ &\geq f(\mathbf{x}) + \frac{1}{4}t^2\lambda^2 |\mathbf{v}|^2 \end{aligned}$$

whenever t is small enough. Thus in the direction $\mathbf{v}$ the function has a local minimum at $\mathbf{x}$. The assertion about the local maximum in some direction follows similarly. This proves the theorem.

This theorem is an analogue of the second derivative test for higher dimensions. As in one dimension, when there is a zero eigenvalue, it may be impossible to determine from the Hessian matrix what the local qualitative behavior of the function is. For example, consider

$$f_1(x,y) = x^4 + y^2,\ f_2(x,y) = -x^4 + y^2.$$

Then $Df_i(0,0) = \mathbf{0}$ and for both functions, the Hessian matrix evaluated at $(0,0)$ equals

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

but the behavior of the two functions is very different near the origin. The second has a saddle point while the first has a minimum there.

7.8 The Estimation Of Eigenvalues

There are ways to estimate the eigenvalues for matrices. The most famous is known as Gerschgorin's theorem. This theorem gives a rough idea where the eigenvalues are just from looking at the matrix.

Theorem 7.8.1 *Let A be an $n \times n$ matrix. Consider the n Gerschgorin discs defined as*

$$D_i \equiv \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Then every eigenvalue is contained in some Gerschgorin disc.

This theorem says to add up the absolute values of the entries of the i^{th} row which are off the main diagonal and form the disc centered at a_{ii} having this radius. The union of these discs contains $\sigma(A)$.

Proof: Suppose $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Then for $A = (a_{ij})$

$$\sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i.$$

Therefore, picking k such that $|x_k| \geq |x_j|$ for all x_j, it follows that $|x_k| \neq 0$ since $|\mathbf{x}| \neq 0$ and

$$|x_k| \sum_{j \neq i} |a_{ij}| \geq \sum_{j \neq i} |a_{ij}| |x_j| \geq |\lambda - a_{ii}| |x_k|.$$

Now dividing by $|x_k|$, it follows λ is contained in the k^{th} Gerschgorin disc.

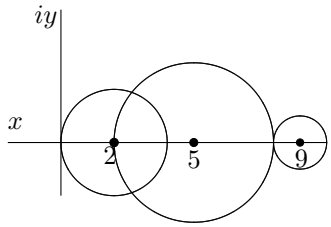
Example 7.8.2 *Here is a matrix. Estimate its eigenvalues.*

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 0 \\ 0 & 1 & 9 \end{pmatrix}$$

According to Gerschgorin's theorem the eigenvalues are contained in the disks

$$\begin{aligned} D_1 &= \{\lambda \in \mathbb{C} : |\lambda - 2| \leq 2\}, \\ D_2 &= \{\lambda \in \mathbb{C} : |\lambda - 5| \leq 3\}, \\ D_3 &= \{\lambda \in \mathbb{C} : |\lambda - 9| \leq 1\} \end{aligned}$$

It is important to observe that these disks are in the complex plane. In general this is the case. If you want to find eigenvalues they will be complex numbers.



So what are the values of the eigenvalues? In this case they are real. You can compute them by graphing the characteristic polynomial, $\lambda^3 - 16\lambda^2 + 70\lambda - 66$ and then zooming in on the zeros. If you do this you find the solution is $\{\lambda = 1.295\,3\}, \{\lambda = 5.590\,5\}, \{\lambda = 9.114\,2\}$. Of course these are only approximations and so this information is useless for finding eigenvectors. However, in many applications, it is the size of the eigenvalues which is important and so these numerical values would be helpful for such applications. In this case, you might think there is no real reason for Gerschgorin's theorem. Why not just compute the characteristic equation and graph and zoom? This is fine up to a point, but what if the matrix was huge? Then it might be hard to find the characteristic polynomial. Remember the difficulties in expanding a big matrix along a row or column. Also, what if the eigenvalues were complex? You don't see these by following this procedure. However, Gerschgorin's theorem will at least estimate them.

7.9 Advanced Theorems

More can be said but this requires some theory from complex variables[1]. The following is a fundamental theorem about counting zeros.

Theorem 7.9.1 *Let U be a region and let $\gamma : [a,b] \to U$ be closed, continuous, bounded variation, and the winding number, $n(\gamma, z) = 0$ for all $z \notin U$. Suppose also that f is analytic on U having zeros $a_1, \cdots, a_m$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on $\gamma([a,b])$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: It is given that $f(z) = \prod_{j=1}^m (z - a_j)\, g(z)$ where $g(z) \neq 0$ on U. Hence using the product rule,

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}$$

where $\frac{g'(z)}{g(z)}$ is analytic on U and so

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \sum_{j=1}^m n(\gamma, a_j) + \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz \\ &= \sum_{j=1}^m n(\gamma, a_j).\end{aligned}$$

Therefore, this proves the theorem.

Now let A be an $n \times n$ matrix. Recall that the eigenvalues of A are given by the zeros of the polynomial, $p_A(z) = \det(zI - A)$ where I is the $n \times n$ identity. You can argue that small changes in A will produce small changes in $p_A(z)$ and $p_A'(z)$. Let γ_k denote a very small closed circle which winds around z_k, one of the eigenvalues of A, in the counter clockwise direction so that $n(\gamma_k, z_k) = 1$. This circle is to enclose only z_k and is to have no other eigenvalue on it. Then apply Theorem 7.9.1. According to this theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{p_A'(z)}{p_A(z)}dz$$

is always an integer equal to the multiplicity of z_k as a root of $p_A(t)$. Therefore, small changes in A result in no change to the above contour integral because it must be an integer and small changes in A result in small changes in the integral. Therefore whenever B is close enough to A, the two matrices have the same number of zeros inside γ_k, the zeros being counted according to multiplicity. By making the radius of the small circle equal to ε where ε is less than the minimum distance between any two distinct eigenvalues of A, this shows that if B is close enough to A, every eigenvalue of B is closer than ε to some eigenvalue of A.

Theorem 7.9.2 *If λ is an eigenvalue of A, then if all the entries of B are close enough to the corresponding entries of A, some eigenvalue of B will be within ε of λ.*

[1] If you haven't studied the theory of a complex variable, you should skip this section because you won't understand any of it.

Consider the situation that $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0,1]$.

Lemma 7.9.3 *Let $\lambda(t) \in \sigma(A(t))$ for $t < 1$ and let $\Sigma_t = \cup_{s \geq t} \sigma(A(s))$. Also let K_t be the connected component of $\lambda(t)$ in Σ_t. Then there exists $\eta > 0$ such that $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$.*

Proof: Denote by $D(\lambda(t), \delta)$ the disc centered at $\lambda(t)$ having radius $\delta > 0$, with other occurrences of this notation being defined similarly. Thus

$$D(\lambda(t), \delta) \equiv \{z \in \mathbb{C} : |\lambda(t) - z| \leq \delta\}.$$

Suppose $\delta > 0$ is small enough that $\lambda(t)$ is the only element of $\sigma(A(t))$ contained in $D(\lambda(t), \delta)$ and that $p_{A(t)}$ has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists $\eta > 0, t + \eta < 1$, such that for $s \in [t, t+\eta]$, $p_{A(s)}$ also has no zeroes on the boundary of this disc and $A(s)$ has the same number of eigenvalues, counted according to multiplicity, in the disc as $A(t)$. Thus $\sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset$ for all $s \in [t, t+\eta]$. Now let

$$H = \bigcup_{s \in [t, t+\eta]} \sigma(A(s)) \cap D(\lambda(t), \delta).$$

It will be shown that H is connected. Suppose not. Then $H = P \cup Q$ where P, Q are separated and $\lambda(t) \in P$. Let $s_0 \equiv \inf\{s : \lambda(s) \in Q \text{ for some } \lambda(s) \in \sigma(A(s))\}$. There exists $\lambda(s_0) \in \sigma(A(s_0)) \cap D(\lambda(t), \delta)$. If $\lambda(s_0) \notin Q$, then from the above discussion there are $\lambda(s) \in \sigma(A(s)) \cap Q$ for $s > s_0$ arbitrarily close to $\lambda(s_0)$. Therefore, $\lambda(s_0) \in Q$ which shows that $s_0 > t$ because $\lambda(t)$ is the only element of $\sigma(A(t))$ in $D(\lambda(t), \delta)$ and $\lambda(t) \in P$. Now let $s_n \uparrow s_0$. Then $\lambda(s_n) \in P$ for any $\lambda(s_n) \in \sigma(A(s_n)) \cap D(\lambda(t), \delta)$ and also it follows from the above discussion that for some choice of $s_n \to s_0$, $\lambda(s_n) \to \lambda(s_0)$ which contradicts P and Q separated and nonempty. Since P is nonempty, this shows $Q = \emptyset$. Therefore, H is connected as claimed. But $K_t \supseteq H$ and so $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$. This proves the lemma.

Theorem 7.9.4 *Suppose $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0,1]$. Let $\lambda(0) \in \sigma(A(0))$ and define $\Sigma \equiv \cup_{t \in [0,1]} \sigma(A(t))$. Let $K_{\lambda(0)} = K_0$ denote the connected component of $\lambda(0)$ in Σ. Then $K_0 \cap \sigma(A(t)) \neq \emptyset$ for all $t \in [0,1]$.*

Proof: Let $S \equiv \{t \in [0,1] : K_0 \cap \sigma(A(s)) \neq \emptyset \text{ for all } s \in [0,t]\}$. Then $0 \in S$. Let $t_0 = \sup(S)$. Say $\sigma(A(t_0)) = \lambda_1(t_0), \cdots, \lambda_r(t_0)$.

Claim: At least one of these is a limit point of K_0 and consequently must be in K_0 which shows that S has a last point. Why is this claim true? Let $s_n \uparrow t_0$ so $s_n \in S$. Now let the discs, $D(\lambda_i(t_0), \delta), i = 1, \cdots, r$ be disjoint with $p_{A(t_0)}$ having no zeroes on γ_i the boundary of $D(\lambda_i(t_0), \delta)$. Then for n large enough it follows from Theorem 7.9.1 and the discussion following it that $\sigma(A(s_n))$ is contained in $\cup_{i=1}^r D(\lambda_i(t_0), \delta)$. It follows that $K_0 \cap (\sigma(A(t_0)) + D(0, \delta)) \neq \emptyset$ for all δ small enough. This requires at least one of the $\lambda_i(t_0)$ to be in $\overline{K_0}$. Therefore, $t_0 \in S$ and S has a last point.

Now by Lemma 7.9.3, if $t_0 < 1$, then $K_0 \cup K_t$ would be a strictly larger connected set containing $\lambda(0)$. (The reason this would be strictly larger is that $K_0 \cap \sigma(A(s)) = \emptyset$ for some $s \in (t, t+\eta)$ while $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$.) Therefore, $t_0 = 1$ and this proves the theorem.

Corollary 7.9.5 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains an eigenvalue of A. Also, if there are n disjoint Gerschgorin discs, then each one contains an eigenvalue of A.*

Proof: Denote by $A(t)$ the matrix $\left(a_{ij}^t\right)$ where if $i \neq j$, $a_{ij}^t = ta_{ij}$ and $a_{ii}^t = a_{ii}$. Thus to get $A(t)$ multiply all non diagonal terms by t. Let $t \in [0,1]$. Then $A(0) = \text{diag}(a_{11}, \cdots, a_{nn})$ and $A(1) = A$. Furthermore, the map, $t \to A(t)$ is continuous. Denote by D_j^t the Gerschgorin disc obtained from the j^{th} row for the matrix, $A(t)$. Then it is clear that $D_j^t \subseteq D_j$ the j^{th} Gerschgorin disc for A. It follows a_{ii} is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point a_{ii} which is contained in D_i. Letting K be the connected component in Σ for Σ defined in Theorem 7.9.4 which is determined by a_{ii}, Gerschgorin's theorem implies that $K \cap \sigma(A(t)) \subseteq \cup_{j=1}^n D_j^t \subseteq \cup_{j=1}^n D_j = D_i \cup (\cup_{j \neq i} D_j)$ and also, since K is connected, there are not points of K in both D_i and $(\cup_{j \neq i} D_j)$. Since at least one point of K is in D_i,(a_{ii}), it follows all of K must be contained in D_i. Now by Theorem 7.9.4 this shows there are points of $K \cap \sigma(A)$ in D_i. The last assertion follows immediately.

This can be improved even more. This involves the following lemma.

Lemma 7.9.6 *In the situation of Theorem 7.9.4 suppose $\lambda(0) = K_0 \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in [0,1]$,*

$$\sigma(A(t)) \cap K_0 = \lambda(t)$$

where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.

Proof: Let $S \equiv \{t \in [0,1] : K_0 \cap \sigma(A(s)) = \lambda(s),$ a simple eigenvalue for all $s \in [0,t]\}$. Then $0 \in S$ so it is nonempty. Let $t_0 = \sup(S)$ and suppose $\lambda_1 \neq \lambda_2$ are two elements of $\sigma(A(t_0)) \cap K_0$. Then choosing $\eta > 0$ small enough, and letting D_i be disjoint discs containing λ_i respectively, similar arguments to those of Lemma 7.9.3 can be used to conclude

$$H_i \equiv \cup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D_i$$

is a connected and nonempty set for $i = 1, 2$ which would require that $H_i \subseteq K_0$. But then there would be two different eigenvalues of $A(s)$ contained in K_0, contrary to the definition of t_0. Therefore, there is at most one eigenvalue, $\lambda(t_0) \in K_0 \cap \sigma(A(t_0))$. Could it be a repeated root of the characteristic equation? Suppose $\lambda(t_0)$ is a repeated root of the characteristic equation. As before, choose a small disc, D centered at $\lambda(t_0)$ and η small enough that

$$H \equiv \cup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since H is connected and contains $\lambda(t_0)$ it must be contained in K_0 which contradicts the condition for $s \in S$ for all these $s \in [t_0 - \eta, t_0]$. Therefore, $t_0 \in S$ as hoped. If $t_0 < 1$, there exists a small disc centered at $\lambda(t_0)$ and $\eta > 0$ such that for all $s \in [t_0, t_0 + \eta]$, $A(s)$ has only simple eigenvalues in D and the only eigenvalues of $A(s)$ which could be in K_0 are in D. (This last assertion follows from noting that $\lambda(t_0)$ is the only eigenvalue of $A(t_0)$ in K_0 and so the others are at a positive distance from K_0. For s close enough to t_0, the eigenvalues of $A(s)$ are either close to these eigenvalues of $A(t_0)$ at a positive distance from K_0 or they are close to the eigenvalue, $\lambda(t_0)$ in which case it can be assumed they are in D.) But this shows that t_0 is not really an upper bound to S. Therefore, $t_0 = 1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be sharpened.

Corollary 7.9.7 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains exactly one eigenvalue of A and this eigenvalue is a simple root to the characteristic polynomial of A.*

Proof: In the proof of Corollary 7.9.5, note that a_{ii} is a simple root of $A(0)$ since otherwise the i^{th} Gerschgorin disc would not be disjoint from the others. Also, K, the connected component determined by a_{ii} must be contained in D_i because it is connected and by Gerschgorin's theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the a_{jj}, are outside D_i, it follows that $K \cap \sigma(A(0)) = a_{ii}$. Therefore, by Lemma 7.9.6, $K \cap \sigma(A(1)) = K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

Example 7.9.8 *Consider the matrix,*

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The Gerschgorin discs are $D(5,1), D(1,2)$, and $D(0,1)$. Observe $D(5,1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5,1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^3 - 6t^2 + 3t + 5 = 0$. The numerical values of these are $-.669\,66, 1.423\,1$, and $5.246\,55$, verifying the predictions of Gerschgorin's theorem.

7.10 Exercises

1. ♠Explain why it is typically impossible to compute the upper triangular matrix whose existence is guaranteed by Schur's theorem.

2. ♠Now recall the QR factorization of Theorem 5.7.5 on Page 133. The QR algorithm is a technique which does compute the upper triangular matrix in Schur's theorem. There is much more to the QR algorithm than will be presented here. In fact, what I am about to show you is not the way it is done in practice. One first obtains what is called a Hessenburg matrix for which the algorithm will work better. However, the idea is as follows. Start with A an $n \times n$ matrix having real eigenvalues. Form $A = QR$ where Q is orthogonal and R is upper triangular. (Right triangular.) This can be done using the technique of Theorem 5.7.5 using Householder matrices. Next take $A_1 \equiv RQ$. Show that $A = QA_1Q^T$. In other words these two matrices, A, A_1 are similar. Explain why they have the same eigenvalues. Continue by letting A_1 play the role of A. Thus the algorithm is of the form $A_n = QR_n$ and $A_{n+1} = R_{n+1}Q$. Explain why $A = Q_nA_nQ_n^T$ for some Q_n orthogonal. Thus A_n is a sequence of matrices each similar to A. The remarkable thing is that often these matrices converge to an upper triangular matrix, T and $A = QTQ^T$ for some orthogonal matrix, the limit of the Q_n where the limit means the entries converge. Then the process computes the upper triangular Schur form of the matrix A. Thus the eigenvalues of A appear on the diagonal of T. You will see approximately what these are as the process continues.

3. Try the QR algorithm on
$$\begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}$$
which has eigenvalues 3 and 2. I suggest you use a computer algebra system to do the computations.

4. ♠Now try the QR algorithm on
$$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

Show that the algorithm cannot converge for this example. **Hint:** Try a few iterations of the algorithm.

5. ♠Show the two matrices $A \equiv \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$ and $B \equiv \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ are similar; that is there exists a matrix S such that $A = S^{-1}BS$ but there is no orthogonal matrix Q such that $Q^TBQ = A$. Show the QR algorithm does converge for the matrix B although it fails to do so for A.

6. ♠Let F be an $m \times n$ matrix. Show that F^*F has all real eigenvalues and furthermore, they are all nonnegative.

7. ♠If A is a real $n \times n$ matrix and λ is a complex eigenvalue of A having eigenvector $\mathbf{z} + i\mathbf{w}$, show that $\mathbf{w} \neq \mathbf{0}$.

8. ♠Suppose $A = Q^TDQ$ where Q is an orthogonal matrix and all the matrices are real. Also D is a diagonal matrix. Show that A must be symmetric.

9. ♠Suppose A is an $n \times n$ matrix and there exists a unitary matrix U such that

$$A = U^*DU$$

where D is a diagonal matrix. Explain why A must be normal.

10. ♠Show that every unitary matrix preserves distance. That is, if U is unitary,

$$|U\mathbf{x}| = |\mathbf{x}|.$$

11. ♠Show that if a matrix does preserve distances, then it must be unitary.

12. ♠Suppose A is an $n \times n$ matrix which is diagonally dominant. Recall this means

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|$$

show A^{-1} must exist.

13. ♠Give some disks in the complex plane whose union contains all the eigenvalues of the matrix

$$\begin{pmatrix} 1+2i & 4 & 2 \\ 0 & i & 3 \\ 5 & 6 & 7 \end{pmatrix}$$

14. ♠Using Schur's theorem, show the trace of an $n \times n$ matrix equals the sum of the eigenvalues and the determinant of an $n \times n$ matrix is the product of the eigenvalues.

15. ♠Show a square matrix is invertible if and only if it has no zero eigenvalues.

16. ♠Here is a matrix.

$$\begin{pmatrix} 1234 & 6 & 5 & 3 \\ 0 & -654 & 9 & 123 \\ 98 & 123 & 10,000 & 11 \\ 56 & 78 & 98 & 400 \end{pmatrix}$$

I know this matrix has an inverse before doing any computations. How do I know?

17. Show the critical points of the following function are

$$(0,-3,0),(2,-3,0), \text{and } \left(1,-3,-\frac{1}{3}\right)$$

and classify them as local minima, local maxima or saddle points.

$f(x,y,z) = -\frac{3}{2}x^4 + 6x^3 - 6x^2 + zx^2 - 2zx - 2y^2 - 12y - 18 - \frac{3}{2}z^2$.

The Hessian is

$$\begin{pmatrix} -12+36x+2z-18x^2 & 0 & -2+2x \\ 0 & -4 & 0 \\ -2+2x & 0 & -3 \end{pmatrix}$$

Now consider the critical point, $\left(1,-3,-\frac{1}{3}\right)$. At this point the Hessian matrix equals

$$\begin{pmatrix} \frac{16}{3} & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

The eigenvalues are $\frac{16}{3}, -3, -4$ and so this point is a saddle point.

Next consider the critical point, $(2,-3,0)$. At this point the Hessian matrix is

$$\begin{pmatrix} -12 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -3 \end{pmatrix}$$

The eigenvalues are $-4, -\frac{15}{2} + \frac{1}{2}\sqrt{97}, -\frac{15}{2} - \frac{1}{2}\sqrt{97}$, all negative so at this point there is a local max.

Finally consider the critical point, $(0,-3,0)$. At this point the Hessian is

$$\begin{pmatrix} -12 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -3 \end{pmatrix}$$

and the eigenvalues are the same as the above, all negative. Therefore, there is a local maximum at this point.

18. Here is a function of three variables.

$$f(x,y,z) = 5yx - 5y + 3y^2 - 5zy + zx - z - z^2$$

change the variables so that in the new variables there are no mixed terms, terms involving xy, yz etc.

19. ♠Here is a function of three variables.

$$f(x,y,z) = 2x^2 - 4x + 2 + 9yx - 9y - 3zx + 3z + 5y^2 - 9zy - 7z^2$$

change the variables so that in the new variables there are no mixed terms, terms involving xy, yz etc.

20. Show the critical points of the function,

$$f(x,y,z) = -2yx^2 - 6yx - 4zx^2 - 12zx + y^2 + 2yz.$$

are points of the form,

$$(x,y,z) = \left(t, 2t^2 + 6t, -t^2 - 3t\right)$$

for $t \in \mathbf{R}$ and classify them as local minima, local maxima or saddle points.

21. Show the critical points of the function

$$f(x,y,z) = \frac{1}{2}x^4 - 4x^3 + 8x^2 - 3zx^2 + 12zx + 2y^2 + 4y + 2 + \frac{1}{2}z^2.$$

are $(0,-1,0), (4,-1,0)$, and $(2,-1,-12)$ and classify them as local minima, local maxima or saddle points.

22. ♠Let $f(x,y) = 3x^4 - 24x^2 + 48 - yx^2 + 4y$. Find and classify the critical points using the second derivative test.

23. Let $f(x,y) = 3x^4 - 5x^2 + 2 - y^2x^2 + y^2$. Find and classify the critical points using the second derivative test.

24. Let $f(x,y) = 5x^4 - 7x^2 - 2 - 3y^2x^2 + 11y^2 - 4y^4$. Find and classify the critical points using the second derivative test.

25. Let $f(x,y,z) = -2x^4 - 3yx^2 + 3x^2 + 5x^2z + 3y^2 - 6y + 3 - 3zy + 3z + z^2$. Find and classify the critical points using the second derivative test.

26. Let $f(x,y,z) = 3yx^2 - 3x^2 - x^2z - y^2 + 2y - 1 + 3zy - 3z - 3z^2$. Find and classify the critical points using the second derivative test.

27. ♠Let Q be orthogonal. Find the possible values of $\det(Q)$.

28. ♠Let U be unitary. Find the possible values of $\det(U)$.

29. ♠If a matrix is nonzero can it have only zero for eigenvalues?

30. ♠A matrix A is called nilpotent if $A^k = 0$ for some positive integer k. Suppose A is a nilpotent matrix. Show it has only 0 for an eigenvalue.

31. ♠If A is a nonzero nilpotent matrix, show it must be defective.

32. ♠Suppose A is a nondefective $n \times n$ matrix and its eigenvalues are all either 0 or 1. Show $A^2 = A$. Could you say anything interesting if the eigenvalues were all either 0,1,or -1? By DeMoivre's theorem, an n^{th} root of unity is of the form

$$\left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

Could you generalize the sort of thing just described to get $A^n = A$? **Hint:** Since A is nondefective, there exists S such that $S^{-1}AS = D$ where D is a diagonal matrix.

33. ♠This and the following problems will present most of a differential equations course. To begin with, consider the scalar initial value problem

$$y' = ay,\ y(t_0) = y_0$$

When a is real, show the unique solution to this problem is $y = y_0 e^{a(t-t_0)}$. Next suppose

$$y' = (a+ib)\,y,\ y(t_0) = y_0 \tag{7.20}$$

where $y(t) = u(t) + iv(t)$. Show there exists a unique solution and it is

$$\begin{aligned} y(t) &= y_0 e^{a(t-t_0)} \left(\cos b(t-t_0) + i \sin b(t-t_0)\right) \\ &\equiv e^{(a+ib)(t-t_0)} y_0. \end{aligned} \tag{7.21}$$

Next show that for a real or complex there exists a unique solution to the initial value problem

$$y' = ay + f,\ y(t_0) = y_0$$

and it is given by

$$y(t) = e^{a(t-t_0)} y_0 + e^{at} \int_{t_0}^{t} e^{-as} f(s)\, ds.$$

Hint: For the first part write as $y' - ay = 0$ and multiply both sides by e^{-at}. Then expain why you get

$$\frac{d}{dt}\left(e^{-at} y(t)\right) = 0,\ y(t_0) = 0.$$

Now you finish the argument. To show uniqueness in the second part, suppose

$$y' = (a+ib)\,y,\ y(0) = 0$$

and verify this requires $y(t) = 0$. To do this, note

$$\overline{y}' = (a-ib)\,\overline{y},\ \overline{y}(0) = 0$$

and that

$$\begin{aligned} \frac{d}{dt}|y(t)|^2 &= y'(t)\,\overline{y}(t) + \overline{y}'(t)\,y(t) \\ &= (a+ib)\,y(t)\,\overline{y}(t) + (a-ib)\,\overline{y}(t)\,y(t) \\ &= 2a\,|y(t)|^2,\ |y|^2(t_0) = 0 \end{aligned}$$

Thus from the first part $|y(t)|^2 = 0e^{-2at} = 0$. Finally observe by a simple computation that 7.20 is solved by 7.21. For the last part, write the equation as

$$y' - ay = f$$

and multiply both sides by e^{-at} and then integrate from t_0 to t using the initial condition.

34. ♠Now consider A an $n \times n$ matrix. By Schur's theorem there exists unitary Q such that

$$Q^{-1} A Q = T$$

where T is upper triangular. Now consider the first order initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{x}_0.$$

Show there exists a unique solution to this first order system. **Hint:** Let $\mathbf{y} = Q^{-1}\mathbf{x}$ and so the system becomes

$$\mathbf{y}' = T\mathbf{y},\ \mathbf{y}(t_0) = Q^{-1}\mathbf{x}_0 \tag{7.22}$$

Now letting $\mathbf{y} = (y_1, \cdots, y_n)^T$, the bottom equation becomes

$$y_n' = t_{nn}y_n,\ y_n(t_0) = \left(Q^{-1}\mathbf{x}_0\right)_n.$$

Then use the solution you get in this to get the solution to the initial value problem which occurs one level up, namely

$$y_{n-1}' = t_{(n-1)(n-1)}y_{n-1} + t_{(n-1)n}y_n,\ y_{n-1}(t_0) = \left(Q^{-1}\mathbf{x}_0\right)_{n-1}$$

Continue doing this to obtain a unique solution to 7.22.

35. ♠Now suppose $\Phi(t)$ is an $n \times n$ matrix of the form

$$\Phi(t) = \begin{pmatrix} \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \tag{7.23}$$

where

$$\mathbf{x}_k'(t) = A\mathbf{x}_k(t).$$

Explain why

$$\Phi'(t) = A\Phi(t)$$

if and only if $\Phi(t)$ is given in the form of 7.23. Also explain why if $\mathbf{c} \in \mathbb{F}^n$,

$$\mathbf{y}(t) \equiv \Phi(t)\mathbf{c}$$

solves the equation

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

36. ♠In the above problem, consider the question whether all solutions to

$$\mathbf{x}' = A\mathbf{x} \tag{7.24}$$

are obtained in the form $\Phi(t)\mathbf{c}$ for some choice of $\mathbf{c} \in \mathbb{F}^n$. In other words, is the general solution to this equation $\Phi(t)\mathbf{c}$ for $\mathbf{c} \in \mathbb{F}^n$? Prove the following theorem using linear algebra.

Theorem 7.10.1 *Suppose $\Phi(t)$ is an $n \times n$ matrix which satisfies*

$$\Phi'(t) = A\Phi(t).$$

Then the general solution to 7.24 is $\Phi(t)\mathbf{c}$ if and only if $\Phi(t)^{-1}$ exists for some t. Furthermore, if $\Phi'(t) = A\Phi(t)$, then either $\Phi(t)^{-1}$ exists for all t or $\Phi(t)^{-1}$ never exists for any t.

($\det(\Phi(t))$ is called the Wronskian and this theorem is sometimes called the Wronskian alternative.)

Hint: Suppose first the general solution is of the form $\Phi(t)\mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector in $\mathbb{F}^n$. You need to verify $\Phi(t)^{-1}$ exists for some t. In fact, show

$\Phi(t)^{-1}$ exists for every t. Suppose then that $\Phi(t_0)^{-1}$ does not exist. Explain why there exists $\mathbf{c} \in \mathbb{F}^n$ such that there is no solution $\mathbf{x}$ to

$$\mathbf{c} = \Phi(t_0)\mathbf{x}$$

By the existence part of Problem 34 there exists a solution to

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{c}$$

but this cannot be in the form $\Phi(t)\mathbf{c}$. Thus for every t, $\Phi(t)^{-1}$ exists. Next suppose for some $t_0, \Phi(t_0)^{-1}$ exists. Let $\mathbf{z}' = A\mathbf{z}$ and choose $\mathbf{c}$ such that

$$\mathbf{z}(t_0) = \Phi(t_0)\mathbf{c}$$

Then both $\mathbf{z}(t), \Phi(t)\mathbf{c}$ solve

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{z}(t_0)$$

Apply uniqueness to conclude $\mathbf{z} = \Phi(t)\mathbf{c}$. Finally, consider that $\Phi(t)\mathbf{c}$ for $\mathbf{c} \in \mathbb{F}^n$ either is the general solution or it is not the general solution. If it is, then $\Phi(t)^{-1}$ exists for all t. If it is not, then $\Phi(t)^{-1}$ cannot exist for any t from what was just shown.

37. ♠Let $\Phi'(t) = A\Phi(t)$. Then $\Phi(t)$ is called a fundamental matrix if $\Phi(t)^{-1}$ exists for all t. Show there exists a unique solution to the equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7.25}$$

and it is given by the formula

$$\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}\mathbf{f}(s)\,ds$$

Now these few problems have done virtually everything of significance in an entire undergraduate differential equations course, illustrating the superiority of linear algebra. The above formula is called the variation of constants formula.

Hint: Uniquenss is easy. If $\mathbf{x}_1, \mathbf{x}_2$ are two solutions then let $\mathbf{u}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ and argue $\mathbf{u}' = A\mathbf{u},\ \mathbf{u}(t_0) = \mathbf{0}$. Then use Problem 34. To verify there exists a solution, you could just differentiate the above formula using the fundamental theorem of calculus and verify it works. Another way is to assume the solution in the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}(t)$$

and find $\mathbf{c}(t)$ to make it all work out. This is called the method of variation of parameters.

38. ♠Show there exists a special Φ such that $\Phi'(t) = A\Phi(t),\ \Phi(0) = I$, and suppose $\Phi(t)^{-1}$ exists for all t. Show using uniqueness that

$$\Phi(-t) = \Phi(t)^{-1}$$

and that for all $t, s \in \mathbb{R}$

$$\Phi(t+s) = \Phi(t)\Phi(s)$$

Explain why with this special Φ, the solution to 7.25 can be written as

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t - s)\mathbf{f}(s)\, ds.$$

Hint: Let $\Phi(t)$ be such that the j^{th} column is $\mathbf{x}_j(t)$ where

$$\mathbf{x}_j' = A\mathbf{x}_j,\ \mathbf{x}_j(0) = \mathbf{e}_j.$$

Use uniqueness as required.

Vector Spaces And Fields

8.1 Vector Space Axioms

It is time to consider the idea of a Vector space.

Definition 8.1.1 *A vector space is an Abelian group of "vectors" satisfying the axioms of an Abelian group,*

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v},$$

the commutative law of addition,

$$(\mathbf{v}+\mathbf{w})+\mathbf{z}=\mathbf{v}+(\mathbf{w}+\mathbf{z}),$$

the associative law for addition,

$$\mathbf{v}+\mathbf{0}=\mathbf{v},$$

the existence of an additive identity,

$$\mathbf{v}+(-\mathbf{v})=\mathbf{0},$$

the existence of an additive inverse, along with a field of "scalars", $\mathbb{F}$ which are allowed to multiply the vectors according to the following rules. (The Greek letters denote scalars.)

$$\alpha(\mathbf{v}+\mathbf{w})=\alpha\mathbf{v}+\alpha\mathbf{w}, \tag{8.1}$$

$$(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}, \tag{8.2}$$

$$\alpha(\beta\mathbf{v})=\alpha\beta(\mathbf{v}), \tag{8.3}$$

$$1\mathbf{v}=\mathbf{v}. \tag{8.4}$$

The field of scalars is usually $\mathbb{R}$ or $\mathbb{C}$ and the vector space will be called real or complex depending on whether the field is $\mathbb{R}$ or $\mathbb{C}$. However, other fields are also possible. For example, one could use the field of rational numbers or even the field of the integers mod p for p a prime. A vector space is also called a linear space.

For example, $\mathbb{R}^n$ with the usual conventions is an example of a real vector space and $\mathbb{C}^n$ is an example of a complex vector space. Up to now, the discussion has been for $\mathbb{R}^n$ or $\mathbb{C}^n$ and all that is taking place is an increase in generality and abstraction.

8.2 Subspaces And Bases

8.2.1 Basic Definitions

Definition 8.2.1 *If* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subseteq V$, *a vector space, then*

$$\text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n) \equiv \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{v}_i : \alpha_i \in \mathbb{F} \right\}.$$

A subset, $W \subseteq V$ *is said to be a subspace if it is also a vector space with the same field of scalars. Thus* $W \subseteq V$ *is a subspace if* $ax + by \in W$ *whenever* $a, b \in \mathbb{F}$ *and* $x, y \in W$. *The span of a set of vectors as just described is an example of a subspace.*

Definition 8.2.2 *If* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subseteq V$, *the set of vectors is linearly independent if*

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0}$$

implies

$$\alpha_1 = \cdots = \alpha_n = 0$$

and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *is called a basis for* V *if*

$$\text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_n) = V$$

and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *is linearly independent. The set of vectors is linearly dependent if it is not linearly independent.*

8.2.2 A Fundamental Theorem

The next theorem is called the exchange theorem. It is very important that you understand this theorem. It is so important that I have given several proofs of it. Some amount to the same thing, just worded differently.

Theorem 8.2.3 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *be a linearly independent set of vectors such that each* $\mathbf{x}_i$ *is in the span*$\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$. *Then* $r \leq s$.

Proof 1: Define $\text{span}\{\mathbf{y}_1, \cdots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \cdots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^{s} c_i \mathbf{y}_i. \tag{8.5}$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = 0$ and so $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = 0$, $1\mathbf{x}_1 + \sum_{i=2}^{r} 0\mathbf{x}_i = \mathbf{x}_1 = 0$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve 8.5 for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \text{span}\left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s}^{\text{s-1 vectors here}}\right).$$

Define $\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s\}$$

Therefore, $\text{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \cdots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors, $\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ to obtain $\mathbf{v} \in \text{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$. The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that $\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$ where the vectors, $\mathbf{z}_1, \cdots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ and $l + p = s$. This has now been done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l + 1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \cdots, \mathbf{x}_l\}$ and since $\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$ there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^{l} c_i \mathbf{x}_i + \sum_{j=1}^{p} d_j \mathbf{z}_j. \tag{8.6}$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (8.6) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\text{span}\left(\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_p}^{\text{p-1 vectors here}}\right) = V.$$

Continue this way, eventually obtaining

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s) = V.$$

But then $\mathbf{x}_r \in \text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_s\}$ contrary to the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Proof 2: Let

$$\mathbf{x}_k = \sum_{j=1}^{s} a_{jk} \mathbf{y}_j$$

If $r > s$, then the matrix $A = (a_{jk})$ has more columns than rows. By Corollary 4.3.9 one of these columns is a linear combination of the others. This implies there exist scalars $c_1, \cdots, c_r$, not all zero such that

$$\sum_{k=1}^{r} a_{jk} c_k = 0, \ j = 1, \cdots, r$$

Then

$$\sum_{k=1}^{r} c_k \mathbf{x}_k = \sum_{k=1}^{r} c_k \sum_{j=1}^{s} a_{jk} \mathbf{y}_j = \sum_{j=1}^{s} \left(\sum_{k=1}^{r} c_k a_{jk}\right) \mathbf{y}_j = \mathbf{0}$$

which contradicts the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Hence $r \leq s$.

Proof 3: Suppose $r > s$. Let $\mathbf{z}_k$ denote a vector of $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$. Thus there exists j as small as possible such that

$$\text{span}(\mathbf{y}_1, \cdots, \mathbf{y}_s) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_m, \mathbf{z}_1, \cdots, \mathbf{z}_j)$$

where $m+j=s$. It is given that $m=0$, corresponding to no vectors of $\{\mathbf{x}_1,\cdots,\mathbf{x}_m\}$ and $j=s$, corresponding to all the $\mathbf{y}_k$ results in the above equation holding. If $j>0$ then $m<s$ and so

$$\mathbf{x}_{m+1}=\sum_{k=1}^{m}a_k\mathbf{x}_k+\sum_{i=1}^{j}b_i\mathbf{z}_i$$

Not all the b_i can equal 0 and so you can solve for one of them in terms of $\mathbf{x}_{m+1},\mathbf{x}_m,\cdots,\mathbf{x}_1$, and the other $\mathbf{z}_k$. Therefore, there exists

$$\{\mathbf{z}_1,\cdots,\mathbf{z}_{j-1}\}\subseteq\{\mathbf{y}_1,\cdots,\mathbf{y}_s\}$$

such that

$$\operatorname{span}(\mathbf{y}_1,\cdots,\mathbf{y}_s)=\operatorname{span}(\mathbf{x}_1,\cdots,\mathbf{x}_{m+1},\mathbf{z}_1,\cdots,\mathbf{z}_{j-1})$$

contradicting the choice of j. Hence $j=0$ and

$$\operatorname{span}(\mathbf{y}_1,\cdots,\mathbf{y}_s)=\operatorname{span}(\mathbf{x}_1,\cdots,\mathbf{x}_s)$$

It follows that

$$\mathbf{x}_{s+1}\in\operatorname{span}(\mathbf{x}_1,\cdots,\mathbf{x}_s)$$

contrary to the assumption the $\mathbf{x}_k$ are linearly independent. Therefore, $r\leq s$ as claimed. This proves the theorem.

Corollary 8.2.4 *If $\{\mathbf{u}_1,\cdots,\mathbf{u}_m\}$ and $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ are two bases for V, then $m=n$.*

Proof: By Theorem 8.2.3, $m\leq n$ and $n\leq m$.

Definition 8.2.5 *A vector space V is of dimension n if it has a basis consisting of n vectors. This is well defined thanks to Corollary 8.2.4. It is always assumed here that $n<\infty$ and in this case, such a vector space is said to be finite dimensional.*

Here is a useful lemma.

Lemma 8.2.6 *Suppose $\mathbf{v}\notin\operatorname{span}(\mathbf{u}_1,\cdots,\mathbf{u}_k)$ and $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}$ is linearly independent. Then $\{\mathbf{u}_1,\cdots,\mathbf{u}_k,\mathbf{v}\}$ is also linearly independent.*

Proof: Suppose $\sum_{i=1}^{k}c_i\mathbf{u}_i+d\mathbf{v}=0$. It is required to verify that each $c_i=0$ and that $d=0$. But if $d\neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}$,

$$\mathbf{v}=-\sum_{i=1}^{k}\left(\frac{c_i}{d}\right)\mathbf{u}_i$$

contrary to assumption. Therefore, $d=0$. But then $\sum_{i=1}^{k}c_i\mathbf{u}_i=0$ and the linear independence of $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}$ implies each $c_i=0$ also. This proves the lemma.

Given a spanning set, you can delete vectors till you end up with a basis. Given a linearly independent set, you can add vectors till you get a basis. This is what the following theorem is about, weeding and planting.

Theorem 8.2.7 *If $V=\operatorname{span}(\mathbf{u}_1,\cdots,\mathbf{u}_n)$ then some subset of $\{\mathbf{u}_1,\cdots,\mathbf{u}_n\}$ is a basis for V. Also, if $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}\subseteq V$ is linearly independent and the vector space is finite dimensional, then the set, $\{\mathbf{u}_1,\cdots,\mathbf{u}_k\}$, can be enlarged to obtain a basis of V.*

Proof: Let

$$S = \{E \subseteq \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \text{ such that } \operatorname{span}(E) = V\}.$$

For $E \in S$, let $|E|$ denote the number of elements of E. Let

$$m \equiv \min\{|E| \text{ such that } E \in S\}.$$

Thus there exist vectors

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$$

such that

$$\operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_m) = V$$

and m is as small as possible for this to happen. If this set is linearly independent, it follows it is a basis for V and the theorem is proved. On the other hand, if the set is not linearly independent, then there exist scalars,

$$c_1, \cdots, c_m$$

such that

$$\mathbf{0} = \sum_{i=1}^{m} c_i \mathbf{v}_i$$

and not all the c_i are equal to zero. Suppose $c_k \neq 0$. Then the vector, $\mathbf{v}_k$ may be solved for in terms of the other vectors. Consequently,

$$V = \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_m)$$

contradicting the definition of m. This proves the first part of the theorem.

To obtain the second part, begin with $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ and suppose a basis for V is

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}.$$

If

$$\operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k) = V,$$

then $k = n$. If not, there exists a vector,

$$\mathbf{u}_{k+1} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k).$$

Then by Lemma 8.2.6, $\{\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is also linearly independent. Continue adding vectors in this way until n linearly independent vectors have been obtained. Then

$$\operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_n) = V$$

because if it did not do so, there would exist $\mathbf{u}_{n+1}$ as just described and $\{\mathbf{u}_1, \cdots, \mathbf{u}_{n+1}\}$ would be a linearly independent set of vectors having $n+1$ elements even though $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis. This would contradict Theorem 8.2.3. Therefore, this list is a basis and this proves the theorem.

8.2.3 The Basis Of A Subspace

Every subspace of a finite dimensional vector space is a span of some vectors and in fact it has a basis. This is the content of the next theorem.

Theorem 8.2.8 *Let V be a nonzero subspace of a finite dimensional vector space, W of dimension, n. Then V has a basis with no more than n vectors.*

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq 0$. If $\text{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V. Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\text{span}\{\mathbf{v}_1\}$. By Lemma 8.2.6 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem, Theorem 8.2.3. This proves the theorem.

8.3 Lots Of Fields

8.3.1 Irreducible Polynomials

I mentioned earlier that most things hold for arbitrary fields. However, I have not bothered to give any examples of other fields. This is the point of this section. It also turns out that showing the algebraic numbers are a field can be understood using vector space concepts and it gives a very convincing application of the abstract theory presented earlier in this chapter.

Here I will give some basic algebra relating to polynomials. This is interesting for its own sake but also provides the basis for constructing many different kinds of fields. The first is the Euclidean algorithm for polynomials.

Definition 8.3.1 *A polynomial is an expression of the form $\sum_{k=0}^{n} a_k \lambda^k$ where as usual λ^0 is defined to equal 1. Two polynomials are said to be equal if their corresponding coefficients are the same. Thus, in particular, $p(\lambda) = 0$ means each of the $a_k = 0$. An element of the field λ is said to be a root of the polynomial if $p(\lambda) = 0$ in the sense that when you plug in λ into the formula and do the indicated operations, you get 0.*

Lemma 8.3.2 *Let $f(\lambda)$ and $g(\lambda)$ be polynomials. Then there exists a polynomial, $q(\lambda)$ such that*

$$f(\lambda) = q(\lambda) g(\lambda) + r(\lambda)$$

where the degree of $r(\lambda)$ is less than the degree of $g(\lambda)$ or $r(\lambda) = 0$.

Proof: Consider the polynomials of the form $f(\lambda) - g(\lambda) l(\lambda)$ and out of all these polynomials, pick one which has the smallest degree. This can be done because of the well ordering of the natural numbers. Let this take place when $l(\lambda) = q_1(\lambda)$ and let

$$r(\lambda) = f(\lambda) - g(\lambda) q_1(\lambda).$$

It is required to show degree of $r(\lambda) <$ degree of $g(\lambda)$ or else $r(\lambda) = 0$.

Suppose $f(\lambda) - g(\lambda) l(\lambda)$ is never equal to zero for any $l(\lambda)$. Then $r(\lambda) \neq 0$. It is required to show the degree of $r(\lambda)$ is smaller than the degree of $g(\lambda)$. If this doesn't happen, then the degree of $r \geq$ the degree of g. Let

$$\begin{aligned} r(\lambda) &= b_m \lambda^m + \cdots + b_1 \lambda + b_0 \\ g(\lambda) &= a_n \lambda^n + \cdots + a_1 \lambda + a_0 \end{aligned}$$

where $m \geq n$ and b_m and a_n are nonzero. Then let $r_1(\lambda)$ be given by

$$r_1(\lambda) = r(\lambda) - \frac{\lambda^{m-n} b_m}{a_n} g(\lambda)$$

$$= (b_m \lambda^m + \cdots + b_1 \lambda + b_0) - \frac{\lambda^{m-n} b_m}{a_n} (a_n \lambda^n + \cdots + a_1 \lambda + a_0)$$

which has smaller degree than m, the degree of $r(\lambda)$. But

$$\begin{aligned} r_1(\lambda) &= \overbrace{f(\lambda) - g(\lambda) q_1(\lambda)}^{r(\lambda)} - \frac{\lambda^{m-n} b_m}{a_n} g(\lambda) \\ &= f(\lambda) - g(\lambda) \left(q_1(\lambda) + \frac{\lambda^{m-n} b_m}{a_n} \right), \end{aligned}$$

and this is not zero by the assumption that $f(\lambda) - g(\lambda) l(\lambda)$ is never equal to zero for any $l(\lambda)$ yet has smaller degree than $r(\lambda)$ which is a contradiction to the choice of $r(\lambda)$. This proves the lemma.

Now with this lemma, here is another one which is very fundamental. First here is a definition. A polynomial is **monic** means it is of the form

$$\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

That is, the leading coefficient is 1. In what follows, the coefficients of polynomials are in $\mathbb{F}$, a field of scalars which is completely arbitrary. Think $\mathbb{R}$ if you need an example.

Definition 8.3.3 *A polynomial f is said to divide a polynomial g if $g(\lambda) = f(\lambda) r(\lambda)$ for some polynomial $r(\lambda)$. Let $\{\phi_i(\lambda)\}$ be a finite set of polynomials. The greatest common divisor will be the monic polynomial q such that $q(\lambda)$ divides each $\phi_i(\lambda)$ and if $p(\lambda)$ divides each $\phi_i(\lambda)$, then $p(\lambda)$ divides $q(\lambda)$. The finite set of polynomials $\{\phi_i\}$ is said to be relatively prime if their greatest common divisor is 1. A polynomial $f(\lambda)$ is irreducible if there is no polynomial with coefficients in $\mathbb{F}$ which divides it except nonzero multiples of $f(\lambda)$ and constants.*

Proposition 8.3.4 *The greatest common divisor is unique.*

Proof: Suppose both $q(\lambda)$ and $q'(\lambda)$ work. Then $q(\lambda)$ divides $q'(\lambda)$ and the other way around and so

$$q'(\lambda) = q(\lambda) l(\lambda), \ q(\lambda) = l'(\lambda) q'(\lambda)$$

Therefore, the two must have the same degree. Hence $l'(\lambda), l(\lambda)$ are both constants. However, this constant must be 1 because both $q(\lambda)$ and $q'(\lambda)$ are monic. This proves the proposition.

Theorem 8.3.5 *Let $\psi(\lambda)$ be the greatest common divisor of $\{\phi_i(\lambda)\}$. Then there exist polynomials $r_i(\lambda)$ such that*

$$\psi(\lambda) = \sum_{i=1}^{p} r_i(\lambda) \phi_i(\lambda).$$

Furthermore, $\psi(\lambda)$ is the monic polynomial of smallest degree which can be written in the above form.

Proof: Let S denote the set of monic polynomials which are of the form

$$\sum_{i=1}^{p} r_i(\lambda) \phi_i(\lambda)$$

where $r_i(\lambda)$ is a polynomial. Then let the r_i be chosen such that the degree of the expression $\sum_{i=1}^{p} r_i(\lambda) \phi_i(\lambda)$ is as small as possible. Letting $\psi(\lambda)$ equal this sum, it remains to verify

it is the greatest common divisor. First, does it divide each $\phi_i(\lambda)$? Suppose it fails to divide $\phi_1(\lambda)$. Then by Lemma 8.3.2

$$\phi_1(\lambda) = \psi(\lambda) l(\lambda) + r(\lambda)$$

where degree of $r(\lambda)$ is less than that of $\psi(\lambda)$. Then dividing $r(\lambda)$ by the leading coefficient if necessary and denoting the result by $\psi_1(\lambda)$, it follows the degree of $\psi_1(\lambda)$ is less than the degree of $\psi(\lambda)$ and $\psi_1(\lambda)$ is of the form

$$\psi_1(\lambda) = (\phi_1(\lambda) - \psi(\lambda) l(\lambda)) a$$

$$\begin{aligned} &= \left(\phi_1(\lambda) - \sum_{i=1}^{p} r_i(\lambda) \phi_i(\lambda) l(\lambda)\right) a \\ &= \left((1 - r_1(\lambda)) \phi_1(\lambda) + \sum_{i=2}^{p} (-r_i(\lambda) l(\lambda)) \phi_i(\lambda)\right) a \end{aligned}$$

for a suitable $a \in \mathbb{F}$. This is one of the polynomials in S. Therefore, $\psi(\lambda)$ does not have the smallest degree after all because the degree of $\psi_1(\lambda)$ is smaller. This is a contradiction. Therefore, $\psi(\lambda)$ divides $\phi_1(\lambda)$. Similarly it divides all the other $\phi_i(\lambda)$.

If $p(\lambda)$ divides all the $\phi_i(\lambda)$, then it divides $\psi(\lambda)$ because of the formula for $\psi(\lambda)$ which equals $\sum_{i=1}^{p} r_i(\lambda) \phi_i(\lambda)$. This proves the theorem.

Lemma 8.3.6 *Let $\psi(\lambda)$ be an irreducible monic polynomial not equal to 1 which divides*

$$\prod_{i=1}^{p} \phi_i(\lambda)^{k_i}, \ k_i \text{ a positive integer,}$$

where each $\phi_i(\lambda)$ is an irreducible monic polynomial. Then $\psi(\lambda)$ equals some $\phi_i(\lambda)$.

Proof: Suppose $\psi(\lambda)$ is an irreducible monic polynomial. Let S denote all products of the form

$$\prod_{i=1}^{p} \phi_i(\lambda)^{k_i}$$

where each $\phi_i(\lambda)$ is an irreducible monic polynomial and $\psi(\lambda)$ divides the product and yet is not equal to any of the $\phi_i(\lambda)$. If the lemma is false, then $S \neq \emptyset$. Out of all such products in S, chose one for which $m \equiv \sum_{i=1}^{p} k_i$ is as small as possible.

First, I claim that $\psi(\lambda)$ and $\phi_1(\lambda)$ are relatively prime. Suppose $\eta(\lambda)$ is a nonconstant polynomial. If $\eta(\lambda)$ divides $\phi_1(\lambda)$, then since $\phi_1(\lambda)$ is irreducible, $\eta(\lambda)$ equals $a\phi_1(\lambda)$ for some $a \in \mathbb{F}$. If $\eta(\lambda)$ divides $\psi(\lambda)$ then it must be of the form $b\psi(\lambda)$ for some $b \in \mathbb{F}$ and so it follows

$$\psi(\lambda) = \frac{a}{b} \phi_1(\lambda)$$

but both $\psi(\lambda)$ and $\phi_1(\lambda)$ are monic polynomials which implies $a = b$ and so $\psi(\lambda) = \phi_1(\lambda)$. This is assumed not to happen. It follows the only polynomials which divide both $\psi(\lambda)$ and $\phi_1(\lambda)$ are constants and so the two polynomials are relatively prime. Thus a polynomial which divides them both must be a constant, and if it is monic, then it must be 1. Thus 1 is the greatest common divisor.

Therefore, from Theorem 8.3.5 there exist polynomials $r_i(\lambda)$ such that

$$1 = r_1(\lambda) \psi(\lambda) + r_2(\lambda) \phi_1(\lambda) \tag{8.7}$$

Since $\psi(\lambda)$ divides $\prod_{i=1}^{p}\phi_i(\lambda)^{k_i}$, there exists $g(\lambda)$ such that

$$\psi(\lambda)\,g(\lambda)=\prod_{i=1}^{p}\phi_i(\lambda)^{k_i}.$$

Multiply both sides of 8.7 by $\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}$. Then you get

$$\begin{aligned}
&\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}\\
=\;& r_1(\lambda)\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}\psi(\lambda)+r_2(\lambda)\prod_{i=1}^{p}\phi_i(\lambda)^{k_i}\\
=\;& r_1(\lambda)\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}\psi(\lambda)+r_2(\lambda)\psi(\lambda)\,g(\lambda)\\
=\;& \psi(\lambda)\left(r_1(\lambda)\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}+r_2(\lambda)\,g(\lambda)\right)
\end{aligned}$$

which shows $\psi(\lambda)$ divides $\prod_{i=2}^{p}\phi_i(\lambda)^{k_i}\phi_1(\lambda)^{k_1-1}$ which is of the same form as $\prod_{i=1}^{p}\phi_i(\lambda)^{k_i}$ except with k_1 replaced with k_1-1. Since the sum of the exponents of the $\phi_i(\lambda)$ is as small as possible, this is a contradiction. It follows $\psi(\lambda)$ must equal one of the $\phi_i(\lambda)$. This proves the lemma.

Now here is a simple lemma about canceling monic polynomials.

Lemma 8.3.7 *Suppose $p(\lambda)$ is a monic polynomial and $q(\lambda)$ is a polynomial such that*

$$p(\lambda)\,q(\lambda)=0.$$

Then $q(\lambda)=0$. Also if

$$p(\lambda)\,q_1(\lambda)=p(\lambda)\,q_2(\lambda)$$

then $q_1(\lambda)=q_2(\lambda)$.

Proof: Let

$$p(\lambda)=\sum_{j=1}^{k}p_j\lambda^j,\ q(\lambda)=\sum_{i=1}^{n}q_i\lambda^i,\ p_k=1.$$

Then the product equals

$$\sum_{j=1}^{k}\sum_{i=1}^{n}p_jq_i\lambda^{i+j}.$$

Then look at those terms involving λ^{k+n}. This is $p_kq_n\lambda^{k+n}$ and is given to be 0. Since $p_k=1$, it follows $q_n=0$. Thus

$$\sum_{j=1}^{k}\sum_{i=1}^{n-1}p_jq_i\lambda^{i+j}=0.$$

Then consider the term involving λ^{n-1+k} and conclude that since $p_k=1$, it follows $q_{n-1}=0$. Continuing this way, each $q_i=0$. This proves the first part. The second follows from

$$p(\lambda)\,(q_1(\lambda)-q_2(\lambda))=0.$$

This proves the lemma.

The following is the analog of the fundamental theorem of arithmetic for polynomials.

Theorem 8.3.8 *Let $f(\lambda)$ be a nonconstant polynomial with coefficients in $\mathbb{F}$. Then there is some $a \in \mathbb{F}$ such that $f(\lambda) = a\prod_{i=1}^{n}\phi_i(\lambda)$ where $\phi_i(\lambda)$ is an irreducible nonconstant monic polynomial and repeats are allowed. Furthermore, this factorization is unique in the sense that any two of these factorizations have the same nonconstant factors in the product, possibly in different order and the same constant a.*

Proof: That such a factorization exists is obvious. If $f(\lambda)$ is irreducible, you are done. Factor out the leading coefficient. If not, then $f(\lambda) = a\phi_1(\lambda)\phi_2(\lambda)$ where these are monic polynomials. Continue doing this with the ϕ_i and eventually arrive at a factorization of the desired form.

It remains to argue the factorization is unique except for order of the factors. Suppose

$$a\prod_{i=1}^{n}\phi_i(\lambda) = b\prod_{i=1}^{m}\psi_i(\lambda)$$

where the $\phi_i(\lambda)$ and the $\psi_i(\lambda)$ are all irreducible monic nonconstant polynomials and $a, b \in \mathbb{F}$. If $n > m$, then by Lemma 8.3.6, each $\psi_i(\lambda)$ equals one of the $\phi_j(\lambda)$. By the above cancelation lemma, Lemma 8.3.7, you can cancel all these $\psi_i(\lambda)$ with appropriate $\phi_j(\lambda)$ and obtain a contradiction because the resulting polynomials on either side would have different degrees. Similarly, it cannot happen that $n < m$. It follows $n = m$ and the two products consist of the same polynomials. Then it follows $a = b$. This proves the theorem.

The following corollary will be well used. This corollary seems rather believable but does require a proof.

Corollary 8.3.9 *Let $q(\lambda) = \prod_{i=1}^{p}\phi_i(\lambda)^{k_i}$ where the k_i are positive integers and the $\phi_i(\lambda)$ are irreducible monic polynomials. Suppose also that $p(\lambda)$ is a monic polynomial which divides $q(\lambda)$. Then*

$$p(\lambda) = \prod_{i=1}^{p}\phi_i(\lambda)^{r_i}$$

where r_i is a nonnegative integer no larger than k_i.

Proof: Using Theorem 8.3.8, let $p(\lambda) = b\prod_{i=1}^{s}\psi_i(\lambda)^{r_i}$ where the $\psi_i(\lambda)$ are each irreducible and monic and $b \in \mathbb{F}$. Then there exists a polynomial $g(\lambda)$ such that

$$p(\lambda)g(\lambda) = g(\lambda)b\prod_{i=1}^{s}\psi_i(\lambda)^{r_i} = \prod_{i=1}^{p}\phi_i(\lambda)^{k_i}$$

Since $p(\lambda)$ is monic, $b = 1$. Also $g(\lambda)$ must be monic. Hence

$$p(\lambda)g(\lambda) = \overbrace{\prod_{i=1}^{s}\psi_i(\lambda)^{r_i}}^{p(\lambda)}\prod_{j=1}^{l}\eta_j(\lambda) = \prod_{i=1}^{p}\phi_i(\lambda)^{k_i}$$

and by uniqueness, each ψ_i equals one of the $\phi_j(\lambda)$ and the same holding true of the $\eta_i(\lambda)$. Therefore, $p(\lambda)$ is of the desired form. This proves the corollary.

8.3.2 Polynomials And Fields

When you have a polynomial like $x^2 - 3$ which has no rational roots, it turns out you can enlarge the field of rational numbers to obtain a larger field such that this polynomial does

have roots in this larger field. I am going to discuss a systematic way to do this. It will turn out that for any polynomial with coefficients in any field, there always exists a possibly larger field such that the polynomial has roots in this larger field. This book has mainly featured the field of real or complex numbers but this procedure will show how to obtain many other fields which could be used in most of what was presented earlier in the book. Here is an important idea concerning equivalence relations which I hope is familiar.

Definition 8.3.10 *Let S be a set. The symbol, $\sim$ is called an equivalence relation on S if it satisfies the following axioms.*

1. *$x \sim x$ for all $x \in S$. (Reflexive)*
2. *If $x \sim y$ then $y \sim x$. (Symmetric)*
3. *If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)*

Definition 8.3.11 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

Also recall the notion of equivalence classes.

Theorem 8.3.12 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

Definition 8.3.13 *Let $\mathbb{F}$ be a field, for example the rational numbers, and denote by $\mathbb{F}[x]$ the polynomials having coefficients in $\mathbb{F}$. Suppose $p(x)$ is a polynomial. Let $a(x) \sim b(x)$ ($a(x)$ is similar to $b(x)$) when*

$$a(x) - b(x) = k(x)\, p(x)$$

for some polynomial $k(x)$.

Proposition 8.3.14 *In the above definition, $\sim$ is an equivalence relation.*

Proof: First of all, note that $a(x) \sim a(x)$ because their difference equals $0p(x)$. If $a(x) \sim b(x)$, then $a(x) - b(x) = k(x)\, p(x)$ for some $k(x)$. But then $b(x) - a(x) = -k(x)\, p(x)$ and so $b(x) \sim a(x)$. Next suppose $a(x) \sim b(x)$ and $b(x) \sim c(x)$. Then $a(x) - b(x) = k(x)\, p(x)$ for some polynomial $k(x)$ and also $b(x) - c(x) = l(x)\, p(x)$ for some polynomial $l(x)$. Then

$$a(x) - c(x) = a(x) - b(x) + b(x) - c(x)$$

$$= k(x)\, p(x) + l(x)\, p(x) = (l(x) + k(x))\, p(x)$$

and so $a(x) \sim c(x)$ and this shows the transitive law. This proves the proposition.

With this proposition, here is another definition which essentially describes the elements of the new field. It will eventually be necessary to assume the polynomial $p(x)$ in the above definition is irreducible so I will begin assuming this.

Definition 8.3.15 *Let $\mathbb{F}$ be a field and let $p(x) \in \mathbb{F}[x]$ be irreducible. This means there is no polynomial which divides $p(x)$ except for itself and constants. For the similarity relation defined in Definition 8.3.13, define the following operations on the equivalence classes. $[a(x)]$ is an equivalence class means that it is the set of all polynomials which are similar to $a(x)$.*

$$\begin{aligned} [a(x)] + [b(x)] &\equiv [a(x) + b(x)] \\ [a(x)]\,[b(x)] &\equiv [a(x)\, b(x)] \end{aligned}$$

This collection of equivalence classes is sometimes denoted by $\mathbb{F}/[p(x)]$.

Proposition 8.3.16 *In the situation of Definition 8.3.15, $p(x)$ and $q(x)$ are relatively prime for any $q(x) \in \mathbb{F}[x]$ which is not a multiple of $p(x)$. Also the definitions of addition and multiplication are well defined. In addition, if $a, b \in \mathbb{F}$ and $[a] = [b]$, then $a = b$.*

Proof: First consider the claim about $p(x), q(x)$ being relatively prime. If $\psi(x)$ is the greatest common divisor, it follows $\psi(x)$ is either equal to $p(x)$ or 1. If it is $p(x)$, then $q(x)$ is a multiple of $p(x)$. If it is 1, then by definition, the two polynomials are relatively prime.

To show the operations are well defined, suppose

$$[a(x)] = [a'(x)], [b(x)] = [b'(x)]$$

It is necessary to show

$$[a(x) + b(x)] = [a'(x) + b'(x)]$$

$$[a(x) b(x)] = [a'(x) b'(x)]$$

Consider the second of the two.

$$\begin{aligned} & a'(x) b'(x) - a(x) b(x) \\ = \; & a'(x) b'(x) - a(x) b'(x) + a(x) b'(x) - a(x) b(x) \\ = \; & b'(x)(a'(x) - a(x)) + a(x)(b'(x) - b(x)) \end{aligned}$$

Now by assumption $(a'(x) - a(x))$ is a multiple of $p(x)$ as is $(b'(x) - b(x))$, so the above is a multiple of $p(x)$ and by definition this shows $[a(x) b(x)] = [a'(x) b'(x)]$. The case for addition is similar.

Now suppose $[a] = [b]$. This means $a - b = k(x) p(x)$ for some polynomial $k(x)$. Then $k(x)$ must equal 0 since otherwise the two polynomials $a - b$ and $k(x) p(x)$ could not be equal because they would have different degree. This proves the proposition.

Note that from this proposition and math induction, if each $a_i \in \mathbb{F}$,

$$\left[a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\right]$$

$$= [a_n][x]^n + [a_{n-1}][x]^{n-1} + \cdots [a_1][x] + [a_0] \tag{8.8}$$

With the above preparation, here is a definition of a field in which the irreducible polynomial $p(x)$ has a root.

Definition 8.3.17 *Let $p(x) \in \mathbb{F}[x]$ be irreducible and let $a(x) \sim b(x)$ when $a(x) - b(x)$ is a multiple of $p(x)$. Let $\mathbb{G}$ denote the set of equivalence classes as described above with the operations also described in Definition 8.3.15.*

Theorem 8.3.18 *The set of all equivalence classes $\mathbb{G} \equiv \mathbb{F}/[p(x)]$ described above with the multiplicative identity given by $[1]$ and the additive identity given by $[0]$ along with the operations of Definition 8.3.15, is a field and $p([x]) = [0]$. (Thus p has a root in this new field.)*

Proof: Everything is obvious except for the existence of the multiplicative inverse and the assertion that $p([x]) = 0$. Suppose then that $[a(x)] \neq [0]$. That is, $a(x)$ is not a multiple of $p(x)$. Why does $[a(x)]^{-1}$ exist? By Theorem 8.3.5, $a(x), p(x)$ are relatively prime and so there exist polynomials $\psi(x), \phi(x)$ such that

$$1 = \psi(x) p(x) + a(x) \phi(x)$$

and so

$$1 - a(x)\phi(x) = \psi(x)p(x)$$

which, by definition implies

$$[1 - a(x)\phi(x)] = [1] - [a(x)\phi(x)] = [1] - [a(x)][\phi(x)] = [0]$$

and so $[\phi(x)] = [a(x)]^{-1}$. This shows $\mathbb{G}$ is a field.

Now if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $p([x]) = 0$ by 8.8 and the definition which says $[p(x)] = [0]$. This proves the theorem.

Usually, people simply write b rather than $[b]$ if $b \in \mathbb{F}$. Then with this convention,

$$[b\phi(x)] = [b][\phi(x)] = b[\phi(x)].$$

This shows how to enlarge a field to get a new one in which the polynomial has a root. By using a succession of such enlargements, called field extensions, there will exist a field in which the given polynomial can be factored into a product of polynomials having degree one. This is called the spliting field.

Theorem 8.3.19 *Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with coefficients in a field of scalars $\mathbb{F}$. There exists a larger field $\mathbb{G}$ such that there exist $\{z_1, \cdots, z_n\}$ listed according to multiplicity such that*

$$p(x) = \prod_{i=1}^{n} (x - z_i)$$

This larger field is called the splitting field.

Proof: From Theorem 8.3.18, there exists a field $\mathbb{F}_1$ such that $p(x)$ has a root, z_1 ($= [x]$ if p is irreducible.) Then by the Euclidean algorithm

$$p(x) = (x - z_1)q_1(x) + r$$

where $r \in \mathbb{F}_1$. Since $p(z_1) = 0$, this requires $r = 0$. Now do the same for $q_1(x)$ that was done for $p(x)$, enlarging the field to $\mathbb{F}_2$ if necessary, such that in this new field

$$q_1(x) = (x - z_2)q_2(x).$$

and so

$$p(x) = (x - z_1)(x - z_2)q_2(x)$$

After n such extensions, you will have obtained the necessary field $\mathbb{G}$.

Example 8.3.20 *The polynomial $x^2 + 1$ is irreducible in $\mathbb{R}(x)$, polynomials having real coefficients. To see this is the case, suppose $\psi(x)$ divides $x^2 + 1$. Then*

$$x^2 + 1 = \psi(x)q(x)$$

If the degree of $\psi(x)$ is less than 2, then it must be either a constant or of the form $ax + b$. In the latter case, $-b/a$ must be a zero of the right side, hence of the left but $x^2 + 1$ has no real zeros. Therefore, the degree of $\psi(x)$ must be two and $q(x)$ must be a constant. Thus the only polynomial which divides $x^2 + 1$ are constants and multiples of $x^2 + 1$. Therefore, this shows $x^2 + 1$ is irreducible. Find the inverse of $\left[x^2 + x + 1\right]$ in the space of equivalence classes, $\mathbb{R}/\left[x^2 + 1\right]$.

You can solve this with partial fractions.

$$\frac{1}{(x^2+1)(x^2+x+1)} = -\frac{x}{x^2+1} + \frac{x+1}{x^2+x+1}$$

and so

$$1 = (-x)\left(x^2+x+1\right) + (x+1)\left(x^2+1\right)$$

which implies

$$1 \sim (-x)\left(x^2+x+1\right)$$

and so the inverse is $[-x]$.

The following proposition is interesting. It gives more information on the things in $\mathbb{F}/[p(x)]$ described above.

Proposition 8.3.21 *Suppose $p(x) \in \mathbb{F}[x]$ is irreducible and has degree n. Then every element of $\mathbb{G} = \mathbb{F}[x]/[p(x)]$ is of the form $[0]$ or $[r(x)]$ where the degree of $r(x)$ is less than n.*

Proof: This follows right away from the Euclidean algorithm for polynomials. If $k(x)$ has degee larger than $n-1$, then

$$k(x) = q(x)\,p(x) + r(x)$$

where $r(x)$ is either equal to 0 or has degree less than n. Hence

$$[k(x)] = [r(x)].$$

This proves the proposition.

Example 8.3.22 *In the situation of the above example, find $[ax+b]^{-1}$ assuming $a^2+b^2 \neq 0$. Note this includes all cases of interest thanks to the above proposition.*

You can do it with partial fractions as above.

$$\frac{1}{(x^2+1)(ax+b)} = \frac{b-ax}{(a^2+b^2)(x^2+1)} + \frac{a^2}{(a^2+b^2)(ax+b)}$$

and so

$$1 = \frac{1}{a^2+b^2}(b-ax)(ax+b) + \frac{a^2}{(a^2+b^2)}\left(x^2+1\right)$$

Thus

$$\frac{1}{a^2+b^2}(b-ax)(ax+b) \sim 1$$

and so

$$[ax+b]^{-1} = \frac{[(b-ax)]}{a^2+b^2} = \frac{b-a[x]}{a^2+b^2}$$

You might find it interesting to recall that $(ai+b)^{-1} = \frac{b-ai}{a^2+b^2}$.

8.3.3 The Algebraic Numbers

Each polynomial having coefficients in a field $\mathbb{F}$ has a splitting field. Here I will consider the case of all polynomials $p(x)$ having coefficients in a field $\mathbb{F} \subseteq \mathbb{G}$ and will look at all roots which are also in $\mathbb{G}$. The theory of vector spaces is very useful in the study of these algebraic numbers.

Definition 8.3.23 *The algebraic numbers $\mathbb{A}$ are those numbers which are in $\mathbb{G}$ and also roots of some polynomial $p(x)$ having coefficients in $\mathbb{F}$. Here is a definition.*

Definition 8.3.24 *Let $a_1, \cdots, a_m$ be in $\mathbb{A}$. A polynomial in $\{a_1, \cdots, a_m\}$ will be an expression of the form*

$$\sum_{k_1 \cdots k_n} a_{k_1 \cdots k_n} a_1^{k_1} \cdots a_n^{k_n}$$

where the $a_{k_1 \cdots k_n}$ are in $\mathbb{F}$, each k_j is a nonnegative integer, and all but finitely many of the $a_{k_1 \cdots k_n}$ equal zero. The collection of such polynomials will be denoted by

$$\mathbb{F}[a_1, \cdots, a_m].$$

Theorem 8.3.25 *Let $a \in \mathbb{A}$. Then there exists a unique monic irreducible polynomial $p(x)$ having coefficients in $\mathbb{F}$ such that $p(a) = 0$. This is called the minimal polynomial for a.*

Proof: By definition, there exists a polynomial $q(x)$ having coefficients in $\mathbb{F}$ such that $q(a) = 0$. If $q(x)$ is irreducible, divide by the leading coefficient and this proves the existence. If $q(x)$ is not irreducible, then there exist nonconstant polynomials $r(x)$ and $k(x)$ such that $q(x) = r(x) k(x)$. Then one of $r(a), k(a)$ equals 0. Pick the one which equals zero and let it play the role of $q(x)$. Continuing this way, in finitely many steps one obtains an irreducible polynomial $p(x)$ such that $p(a) = 0$. Now divide by the leading coefficient and this proves existence. Suppose $p_i, i = 1, 2$ both work and one is not a multiple of the other. Then they must be relatively prime because they are both assumed to be irreducible and so there exist polynomials $l(x), k(x)$ such that

$$1 = l(x) p_1(x) + k(x) p_2(x)$$

But now when a is substituted for x, this yields $0 = 1$, a contradiction. Hence one must be a multiple of the other. However, since they are both monic, this constant must be 1 and so they are the same polynomial. This proves the theorem.

Definition 8.3.26 *For a an algebraic number, let $\deg(a)$ denote the degree of the minimal polynomial of a.*

Now notice that for a an algebraic number, $\mathbb{F}[a]$ is a vector space with field of scalars $\mathbb{F}$. Similarly, for $\{a_1, \cdots, a_m\}$ algebraic numbers, $\mathbb{F}[a_1, \cdots, a_m]$ is a vector space with field of scalars $\mathbb{F}$. The following fundamental proposition is important.

Proposition 8.3.27 *Let $\{a_1, \cdots, a_m\}$ be algebraic numbers. Then*

$$\dim \mathbb{F}[a_1, \cdots, a_m] \leq \prod_{j=1}^{m} \deg(a_j)$$

and for an algebraic number a,

$$\dim \mathbb{F}[a] = \deg(a)$$

Every element of $\mathbb{F}[a_1, \cdots, a_m]$ is in $\mathbb{A}$ and $\mathbb{F}[a_1, \cdots, a_m]$ is a field.

Proof: First consider the second assertion. Let the minimal polynomial of a be

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Since $p(a) = 0$, it follows $\{1, a, a^2, \cdots, a^n\}$ is linearly dependent. However, if the degree of $q(x)$ is less than the degree of $p(x)$, then if $q(x)$ is not a constant, the two must be relatively prime because $p(x)$ is irreducible and so there exist polynomials $k(x), l(x)$ such that

$$1 = l(x)\, q(x) + k(x)\, p(x)$$

and this is a contradiction if $q(a) = 0$ because it would imply upon replacing x with a that $1 = 0$. Therefore, no polynomial having degree less than n can have a as a root. It follows

$$\{1, a, a^2, \cdots, a^{n-1}\}$$

is linearly independent. Thus $\dim \mathbb{F}[a] = \deg(a) = n$. Here is why this is. If $q(a)$ is any element of $\mathbb{F}[a]$,

$$q(x) = p(x)\, k(x) + r(x)$$

where $\deg r(x) < \deg p(x)$ and so $q(a) = r(a)$ and $r(a) \in \operatorname{span}\left(1, a, a^2, \cdots, a^{n-1}\right)$.

Now consider the first claim. By definition, $\mathbb{F}[a_1, \cdots, a_m]$ is obtained from all linear combinations of $\left\{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}\right\}$ where the k_i are nonnegative integers. From the first part, it suffices to consider only $k_j \leq \deg(a_j)$. Therefore, there exists a spanning set for $\mathbb{F}[a_1, \cdots, a_m]$ which has

$$\prod_{i=1}^{m} \deg(a_i)$$

entries. By Theorem 8.2.3 this proves the first claim.

Finally consider the last claim. Let $g(a_1, \cdots, a_m)$ be a polynomial in $\{a_1, \cdots, a_m\}$ in $\mathbb{F}[a_1, \cdots, a_m]$. Since

$$\dim \mathbb{F}[a_1, \cdots, a_m] = p \leq \prod_{j=1}^{m} \deg(a_j) < \infty,$$

it follows

$$1, g(a_1, \cdots, a_m), g(a_1, \cdots, a_m)^2, \cdots, g(a_1, \cdots, a_m)^p$$

are dependent. It follows $g(a_1, \cdots, a_m)$ is the root of some polynomial having coefficients in $\mathbb{F}$. Thus everything in $\mathbb{F}[a_1, \cdots, a_m]$ is algebraic. Why is $\mathbb{F}[a_1, \cdots, a_m]$ a field? Let $g(a_1, \cdots, a_m)$ be as just mentioned. Then it has a minimal polynomial,

$$p(x) = x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$$

where the $a_i \in \mathbb{F}$. Then $a_0 \neq 0$ or else the polynomial would not be minimal. Therefore,

$$g(a_1, \cdots, a_m)\left(g(a_1, \cdots, a_m)^{p-1} + a_{p-1} g(a_1, \cdots, a_m)^{p-2} + \cdots + a_1\right) = -a_0$$

and so the multiplicative inverse for $g(a_1, \cdots, a_m)$ is

$$\frac{g(a_1, \cdots, a_m)^{p-1} + a_{p-1} g(a_1, \cdots, a_m)^{p-2} + \cdots + a_1}{-a_0} \in \mathbb{F}[a_1, \cdots, a_m].$$

The other axioms of a field are obvious. This proves the proposition.

Now from this proposition, it is easy to obtain the following interesting result about the algebraic numbers.

Theorem 8.3.28 *The algebraic numbers $\mathbb{A}$, those roots of polynomials in $\mathbb{F}[x]$ which are in $\mathbb{G}$, are a field.*

Proof: Let a be an algebraic number and let $p(x)$ be its minimal polynomial. Then $p(x)$ is of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0 \neq 0$. Then plugging in a yields

$$a\frac{\left(a^{n-1} + a_{n-1}a^{n-2} + \cdots + a_1\right)(-1)}{a_0} = 1.$$

and so $a^{-1} = \frac{\left(a^{n-1}+a_{n-1}a^{n-2}+\cdots+a_1\right)(-1)}{a_0} \in \mathbb{F}[a]$. By the proposition, every element of $\mathbb{F}[a]$ is in $\mathbb{A}$ and this shows that for every element of $\mathbb{A}$, its inverse is also in $\mathbb{A}$. What about products and sums of things in $\mathbb{A}$? Are they still in $\mathbb{A}$? Yes. If $a, b \in \mathbb{A}$, then both $a+b$ and $ab \in \mathbb{F}[a,b]$ and from the proposition, each element of $\mathbb{F}[a,b]$ is in $\mathbb{A}$. This proves the theorem.

A typical example of what is of interest here is when the field $\mathbb{F}$ of scalars is $\mathbb{Q}$, the rational numbers and the field $\mathbb{G}$ is $\mathbb{R}$. However, you can certainly conceive of many other examples by considering the integers mod a prime, for example (See Problem 42 on Page 224 for example.) or any of the fields which occur as field extensions in the above.

There is a very interesting thing about $\mathbb{F}[a_1 \cdots a_n]$ in the case where $\mathbb{F}$ is infinite which says that there exists a single algebraic γ such that $\mathbb{F}[a_1 \cdots a_n] = \mathbb{F}[\gamma]$. In other words, every field extension of this sort is a simple field extension. I found this fact in an early version of [4].

Proposition 8.3.29 *There exists γ such that $\mathbb{F}[a_1 \cdots a_n] = \mathbb{F}[\gamma]$.*

Proof: To begin with, consider $\mathbb{F}[\alpha, \beta]$. Let $\gamma = \alpha + \lambda\beta$. Then by Proposition 8.3.27 γ is an algebraic number and it is also clear

$$\mathbb{F}[\gamma] \subseteq \mathbb{F}[\alpha, \beta]$$

I need to show the other inclusion. This will be done for a suitable choice of λ. To do this, it suffices to verify that both α and β are in $\mathbb{F}[\gamma]$.

Let the minimal polynomials of α and β be $f(x)$ and $g(x)$ respectively. Let the distinct roots of $f(x)$ and $g(x)$ be $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\{\beta_1, \beta_2, \cdots, \beta_m\}$ respectively. These roots are in a field which contains the splitting fields of both $f(x)$ and $g(x)$. Let $\alpha = \alpha_1$ and $\beta = \beta_1$. Now define

$$h(x) \equiv f(\alpha + \lambda\beta - \lambda x) \equiv f(\gamma - \lambda x)$$

so that $h(\beta) = g(\beta) = 0$. It follows $(x - \beta)$ divides both $h(x)$ and $g(x)$. If $(x - \eta)$ is a different linear factor of both $g(x)$ and $h(x)$ then it must be $\left(x - \beta_j\right)$ for some β_j for some $j > 1$ because these are the only factors of $g(x)$. Therefore, this would require

$$0 = h\left(\beta_j\right) = f\left(\alpha_1 + \lambda\beta_1 - \lambda\beta_j\right)$$

and so it would be the case that $\alpha_1 + \lambda\beta_1 - \lambda\beta_j = \alpha_k$ for some k. Hence

$$\lambda = \frac{\alpha_k - \alpha_1}{\beta_1 - \beta_j}$$

Now there are finitely many quotients of the above form and if λ is chosen to not be any of them, then the above cannot happen and so in this case, the only linear factor of both $g(x)$ and $h(x)$ will be $(x - \beta)$. Choose such a λ.

Let $\phi(x)$ be the minimal polynomial of β with respect to the field $\mathbb{F}[\gamma]$. Then this minimal polynomial must divide both $h(x)$ and $g(x)$ because $h(\beta) = g(\beta) = 0$. However, the only factor these two have in common is $x - \beta$ and so $\phi(x) = x - \beta$ which requires $\beta \in \mathbb{F}[\gamma]$. Now also $\alpha = \gamma - \lambda\beta$ and so $\alpha \in \mathbb{F}[\gamma]$ also. Therefore, both $\alpha, \beta \in \mathbb{F}[\gamma]$ which forces $\mathbb{F}[\alpha, \beta] \subseteq \mathbb{F}[\gamma]$. This proves the proposition in the case that $n = 2$. The general result follows right away by observing that

$$\mathbb{F}[a_1 \cdots a_n] = \mathbb{F}[a_1 \cdots a_{n-1}][a_n]$$

and using induction. This proves the proposition.

8.3.4 The Lindemann Weierstrass Theorem And Vector Spaces

As another application of the abstract concept of vector spaces, there is an amazing theorem due to Weierstrass and Lindemann.

Theorem 8.3.30 *Suppose $a_1, \cdots, a_n$ are algebraic numbers and suppose $\alpha_1, \cdots, \alpha_n$ are distinct algebraic numbers. Then*

$$\sum_{i=1}^{n} a_i e^{\alpha_i} \neq 0$$

In other words, the $\{e^{\alpha_1}, \cdots, e^{\alpha_n}\}$ are independent as vectors with field of scalars equal to the algebraic numbers.

There is a proof of this in the appendix. It is long and hard but only depends on elementary considerations other than some algebra involving symmetric polynomials. See Theorem D.3.5.

A number is transcendental if it is not a root of a polynomial which has integer coefficients. Most numbers are this way but it is hard to verify that specific numbers are transcendental. That π is transcendental follows from

$$e^0 + e^{i\pi} = 0.$$

By the above theorem, this could not happen if π were algebraic because then $i\pi$ would also be algebraic. Recall these algebraic numbers form a field and i is clearly algebraic, being a root of $x^2 + 1$. This fact about π was first proved by Lindemann in 1882 and then the general theorem above was proved by Weierstrass in 1885. This fact that π is transcendental solved an old problem called squaring the circle which was to construct a square with the same area as a circle using a straight edge and compass. It can be shown that the fact π is transcendental implies this problem is impossible.[1]

8.4 Exercises

1. ♠Let H denote span $\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$. Find the dimension of H and determine a basis.

2. ♠Let $M = \{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 = u_1 = 0\}$. Is M a subspace? Explain.

[1] Gilbert, the librettist of the Savoy operas, may have heard about this great achievement. In Princess Ida which opened in 1884 he has the following lines. "As for fashion they forswear it, so the say - so they say; and the circle - they will square it some fine day some fine day." Of course it had been proved impossible to do this a couple of years before.

3. Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 \geq u_1\right\}$. Is M a subspace? Explain.

4. ♠Let $\mathbf{w} \in \mathbb{R}^4$ and let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0\right\}$. Is M a subspace? Explain.

5. Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4\right\}$. Is M a subspace? Explain.

6. Let $\mathbf{w}, \mathbf{w}_1$ be given vectors in $\mathbb{R}^4$ and define

$$M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \text{ and } \mathbf{w}_1 \cdot \mathbf{u} = 0\right\}.$$

   Is M a subspace? Explain.

7. Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : |u_1| \leq 4\right\}$. Is M a subspace? Explain.

8. ♠Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \sin(u_1) = 1\right\}$. Is M a subspace? Explain.

9. Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is a set of vectors from $\mathbb{F}^n$. Show that $\mathbf{0}$ is in $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k)$.

10. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

11. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

12. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

13. Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

14. Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

15. ♠Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

16. ♠Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+3s \\ s-t \\ t+s \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

17. Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+3s+u \\ s-t \\ t+s \\ u \end{pmatrix} : s,t,u \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

18. Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+u+1 \\ t+3u \\ t+s+v \\ u \end{pmatrix} : s,t,u,v \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

19. ♠Let V denote the set of functions defined on $[0,1]$. Vector addition is defined as $(f+g)(x) \equiv f(x)+g(x)$ and scalar multiplication is defined as $(\alpha f)(x) \equiv \alpha(f(x))$. Verify V is a vector space. What is its dimension, finite or infinite?

20. Let V denote the set of polynomial functions defined on $[0,1]$. Vector addition is defined as $(f+g)(x) \equiv f(x)+g(x)$ and scalar multiplication is defined as $(\alpha f)(x) \equiv \alpha(f(x))$. Verify V is a vector space. What is its dimension, finite or infinite?

21. Let V denote the set of continuous functions defined on $\mathbb{R}$. Vector addition is defined as $(f+g)(x) \equiv f(x)+g(x)$ and scalar multiplication is defined as $(\alpha f)(x) \equiv \alpha(f(x))$. Verify V is a vector space. Show the set of polynomials is a subspace of V. That is, this set is closed with respect to the vector space axioms. Therefore, this set of polynomials is itself a vector space.

22. Let V be the set of polynomials defined on $\mathbb{R}$ having degree no more than 4. Give a basis for this vector space.

23. Let the vectors be of the form $a+b\sqrt{2}$ where a,b are rational numbers and let the field of scalars be $\mathbb{F}=\mathbb{Q}$, the rational numbers. Show directly this is a vector space. What is its dimension? What is a basis for this vector space?

24. ♠Let V be a vector space with field of scalars $\mathbb{F}$ and suppose $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ is a basis for V. Now let W also be a vector space with field of scalars $\mathbb{F}$. Let $L:\{\mathbf{v}_1,\cdots,\mathbf{v}_n\} \to W$ be a function such that $L\mathbf{v}_j = \mathbf{w}_j$. Explain how L can be extended to a linear transformation mapping V to W in a unique way.

25. ♠If you have 5 vectors in $\mathbb{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{F}^5$? Explain.

26. If you have 6 vectors in $\mathbb{F}^5$, is it possible they are linearly independent? Explain.

27. Suppose A is an $m \times n$ matrix and $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{F}^n) \subseteq \mathbb{F}^m$. Now suppose $A(\mathbf{z}_i) = \mathbf{w}_i$. Show $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is also independent.

28. ♠Suppose V, W are subspaces of $\mathbb{F}^n$. Show $V \cap W$ defined to be all vectors which are in both V and W is a subspace also.

29. ♠Suppose V and W both have dimension equal to 7 and they are subspaces of a vector space of dimension 10. What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linear independent set can be extended to form a basis.

30. Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linear independent set can be extended to form a basis.

31. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

32. Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

33. Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

34. Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

35. Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix.

36. State whether each of the following sets of data are possible for the matrix equation $A\mathbf{x} = \mathbf{b}$. If possible, describe the solution set. That is, tell whether there exists a unique solution no solution or infinitely many solutions.

    (a) A is a 5×6 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint:** This says $\mathbf{b}$ is in the span of four of the columns. Thus the columns are not independent.

    (b) A is a 3×4 matrix, $\text{rank}(A) = 3$ and $\text{rank}(A|\mathbf{b}) = 2$.

    (c) A is a 4×2 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint:** This says $\mathbf{b}$ is in the span of the columns and the columns must be independent.

    (d) A is a 5×5 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 5$. **Hint:** This says $\mathbf{b}$ is not in the span of the columns.

    (e) A is a 4×2 matrix, $\text{rank}(A) = 2$ and $\text{rank}(A|\mathbf{b}) = 2$.

37. Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that A maps $\mathbb{F}^n$ onto $\mathbb{F}^m$. **Hint:** The vectors $\mathbf{e}_1, \cdots, \mathbf{e}_m$ occur as columns in the row reduced echelon form for A.

38. Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. **Hint:** If not, there exists a vector, $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and this implies at least one column of A is a linear combination of the others. Show this would require the column rank to be less than n.

39. Explain why an $n \times n$ matrix, A is both one to one and onto if and only if its rank is n.

40. Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Hint: Consider the subspace, $B(\mathbb{F}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$. Now suppose $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is a basis for $\ker(B)$. Let $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ be such that $B\mathbf{z}_i = \mathbf{w}_i$ and argue that

$$\ker(AB) \subseteq \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_r, \mathbf{z}_1, \cdots, \mathbf{z}_k).$$

Here is how you do this. Suppose $AB\mathbf{x} = \mathbf{0}$. Then $B\mathbf{x} \in \ker(A) \cap B(\mathbb{F}^p)$ and so $B\mathbf{x} = \sum_{i=1}^{k} B\mathbf{z}_i$ showing that

$$\mathbf{x} - \sum_{i=1}^{k} \mathbf{z}_i \in \ker(B).$$

41. ♠Recall that every positive integer can be factored into a product of primes in a unique way. Show there must be infinitely many primes. **Hint:** Show that if you have any finite set of primes and you multiply them and then add 1, the result cannot be divisible by any of the primes in your finite set. This idea in the hint is due to Euclid who lived about 300 B.C.

42. ♠There are lots of fields. This will give an example of a finite field. Let $\mathbb{Z}$ denote the set of integers. Thus $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. Also let p be a prime number. We will say that two integers, a, b are equivalent and write $a \sim b$ if $a - b$ is divisible by p. Thus they are equivalent if $a - b = px$ for some integer x. First show that $a \sim a$. Next show that if $a \sim b$ then $b \sim a$. Finally show that if $a \sim b$ and $b \sim c$ then $a \sim c$. For a an integer, denote by $[a]$ the set of all integers which is equivalent to a, the equivalence class of a. Show first that is suffices to consider only $[a]$ for $a = 0, 1, 2, \cdots, p-1$ and that for $0 \leq a < b \leq p-1$, $[a] \neq [b]$. That is, $[a] = [r]$ where $r \in \{0, 1, 2, \cdots, p-1\}$. Thus there are exactly p of these equivalence classes. **Hint:** Recall the Euclidean algorithm. For $a > 0$, $a = mp + r$ where $r < p$. Next define the following operations.

$$\begin{aligned} [a] + [b] &\equiv [a+b] \\ [a][b] &\equiv [ab] \end{aligned}$$

Show these operations are well defined. That is, if $[a] = [a']$ and $[b] = [b']$, then $[a] + [b] = [a'] + [b']$ with a similar conclusion holding for multiplication. Thus for addition you need to verify $[a+b] = [a'+b']$ and for multiplication you need to verify $[ab] = [a'b']$. For example, if $p = 5$ you have $[3] = [8]$ and $[2] = [7]$. Is $[2 \times 3] = [8 \times 7]$? Is $[2+3] = [8+7]$? Clearly so in this example because when you subtract, the result is divisible by 5. So why is this so in general? Now verify that $\{[0], [1], \cdots, [p-1]\}$ with these operations is a Field. This is called the integers modulo a prime and is written $\mathbb{Z}_p$. Since there are infinitely many primes p, it follows there are infinitely many of these finite fields. **Hint:** Most of the axioms are easy once you have shown the operations are well defined. The only two which are tricky are the ones which give the existence of the additive inverse and the multiplicative inverse. Of these, the first is not hard. $-[x] = [-x]$. Since p is prime, there exist integers x, y such that $1 = px + ky$ and so $1 - ky = px$ which says $1 \sim ky$ and so $[1] = [ky]$. Now you finish the argument. What is the multiplicative identity in this collection of equivalence classes? Of course you could now consider field extensions based on these fields.

43. Explain why $A\mathbf{x} = \mathbf{0}$ always has a solution even when A^{-1} does not exist.

    (a) What can you conclude about A if the solution is unique?

    (b) What can you conclude about A if the solution is not unique?

44. ♠Suppose V is a vector space with field of scalars $\mathbb{F}$. Let $T \in \mathcal{L}(V, W)$, the space of linear transformations mapping V onto W where W is another vector space. Define an equivalence relation on V as follows. $\mathbf{v} \sim \mathbf{w}$ means $\mathbf{v} - \mathbf{w} \in \ker(T)$. Recall that $\ker(T) \equiv \{\mathbf{v} : T\mathbf{v} = \mathbf{0}\}$. Show this is an equivalence relation. Now for $[\mathbf{v}]$ an equivalence class define $T'[\mathbf{v}] \equiv T\mathbf{v}$. Show this is well defined. Also show that with the operations

$$\begin{aligned} [\mathbf{v}] + [\mathbf{w}] &\equiv [\mathbf{v} + \mathbf{w}] \\ \alpha[\mathbf{v}] &\equiv [\alpha\mathbf{v}] \end{aligned}$$

    this set of equivalence classes, denoted by $V/\ker(T)$ is a vector space. Show next that $T' : V/\ker(T) \to W$ is one to one, linear, and onto. This new vector space, $V/\ker(T)$ is called a quotient space. Show its dimension equals the difference between the dimension of V and the dimension of $\ker(T)$.

45. ♠Let V be an n dimensional vector space and let W be a subspace. Generalize the above problem to define and give properties of V/W. What is its dimension? What is a basis?

46. ♠If $\mathbb{F}$ and $\mathbb{G}$ are two fields and $\mathbb{F} \subseteq \mathbb{G}$, can you consider $\mathbb{G}$ as a vector space with field of scalars $\mathbb{F}$? Explain.

47. ♠Let $\mathbb{A}$ denote the algebraic numbers, those numbers which are roots of polynomials having rational coefficients which are in $\mathbb{R}$. Show $\mathbb{A}$ can be considered a vector space with field of scalars $\mathbb{Q}$. What is the dimension of this vector space, finite or infinite?

48. As mentioned, for distinct algebraic numbers α_i, the complex numbers $\{e^{\alpha_i}\}_{i=1}^n$ are linearly independent over the field of scalars $\mathbb{A}$ where $\mathbb{A}$ denotes the algebraic numbers, those which are roots of a polynomial having integer (rational) coefficients. What is the dimension of the vector space $\mathbb{C}$ with field of scalars $\mathbb{A}$, finite or infinite? If the field of scalars were $\mathbb{C}$ instead of $\mathbb{A}$, would this change? What if the field of scalars were $\mathbb{R}$?

49. ♠Suppose $\mathbb{F}$ is a countable field and let $\mathbb{A}$ be the algebraic numbers, those numbers which are roots of a polynomial having coefficients in $\mathbb{F}$ which are in $\mathbb{G}$, some other field containing $\mathbb{F}$. Show $\mathbb{A}$ is also countable.

50. ♠This problem is on partial fractions. Suppose you have

$$R(x) = \frac{p(x)}{q_1(x) \cdots q_m(x)}$$

    where the polynomials $q_i(x)$ are relatively prime and all the polynomials $p(x)$ and $q_i(x)$ have coefficients in a field of scalars $\mathbb{F}$. Thus there exist polynomials $a_i(x)$ having coefficients in $\mathbb{F}$ such that

$$1 = \sum_{i=1}^{m} a_i(x)\, q_i(x)$$

Explain why

$$R(x) = \frac{p(x)\sum_{i=1}^{m} a_i(x)\, q_i(x)}{q_1(x)\cdots q_m(x)} = \sum_{i=1}^{m} \frac{a_i(x)\, p(x)}{\prod_{j\neq i} q_j(x)}$$

Now continue doing this on each term in the above sum till finally you obtain an expression of the form

$$\sum_{i=1}^{m} \frac{b_i(x)}{q_i(x)}$$

Using the Euclidean algorithm for polynomials, explain why the above is of the form

$$M(x) + \sum_{i=1}^{m} \frac{r_i(x)}{q_i(x)}$$

where the degree of each $r_i(x)$ is less than the degree of $q_i(x)$ and $M(x)$ is a polynomial. Now argue that $M(x) = 0$. From this explain why the usual partial fractions expansion of calculus must be true. You can use the fact that every polynomial having real coefficients factors into a product of irreducible quadratic polynomials and linear polynomials having real coefficients. This follows from the fundamental theorem of algebra in the appendix.

Linear Transformations

9.1 Matrix Multiplication As A Linear Transformation

Definition 9.1.1 *Let V and W be two finite dimensional vector spaces. A function, L which maps V to W is called a linear transformation and written $L \in \mathcal{L}(V,W)$ if for all scalars α and β, and vectors v,w,*

$$L(\alpha v+\beta w) = \alpha L(v) + \beta L(w).$$

An example of a linear transformation is familiar matrix multiplication. Let $A = (a_{ij})$ be an $m \times n$ matrix. Then an example of a linear transformation $L : \mathbb{F}^n \to \mathbb{F}^m$ is given by

$$(L\mathbf{v})_i \equiv \sum_{j=1}^{n} a_{ij} v_j.$$

Here

$$\mathbf{v} \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{F}^n.$$

9.2 $\mathcal{L}(V,W)$ As A Vector Space

Definition 9.2.1 *Given $L, M \in \mathcal{L}(V,W)$ define a new element of $\mathcal{L}(V,W)$, denoted by $L+M$ according to the rule*

$$(L+M)v \equiv Lv + Mv.$$

For α a scalar and $L \in \mathcal{L}(V,W)$, define $\alpha L \in \mathcal{L}(V,W)$ by

$$\alpha L(v) \equiv \alpha (Lv).$$

You should verify that all the axioms of a vector space hold for $\mathcal{L}(V,W)$ with the above definitions of vector addition and scalar multiplication. What about the dimension of $\mathcal{L}(V,W)$?

Before answering this question, here is a useful lemma. It gives a way to define linear transformations and a way to tell when two of them are equal.

Lemma 9.2.2 *Let V and W be vector spaces and suppose $\{v_1, \cdots, v_n\}$ is a basis for V. Then if $L : V \to W$ is given by $Lv_k = w_k \in W$ and*

$$L\left(\sum_{k=1}^{n} a_k v_k\right) \equiv \sum_{k=1}^{n} a_k L v_k = \sum_{k=1}^{n} a_k w_k$$

then L is well defined and is in $\mathcal{L}(V,W)$. Also, if L, M are two linear transformations such that $Lv_k = Mv_k$ for all k, then $M = L$.

Proof: L is well defined on V because, since $\{v_1, \cdots, v_n\}$ is a basis, there is exactly one way to write a given vector of V as a linear combination. Next, observe that L is obviously linear from the definition. If L, M are equal on the basis, then if $\sum_{k=1}^n a_k v_k$ is an arbitrary vector of V,

$$\begin{aligned} L\left(\sum_{k=1}^n a_k v_k\right) &= \sum_{k=1}^n a_k L v_k \\ &= \sum_{k=1}^n a_k M v_k = M\left(\sum_{k=1}^n a_k v_k\right) \end{aligned}$$

and so $L = M$ because they give the same result for every vector in V.

The message is that when you define a linear transformation, it suffices to tell what it does to a basis.

Theorem 9.2.3 *Let V and W be finite dimensional linear spaces of dimension n and m respectively Then* $\dim(\mathcal{L}(V,W)) = mn$.

Proof: Let two sets of bases be

$$\{v_1, \cdots, v_n\} \text{ and } \{w_1, \cdots, w_m\}$$

for V and W respectively. Using Lemma 9.2.2, let $w_i v_j \in \mathcal{L}(V,W)$ be the linear transformation defined on the basis, $\{v_1, \cdots, v_n\}$, by

$$w_i v_k(v_j) \equiv w_i \delta_{jk}$$

where $\delta_{ik} = 1$ if $i = k$ and 0 if $i \neq k$. I will show that $L \in \mathcal{L}(V,W)$ is a linear combination of these special linear transformations called dyadics.

Then let $L \in \mathcal{L}(V,W)$. Since $\{w_1, \cdots, w_m\}$ is a basis, there exist constants, d_{jk} such that

$$Lv_r = \sum_{j=1}^m d_{jr} w_j$$

Now consider the following sum of diadics.

$$\sum_{j=1}^m \sum_{i=1}^n d_{ji} w_j v_i$$

Apply this to v_r. This yields

$$\sum_{j=1}^m \sum_{i=1}^n d_{ji} w_j v_i(v_r) = \sum_{j=1}^m \sum_{i=1}^n d_{ji} w_j \delta_{ir} = \sum_{j=1}^m d_{jr} w_i = Lv_r$$

Therefore, $L = \sum_{j=1}^m \sum_{i=1}^n c_{ji} w_j v_i$ showing the span of the dyadics is all of $\mathcal{L}(V,W)$.

Now consider whether these dyadics form a linearly independent set. Suppose

$$\sum_{i,k} d_{ik} w_i v_k = \mathbf{0}.$$

Are all the scalars d_{ik} equal to 0?

$$\mathbf{0} = \sum_{i,k} d_{ik} w_i v_k (v_l) = \sum_{i=1}^{m} d_{il} w_i$$

and so, since $\{w_1, \cdots, w_m\}$ is a basis, $d_{il} = 0$ for each $i = 1, \cdots, m$. Since l is arbitrary, this shows $d_{il} = 0$ for all i and l. Thus these linear transformations form a basis and this shows that the dimension of $\mathcal{L}(V, W)$ is mn as claimed because there are m choices for the w_i and n choices for the v_j. This proves the theorem.

9.3 The Matrix Of A Linear Transformation

If V is an n dimensional vector space and $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, there exists a linear map

$$q_\beta : \mathbb{F}^n \to V$$

defined as

$$q_\beta (\mathbf{a}) \equiv \sum_{i=1}^{n} a_i v_i$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} a_i \mathbf{e}_i,$$

for $\mathbf{e}_i$ the standard basis vectors for $\mathbb{F}^n$ consisting of

$$\mathbf{e}_i \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

where the one is in the i^{th} slot. It is clear that q defined in this way, is one to one, onto, and linear. For $v \in V$, $q_\beta^{-1}(v)$ is a list of scalars called the components of v with respect to the basis $\{v_1, \cdots, v_n\}$.

Definition 9.3.1 *Given a linear transformation L, mapping V to W, where $\beta \equiv \{v_1, \cdots, v_n\}$ is a basis of V and $\gamma \equiv \{w_1, \cdots, w_m\}$ is a basis for W, an $m \times n$ matrix $A \equiv [L]_{\gamma\beta} \equiv (a_{ij})$ is called the matrix of the transformation L with respect to the given choice of bases for V and W, if whenever $v \in V$, then multiplication of the components of v by $A = (a_{ij})$ yields the components of Lv.*

The following diagram is descriptive of the definition. Here q_β and q_γ are the maps defined above with reference to the bases, $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$ respectively.

$$\begin{array}{ccccc} & & L & & \\ \beta = \{v_1, \cdots, v_n\} & V & \to & W & \{w_1, \cdots, w_m\} = \gamma \\ & q_\beta \uparrow & \circ & \uparrow q_\gamma & \\ & \mathbb{F}^n & \to & \mathbb{F}^m & \\ & & [L]_{\gamma\beta} & & \end{array} \tag{9.1}$$

Now since γ is a basis for W, there exist unique scalars a_{ij} such that

$$Lv_j = \sum_{i=1}^{m} a_{ij} w_i \tag{9.2}$$

Then if $\mathbf{b} \in \mathbb{F}^n$,

$$Lq_\beta \mathbf{b} = L \sum_{j=1}^{n} b_j v_j = \sum_{j=1}^{n} b_j L v_j = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j a_{ij} w_i \tag{9.3}$$

and letting $A = (a_{ij})$,

$$q_\gamma A\mathbf{b} = q_\gamma \sum_{j=1}^{n} a_{ij} b_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} b_j \right) w_i$$

which is the same thing as 9.3. Therefore, $[L]_{\gamma\beta} = A = (a_{ij})$ causes the diagram to commute. Could there be any other choice of $m \times n$ matrix? If A' is another, then you would have

$$q_\gamma A\mathbf{b} = q_\gamma A'\mathbf{b}$$

for all $\mathbf{b}$. Since q_γ is one to one, it follows $A\mathbf{b} = A'\mathbf{b}$ for all $\mathbf{b}$ which can only happen if $A = A'$. Therefore, (a_{ij}) given above, is **the** matrix of the linear transformation L with respect to the two bases.

It may help to write 9.2 in the form

$$\begin{pmatrix} Lv_1 & \cdots & Lv_n \end{pmatrix} = \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix} [L]_{\gamma\beta} \tag{9.4}$$

with the understanding that you do the multiplications in a formal manner just as you would if everything were numbers. If it helps, use it. If it does not help, ignore it.

Example 9.3.2 *Let*

$$V \equiv \{ \textit{ polynomials of degree 3 or less} \},$$

$$W \equiv \{ \textit{ polynomials of degree 2 or less} \},$$

and $L \equiv D$ where D is the differentiation operator. A basis for V is $\beta = \{1, x, x^2, x^3\}$ and a basis for W is $\gamma = \{1, x, x^2\}$.

What is the matrix of this linear transformation with respect to this basis? Using 9.4,

$$\begin{pmatrix} 0 & 1 & 2x & 3x^2 \end{pmatrix} = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} [D]_{\gamma\beta}.$$

It follows from this that the first column of C is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The next three columns of $[D]_{\gamma\beta}$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

and so

$$[D]_{\gamma\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Now consider the important case where $V = \mathbb{F}^n$, $W = \mathbb{F}^m$, and the basis chosen is the standard basis of vectors $\mathbf{e}_i$ described above.

$$\beta = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}, \ \gamma = \{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$$

Let L be a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$ and let A be the matrix of the transformation with respect to these bases. In this case the coordinate maps q_β and q_γ are simply the identity maps on $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively, and can be accomplished by simply multiplying by the appropriate sized identity matrix. The requirement that A is the matrix of the transformation amounts to

$$L\mathbf{b} = A\mathbf{b}$$

What about the situation where different pairs of bases are chosen for V and W? How are the two matrices with respect to these choices related? Consider the following diagram which illustrates the situation.

$$\begin{array}{ccc} \mathbb{F}^n & \overrightarrow{A_2} & \mathbb{F}^m \\ q_{\beta_2}\uparrow & \circ & q_{\gamma_2}\uparrow \\ V & \overrightarrow{L} & W \\ q_{\beta_1}\uparrow & \circ & q_{\gamma_1}\uparrow \\ \mathbb{F}^n & \overrightarrow{A_1} & \mathbb{F}^m \end{array}$$

In this diagram q_{β_i} and q_{γ_i} are coordinate maps as described above. From the diagram,

$$q_{\gamma_1}^{-1} q_{\gamma_2} A_2 q_{\beta_2}^{-1} q_{\beta_1} = A_1,$$

where $q_{\beta_2}^{-1} q_{\beta_1}$ and $q_{\gamma_1}^{-1} q_{\gamma_2}$ are one to one, onto, and linear maps which may be accomplished by multiplication by a square matrix. Thus there exist matrices P, Q such that $P : \mathbb{F}^n \to \mathbb{F}^n$ and $Q : \mathbb{F}^m \to \mathbb{F}^m$ are invertible and

$$PA_2Q = A_1.$$

Definition 9.3.3 *In the special case where $V = W$ and only one basis is used for $V = W$, this becomes*

$$q_{\beta_1}^{-1} q_{\beta_2} A_2 q_{\beta_2}^{-1} q_{\beta_1} = A_1.$$

Letting S be the matrix of the linear transformation $q_{\beta_2}^{-1} q_{\beta_1}$ with respect to the standard basis vectors in $\mathbb{F}^n$,

$$S^{-1} A_2 S = A_1. \tag{9.5}$$

When this occurs, A_1 is said to be similar to A_2 and $A \to S^{-1}AS$ is called a similarity transformation.

Recall the following.

Definition 9.3.4 *Let S be a set. The symbol, $\sim$ is called an equivalence relation on S if it satisfies the following axioms.*

1. *$x \sim x$ for all $x \in S$. (Reflexive)*
2. *If $x \sim y$ then $y \sim x$. (Symmetric)*

3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)

Definition 9.3.5 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

Also recall the notion of equivalence classes.

Theorem 9.3.6 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

Theorem 9.3.7 *In the vector space of $n \times n$ matrices, define*

$$A \sim B$$

if there exists an invertible matrix S such that

$$A = S^{-1}BS.$$

Then $\sim$ is an equivalence relation and $A \sim B$ if and only if whenever V is an n dimensional vector space, there exists $L \in \mathcal{L}(V,V)$ and bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ such that A is the matrix of L with respect to $\{v_1, \cdots, v_n\}$ and B is the matrix of L with respect to $\{w_1, \cdots, w_n\}$.

Proof: $A \sim A$ because $S = I$ works in the definition. If $A \sim B$, then $B \sim A$, because

$$A = S^{-1}BS$$

implies

$$B = SAS^{-1}.$$

If $A \sim B$ and $B \sim C$, then

$$A = S^{-1}BS,\ B = T^{-1}CT$$

and so

$$A = S^{-1}T^{-1}CTS = (TS)^{-1}CTS$$

which implies $A \sim C$. This verifies the first part of the conclusion.

Now let V be an n dimensional vector space, $A \sim B$ so $A = S^{-1}BS$ and pick a basis for V,

$$\beta \equiv \{v_1, \cdots, v_n\}.$$

Define $L \in \mathcal{L}(V,V)$ by

$$Lv_i \equiv \sum_j a_{ji} v_j$$

where $A = (a_{ij})$. Thus A is the matrix of the linear transformation L. Consider the diagram

$$\begin{array}{ccc} \mathbb{F}^n & \underrightarrow{B} & \mathbb{F}^n \\ q_\gamma \downarrow & \circ & q_\gamma \downarrow \\ V & \underrightarrow{L} & V \\ q_\beta \uparrow & \circ & q_\beta \uparrow \\ \mathbb{F}^n & \underrightarrow{A} & \mathbb{F}^n \end{array}$$

where q_γ is chosen to make the diagram commute. Thus we need $S = q_\gamma^{-1} q_\beta$ which requires

$$q_\gamma = q_\beta S^{-1}$$

Then it follows that B is the matrix of L with respect to the basis

$$\{q_\gamma \mathbf{e}_1, \cdots, q_\gamma \mathbf{e}_n\} \equiv \{w_1, \cdots, w_n\}.$$

That is, A and B are matrices of the same linear transformation L. This proves the theorem because the if part of the conclusion was established earlier.

What if the linear transformation consists of multiplication by a matrix A and you want to find the matrix of this linear transformation with respect to another basis? Is there an easy way to do it? The next proposition considers this.

Proposition 9.3.8 *Let A be an $m \times n$ matrix and let L be the linear transformation which is defined by*

$$L\left(\sum_{k=1}^{n} x_k \mathbf{e}_k\right) \equiv \sum_{k=1}^{n} (A\mathbf{e}_k)\, x_k \equiv \sum_{i=1}^{m}\sum_{k=1}^{n} A_{ik} x_k \mathbf{e}_i$$

In simple language, to find $L\mathbf{x}$, you multiply on the left of $\mathbf{x}$ by A. (A is the matrix of L with respect to the standard basis.) Then the matrix M of this linear transformation with respect to the bases $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ for $\mathbb{F}^n$ and $\gamma = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ for $\mathbb{F}^m$ is given by

$$M = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_m \end{pmatrix}^{-1} A \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$$

where $\begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_m \end{pmatrix}$ is the $m \times m$ matrix which has $\mathbf{w}_j$ as its j^{th} column.

Proof: Consider the following diagram.

$$\begin{array}{ccccc} & & L & & \\ \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} & \mathbb{F}^n & \rightarrow & \mathbb{F}^m & \{\mathbf{w}_1, \cdots, \mathbf{w}_m\} \\ & q_\beta \uparrow & \circ & \uparrow q_\gamma & \\ & \mathbb{F}^n & \rightarrow & \mathbb{F}^m & \\ & & M & & \end{array}$$

Here the coordinate maps are defined in the usual way. Thus

$$q_\beta \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T \equiv \sum_{i=1}^{n} x_i \mathbf{u}_i.$$

Therefore, q_β can be considered the same as multiplication of a vector in $\mathbb{F}^n$ on the left by the matrix

$$\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}.$$

Similar considerations apply to q_γ. Thus it is desired to have the following for an arbitrary $\mathbf{x} \in \mathbb{F}^n$.

$$A \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{pmatrix} M\mathbf{x}$$

Therefore, the conclusion of the proposition follows. This proves the proposition.

Definition 9.3.9 *An $n \times n$ matrix, A, is diagonalizable if there exists an invertible $n \times n$ matrix, S such that $S^{-1}AS = D$, where D is a diagonal matrix. Thus D has zero entries everywhere except on the main diagonal. Write* $\operatorname{diag}(\lambda_1 \cdots, \lambda_n)$ *to denote the diagonal matrix having the λ_i down the main diagonal.*

The following theorem is of great significance.

Theorem 9.3.10 *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if $\mathbb{F}^n$ has a basis of eigenvectors of A. In this case, S of Definition 9.3.9 consists of the $n \times n$ matrix whose columns are the eigenvectors of A and $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$.*

Proof: Suppose first that $\mathbb{F}^n$ has a basis of eigenvectors, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ where $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Then let S denote the matrix $(\mathbf{v}_1 \cdots \mathbf{v}_n)$ and let $S^{-1} \equiv \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{pmatrix}$ where $\mathbf{u}_i^T \mathbf{v}_j = \delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$. S^{-1} exists because S has rank n. Then from block multiplication,

$$S^{-1}AS = \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{pmatrix} (A\mathbf{v}_1 \cdots A\mathbf{v}_n)$$

$$= \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{pmatrix} (\lambda_1 \mathbf{v}_1 \cdots \lambda_n \mathbf{v}_n)$$

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = D.$$

Next suppose A is diagonalizable so $S^{-1}AS = D \equiv \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. Then the columns of S form a basis because S^{-1} is given to exist. It only remains to verify that these columns of A are eigenvectors. But letting $S = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$, $AS = SD$ and so $\begin{pmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \mathbf{v}_1 & \cdots & \lambda_n \mathbf{v}_n \end{pmatrix}$ which shows that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. This proves the theorem.

It makes sense to speak of the determinant of a linear transformation as described in the following corollary.

Corollary 9.3.11 *Let $L \in \mathcal{L}(V, V)$ where V is an n dimensional vector space and let A be the matrix of this linear transformation with respect to a basis on V. Then it is possible to define*

$$\det(L) \equiv \det(A).$$

Proof: Each choice of basis for V determines a matrix for L with respect to the basis. If A and B are two such matrices, it follows from Theorem 9.3.7 that

$$A = S^{-1}BS$$

and so

$$\det(A) = \det\left(S^{-1}\right) \det(B) \det(S).$$

But

$$1 = \det(I) = \det\left(S^{-1}S\right) = \det(S) \det\left(S^{-1}\right)$$

and so

$$\det(A) = \det(B)$$

which proves the corollary.

Definition 9.3.12 *Let $A \in \mathcal{L}(X, Y)$ where X and Y are finite dimensional vector spaces. Define* $\operatorname{rank}(A)$ *to equal the dimension of $A(X)$.*

The following theorem explains how the rank of A is related to the rank of the matrix of A.

Theorem 9.3.13 *Let $A \in \mathcal{L}(X,Y)$. Then $\operatorname{rank}(A) = \operatorname{rank}(M)$ where M is the matrix of A taken with respect to a pair of bases for the vector spaces X, and Y.*

Proof: Recall the diagram which describes what is meant by the matrix of A. Here the two bases are as indicated.

$$\begin{array}{ccccc} \beta = \{v_1, \cdots, v_n\} & X & \underrightarrow{A} & Y & \{w_1, \cdots, w_m\} = \gamma \\ & q_\beta \uparrow & \circ & \uparrow q_\gamma & \\ & \mathbb{F}^n & \underrightarrow{M} & \mathbb{F}^m & \end{array}$$

Let $\{Ax_1, \cdots, Ax_r\}$ be a basis for AX. Thus

$$\left\{q_\gamma M q_\beta^{-1} x_1, \cdots, q_\gamma M q_\beta^{-1} x_r\right\}$$

is a basis for AX. Since q_γ is one to one and onto, it follows

$$\left\{M q_X^{-1} x_1, \cdots, M q_X^{-1} x_r\right\}$$

is linearly independent and so $\operatorname{rank}(A) \leq \operatorname{rank}(M)$. However, one could interchange the roles of M and A in the above argument and thereby turn the inequality around. This proves the theorem.

The following result is a summary of many concepts.

Theorem 9.3.14 *Let $L \in \mathcal{L}(V,V)$ where V is a finite dimensional vector space. Then the following are equivalent.*

1. *L is one to one.*
2. *L maps a basis to a basis.*
3. *L is onto.*
4. *$\det(L) \neq 0$*
5. *If $Lv = 0$ then $v = 0$.*

Proof: Suppose first L is one to one and let $\beta = \{v_i\}_{i=1}^n$ be a basis. Then if $\sum_{i=1}^n c_i L v_i = 0$ it follows $L\left(\sum_{i=1}^n c_i v_i\right) = 0$ which means that since $L(0) = 0$, and L is one to one, it must be the case that $\sum_{i=1}^n c_i v_i = 0$. Since $\{v_i\}$ is a basis, each $c_i = 0$ which shows $\{Lv_i\}$ is a linearly independent set. Since there are n of these, it must be that this is a basis.

Now suppose 2.). Then letting $\{v_i\}$ be a basis, and $y \in V$, it follows from part 2.) that there are constants, $\{c_i\}$ such that $y = \sum_{i=1}^n c_i L v_i = L\left(\sum_{i=1}^n c_i v_i\right)$. Thus L is onto. It has been shown that 2.) implies 3.).

Now suppose 3.). Then the operation consisting of multiplication by the matrix of L, $[L]$, must be onto. However, the vectors in $\mathbb{F}^n$ so obtained, consist of linear combinations of the columns of $[L]$. Therefore, the column rank of $[L]$ is n. By Theorem 3.3.20 this equals the determinant rank and so $\det([L]) \equiv \det(L) \neq 0$.

Now assume 4.) If $Lv = 0$ for some $v \neq 0$, it follows that $[L]\mathbf{x} = 0$ for some $\mathbf{x} \neq \mathbf{0}$. Therefore, the columns of $[L]$ are linearly dependent and so by Theorem 3.3.20, $\det([L]) = \det(L) = 0$ contrary to 4.). Therefore, 4.) implies 5.).

Now suppose 5.) and suppose $Lv = Lw$. Then $L(v - w) = 0$ and so by 5.), $v - w = 0$ showing that L is one to one. This proves the theorem.

Also it is important to note that composition of linear transformations corresponds to multiplication of the matrices. Consider the following diagram.

$$\begin{array}{ccccc} X & \underrightarrow{A} & Y & \underrightarrow{B} & Z \\ q_\beta \uparrow & \circ & \uparrow q_\gamma & \circ & \uparrow q_\delta \\ \mathbb{F}^n & \underrightarrow{[A]_{\gamma\beta}} & \mathbb{F}^m & \underrightarrow{[B]_{\delta\gamma}} & \mathbb{F}^p \end{array}$$

where A and B are two linear transformations, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$. Then $B \circ A \in \mathcal{L}(X,Z)$ and so it has a matrix with respect to bases given on X and Z, the coordinate maps for these bases being q_β and q_δ respectively. Then

$$B \circ A = q_\delta [B]_{\delta\gamma} q_\gamma q_\gamma^{-1} [A]_{\gamma\beta} q_\beta^{-1} = q_\delta [B]_{\delta\gamma} [A]_{\gamma\beta} q_\beta^{-1}.$$

But this shows that $[B]_{\delta\gamma}[A]_{\gamma\beta}$ plays the role of $[B \circ A]_{\delta\beta}$, the matrix of $B \circ A$. Hence the matrix of $B \circ A$ equals the product of the two matrices $[A]_{\gamma\beta}$ and $[B]_{\delta\gamma}$. Of course it is interesting to note that although $[B \circ A]_{\delta\beta}$ must be unique, the matrices, $[A]_{\gamma\beta}$ and $[B]_{\delta\gamma}$ are not unique, depending on γ, the basis chosen for Y.

Theorem 9.3.15 *The matrix of the composition of linear transformations equals the product of the the matrices of these linear transformations.*

9.3.1 Some Geometrically Defined Linear Transformations

If T is any linear transformation which maps $\mathbb{F}^n$ to $\mathbb{F}^m$, there is always an $m \times n$ matrix, A with the property that

$$A\mathbf{x} = T\mathbf{x} \tag{9.6}$$

for all $\mathbf{x} \in \mathbb{F}^n$. You simply take the matrix of the linear transformation with respect to the standard basis. What is the form of A? Suppose $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation and you want to find the matrix defined by this linear transformation as described in 9.6. Then if $\mathbf{x} \in \mathbb{F}^n$ it follows

$$\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$$

where $\mathbf{e}_i$ is the vector which has zeros in every slot but the i^{th} and a 1 in this slot. Then since T is linear,

$$\begin{aligned} T\mathbf{x} &= \sum_{i=1}^n x_i T(\mathbf{e}_i) \\ &= \begin{pmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &\equiv A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$

and so you see that the matrix desired is obtained from letting the i^{th} column equal $T(\mathbf{e}_i)$. This proves the following theorem.

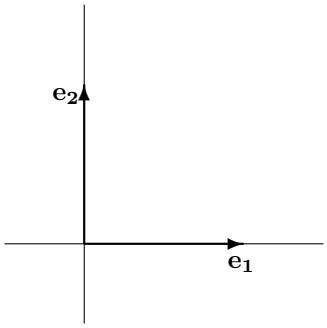
Theorem 9.3.16 *Let T be a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. Then the matrix, A satisfying 9.6 is given by*

$$\begin{pmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{pmatrix}$$

where $T\mathbf{e}_i$ is the i^{th} column of A.

Example 9.3.17 *Determine the matrix for the transformation mapping $\mathbb{R}^2$ to $\mathbb{R}^2$ which consists of rotating every vector counter clockwise through an angle of θ.*

Let $\mathbf{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.



From Theorem 9.3.16, you only need to find $T\mathbf{e}_1$ and $T\mathbf{e}_2$, the first being the first column of the desired matrix, A and the second being the second column. From drawing a picture and doing a little geometry, you see that

$$T\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, T\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Therefore, from Theorem 9.3.16,

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example 9.3.18 *Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ. Thus you want the linear transformation which rotates all angles through an angle of $\theta + \phi$.*

Let $T_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta + \phi$. Then to get $T_{\theta+\phi}$, you could first do T_ϕ and then do T_θ where T_ϕ is the linear transformation which rotates through an angle of ϕ and T_θ is the linear transformation which rotates through an angle of θ. Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ, and A_θ, you must have for every $\mathbf{x}$

$$A_{\theta+\phi}\mathbf{x} = T_{\theta+\phi}\mathbf{x} = T_\theta T_\phi \mathbf{x} = A_\theta A_\phi \mathbf{x}.$$

Consequently, you must have

$$\begin{aligned} A_{\theta+\phi} &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = A_\theta A_\phi \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}.$$

Don't these look familiar? They are the usual trig. identities for the sum of two angles derived here using linear algebra concepts.

Example 9.3.19 *Find the matrix of the linear transformation which rotates vectors in* $\mathbb{R}^3$ *counterclockwise about the positive* z *axis.*

Let T be the name of this linear transformation. In this case, $T\mathbf{e}_3 = \mathbf{e}_3, T\mathbf{e}_1 = (\cos\theta, \sin\theta, 0)^T$, and $T\mathbf{e}_2 = (-\sin\theta, \cos\theta, 0)^T$. Therefore, the matrix of this transformation is just

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9.7}$$

In Physics it is important to consider the work done by a force field on an object. This involves the concept of projection onto a vector. Suppose you want to find the projection of a vector, $\mathbf{v}$ onto the given vector, $\mathbf{u}$, denoted by $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ This is done using the dot product as follows.

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

Because of properties of the dot product, the map $\mathbf{v} \to \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is linear,

$$\begin{aligned} \mathrm{proj}_{\mathbf{u}}(\alpha\mathbf{v}+\beta\mathbf{w}) &= \left(\frac{\alpha\mathbf{v}+\beta\mathbf{w}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \alpha\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} + \beta\left(\frac{\mathbf{w}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\ &= \alpha\,\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) + \beta\,\mathrm{proj}_{\mathbf{u}}(\mathbf{w}). \end{aligned}$$

Example 9.3.20 *Let the projection map be defined above and let* $\mathbf{u} = (1,2,3)^T$. *Find the matrix of this linear transformation with respect to the usual basis.*

You can find this matrix in the same way as in earlier examples. $\mathrm{proj}_{\mathbf{u}}(\mathbf{e}_i)$ gives the i^{th} column of the desired matrix. Therefore, it is only necessary to find

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{e}_i) \equiv \left(\frac{\mathbf{e}_i\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

For the given vector in the example, this implies the columns of the desired matrix are

$$\frac{1}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \frac{2}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \frac{3}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Hence the matrix is

$$\frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

Example 9.3.21 *Find the matrix of the linear transformation which reflects all vectors in* $\mathbb{R}^3$ *through the* xz *plane.*

As illustrated above, you just need to find $T\mathbf{e}_i$ where T is the name of the transformation. But $T\mathbf{e}_1 = \mathbf{e}_1, T\mathbf{e}_3 = \mathbf{e}_3$, and $T\mathbf{e}_2 = -\mathbf{e}_2$ so the matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 9.3.22 *Find the matrix of the linear transformation which first rotates counter clockwise about the positive z axis and then reflects through the xz plane.*

This linear transformation is just the composition of two linear transformations having matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

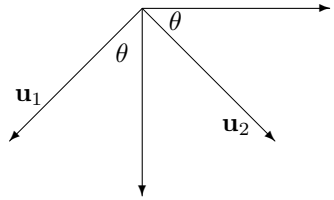
respectively. Thus the matrix desired is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

9.3.2 Rotations About A Given Vector

As an application, I will consider the problem of rotating counter clockwise about a given unit vector which is possibly not one of the unit vectors in coordinate directions. First consider a pair of perpendicular unit vectors, $\mathbf{u}_1$ and $\mathbf{u}_2$ and the problem of rotating in the counterclockwise direction about $\mathbf{u}_3$ where $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$ so that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ forms a right handed orthogonal coordinate system. Thus the vector $\mathbf{u_3}$ is coming out of the page.



Let T denote the desired rotation. Then

$$T(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3) = aT\mathbf{u}_1 + bT\mathbf{u}_2 + cT\mathbf{u}_3$$

$$= (a\cos\theta - b\sin\theta)\,\mathbf{u}_1 + (a\sin\theta + b\cos\theta)\,\mathbf{u}_2 + c\mathbf{u}_3.$$

Thus in terms of the basis $\gamma \equiv \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, the matrix of this transformation is

$$[T]_\gamma \equiv \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

I want to obtain the matrix of the transformation in terms of the usual basis $\beta \equiv \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ because it is in terms of this basis that we usually deal with vectors. From Proposition 9.3.8, if $[T]_\beta$ is this matrix,

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}^{-1} [T]_\beta \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$$

and so you can solve for A if you know the $\mathbf{u}_i$.

Recall why this is so.

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{[T]_\gamma} & \mathbb{R}^3 \\ q_\gamma \downarrow & \circ & q_\gamma \downarrow \\ \mathbb{R}^3 & \xrightarrow{T} & \mathbb{R}^3 \\ I \uparrow & \circ & I \uparrow \\ \mathbb{R}^3 & \xrightarrow{[T]_\beta} & \mathbb{R}^3 \end{array}$$

The map q_γ is accomplished by a multiplication on the left by $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$. Thus

$$[T]_\beta = q_\gamma [T]_\gamma q_\gamma^{-1} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} [T]_\gamma \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}^{-1}.$$

Suppose the unit vector $\mathbf{u}_3$ about which the counterclockwise rotation takes place is (a, b, c). Then I obtain vectors, $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right handed orthonormal system with $\mathbf{u}_3 = (a, b, c)$ and then use the above result. It is of course somewhat arbitrary how this is accomplished. I will assume, however, that $|c| \neq 1$ since otherwise you are looking at either clockwise or counter clockwise rotation about the positive z axis and this is a problem which has been dealt with earlier. (If $c = -1$, it amounts to clockwise rotation about the positive z axis while if $c = 1$, it is counterclockwise rotation about the positive z axis.)

Then let $\mathbf{u}_3 = (a, b, c)$ and $\mathbf{u}_2 \equiv \frac{1}{\sqrt{a^2+b^2}} (b, -a, 0)$. This one is perpendicular to $\mathbf{u}_3$. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is to be a right hand system it is necessary to have

$$\mathbf{u}_1 = \mathbf{u}_2 \times \mathbf{u}_3 = \frac{1}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \left(-ac, -bc, a^2+b^2\right)$$

Now recall that $\mathbf{u}_3$ is a unit vector and so the above equals

$$\frac{1}{\sqrt{(a^2+b^2)}} \left(-ac, -bc, a^2+b^2\right)$$

Then from the above, A is given by

$$\begin{pmatrix} \frac{-ac}{\sqrt{(a^2+b^2)}} & \frac{b}{\sqrt{a^2+b^2}} & a \\ \frac{-bc}{\sqrt{(a^2+b^2)}} & \frac{-a}{\sqrt{a^2+b^2}} & b \\ \sqrt{a^2+b^2} & 0 & c \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-ac}{\sqrt{(a^2+b^2)}} & \frac{b}{\sqrt{a^2+b^2}} & a \\ \frac{-bc}{\sqrt{(a^2+b^2)}} & \frac{-a}{\sqrt{a^2+b^2}} & b \\ \sqrt{a^2+b^2} & 0 & c \end{pmatrix}^{-1}$$

Of course the matrix is an orthogonal matrix so it is easy to take the inverse by simply taking the transpose. Then doing the computation and then some simplification yields

$$= \begin{pmatrix} a^2 + \left(1-a^2\right)\cos\theta & ab(1-\cos\theta) - c\sin\theta & ac(1-\cos\theta) + b\sin\theta \\ ab(1-\cos\theta) + c\sin\theta & b^2 + \left(1-b^2\right)\cos\theta & bc(1-\cos\theta) - a\sin\theta \\ ac(1-\cos\theta) - b\sin\theta & bc(1-\cos\theta) + a\sin\theta & c^2 + \left(1-c^2\right)\cos\theta \end{pmatrix}. \quad (9.8)$$

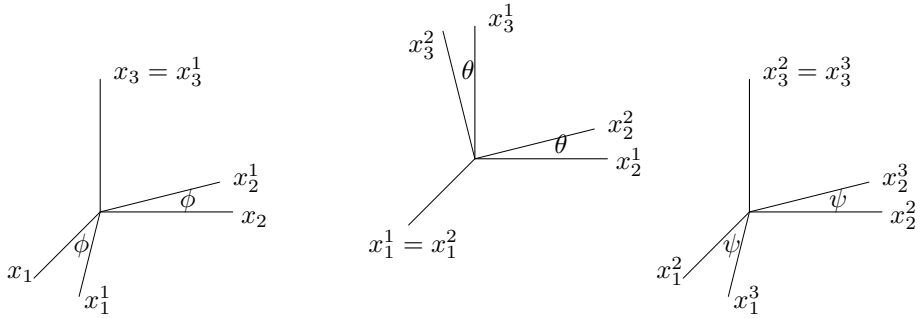
With this, it is clear how to rotate clockwise about the the unit vector, (a,b,c). Just rotate counter clockwise through an angle of $-\theta$. Thus the matrix for this clockwise roation is just

$$= \begin{pmatrix} a^2 + \left(1-a^2\right)\cos\theta & ab\left(1-\cos\theta\right) + c\sin\theta & ac\left(1-\cos\theta\right) - b\sin\theta \\ ab\left(1-\cos\theta\right) - c\sin\theta & b^2 + \left(1-b^2\right)\cos\theta & bc\left(1-\cos\theta\right) + a\sin\theta \\ ac\left(1-\cos\theta\right) + b\sin\theta & bc\left(1-\cos\theta\right) - a\sin\theta & c^2 + \left(1-c^2\right)\cos\theta \end{pmatrix}.$$

In deriving 9.8 it was assumed that $c \neq \pm 1$ but even in this case, it gives the correct answer. Suppose for example that $c = 1$ so you are rotating in the counter clockwise direction about the positive z axis. Then a, b are both equal to zero and 9.8 reduces to 9.7.

9.3.3 The Euler Angles

An important application of the above theory is to the Euler angles, important in the mechanics of rotating bodies. Lagrange studied these things back in the 1700's. To describe the Euler angles consider the following picture in which x_1, x_2 and x_3 are the usual coordinate axes fixed in space and the axes labeled with a superscript denote other coordinate axes. Here is the picture.



We obtain ϕ by rotating counter clockwise about the fixed x_3 axis. Thus this rotation has the matrix

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv M_1(\phi)$$

Next rotate counter clockwise about the x_1^1 axis which results from the first rotation through an angle of θ. Thus it is desired to rotate counter clockwise through an angle θ about the unit vector

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}.$$

Therefore, in 9.8, $a = \cos\phi, b = \sin\phi$, and $c = 0$. It follows the matrix of this transformation with respect to the usual basis is

$$\begin{pmatrix} \cos^2\phi + \sin^2\phi\cos\theta & \cos\phi\sin\phi\left(1-\cos\theta\right) & \sin\phi\sin\theta \\ \cos\phi\sin\phi\left(1-\cos\theta\right) & \sin^2\phi + \cos^2\phi\cos\theta & -\cos\phi\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{pmatrix} \equiv M_2(\phi,\theta)$$

Finally, we rotate counter clockwise about the positive x_3^2 axis by ψ. The vector in the positive x_3^1 axis is the same as the vector in the fixed x_3 axis. Thus the unit vector in the

positive direction of the x_3^2 axis is

$$\begin{pmatrix} \cos^2\phi + \sin^2\phi\cos\theta & \cos\phi\sin\phi\left(1-\cos\theta\right) & \sin\phi\sin\theta \\ \cos\phi\sin\phi\left(1-\cos\theta\right) & \sin^2\phi + \cos^2\phi\cos\theta & -\cos\phi\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi\cos\theta \\ \cos\phi\sin\phi\left(1-\cos\theta\right) \\ -\sin\phi\sin\theta \end{pmatrix} = \begin{pmatrix} \cos^2\phi + \sin^2\phi\cos\theta \\ \cos\phi\sin\phi\left(1-\cos\theta\right) \\ -\sin\phi\sin\theta \end{pmatrix}$$

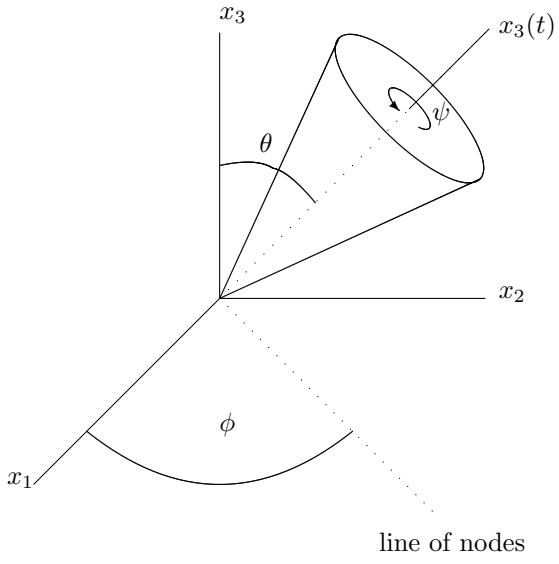
and it is desired to rotate counter clockwise through an angle of ψ about this vector. Thus, in this case,

$$a = \cos^2\phi + \sin^2\phi\cos\theta, b = \cos\phi\sin\phi\left(1-\cos\theta\right), c = -\sin\phi\sin\theta.$$

and you could substitute in to the formula of Theorem 9.8 and obtain a matrix which represents the linear transformation obtained by rotating counter clockwise about the positive x_3^2 axis, $M_3\left(\phi,\theta,\psi\right)$. Then what would be the matrix with respect to the usual basis for the linear transformation which is obtained as a composition of the three just described? By Theorem 9.3.15, this matrix equals the product of these three,

$$M_3\left(\phi,\theta,\psi\right) M_2\left(\phi,\theta\right) M_1\left(\phi\right).$$

I leave the details to you. There are procedures due to Lagrange which will allow you to write differential equations for the Euler angles in a rotating body. To give an idea how these angles apply, consider the following picture.



This is as far as I will go on this topic. The point is, it is possible to give a systematic description in terms of matrix multiplication of a very elaborate geometrical description of a composition of linear transformations. You see from the picture it is possible to describe the motion of the spinning top shown in terms of these Euler angles.

9.4 Eigenvalues And Eigenvectors Of Linear Transformations

Let V be a finite dimensional vector space. For example, it could be a subspace of $\mathbb{C}^n$or $\mathbb{R}^n$. Also suppose $A \in \mathcal{L}\left(V,V\right)$.

Definition 9.4.1 *The characteristic polynomial of A is defined as $q(\lambda) \equiv \det(\lambda I - A)$. The zeros of $q(\lambda)$ in $\mathbb{C}$ are called the eigenvalues of A.*

Lemma 9.4.2 *When λ is an eigenvalue of A which is also in $\mathbb{F}$, the field of scalars, then there exists $v \neq 0$ such that $Av = \lambda v$.*

Proof: This follows from Theorem 9.3.14. Since $\lambda \in \mathbb{F}$,

$$\lambda I - A \in \mathcal{L}(V, V)$$

and since it has zero determinant, it is not one to one.

The following lemma gives the existence of something called the minimal polynomial.

Lemma 9.4.3 *Let $A \in \mathcal{L}(V, V)$ where V is a finite dimensional vector space of dimension n with arbitrary field of scalars. Then there exists a unique polynomial of the form*

$$p(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$$

such that $p(A) = 0$ and m is as small as possible for this to occur.

Proof: Consider the linear transformations, $I, A, A^2, \cdots, A^{n^2}$. There are $n^2 + 1$ of these transformations and so by Theorem 9.2.3 the set is linearly dependent. Thus there exist constants, $c_i \in \mathbb{F}$ such that

$$c_0 I + \sum_{k=1}^{n^2} c_k A^k = 0.$$

This implies there exists a polynomial, $q(\lambda)$ which has the property that $q(A) = 0$. In fact, one example is $q(\lambda) \equiv c_0 + \sum_{k=1}^{n^2} c_k \lambda^k$. Dividing by the leading term, it can be assumed this polynomial is of the form $\lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$, a monic polynomial. Now consider all such monic polynomials, q such that $q(A) = 0$ and pick the one which has the smallest degree m. This is called the minimal polynomial and will be denoted here by $p(\lambda)$. If there were two minimal polynomials, the one just found and another,

$$\lambda^m + d_{m-1}\lambda^{m-1} + \cdots + d_1\lambda + d_0.$$

Then subtracting these would give the following polynomial,

$$q'(\lambda) = (d_{m-1} - c_{m-1})\lambda^{m-1} + \cdots + (d_1 - c_1)\lambda + d_0 - c_0$$

Since $q'(A) = 0$, this requires each $d_k = c_k$ since otherwise you could divide by $d_k - c_k$ where k is the largest one which is nonzero. Thus the choice of m would be contradicted. This proves the lemma.

Theorem 9.4.4 *Let V be a nonzero finite dimensional vector space of dimension n with the field of scalars equal to $\mathbb{F}$. Suppose $A \in \mathcal{L}(V, V)$ and for $p(\lambda)$ the minimal polynomial defined above, let $\mu \in \mathbb{F}$ be a zero of this polynomial. Then there exists $v \neq 0$,$v \in V$ such that*

$$Av = \mu v.$$

If $\mathbb{F} = \mathbb{C}$, then A always has an eigenvector and eigenvalue. Furthermore, if $\{\lambda_1, \cdots, \lambda_m\}$ are the zeros of $p(\lambda)$ in $\mathbb{F}$, these are exactly the eigenvalues of A for which there exists an eigenvector in V.

Proof: Suppose first μ is a zero of $p(\lambda)$. Since $p(\mu) = 0$, it follows

$$p(\lambda) = (\lambda - \mu) k(\lambda)$$

where $k(\lambda)$ is a polynomial having coefficients in $\mathbb{F}$. Since p has minimal degree, $k(A) \neq 0$ and so there exists a vector, $u \neq 0$ such that $k(A) u \equiv v \neq 0$. But then

$$(A - \mu I) v = (A - \mu I) k(A)(u) = \mathbf{0}.$$

The next claim about the existence of an eigenvalue follows from the fundamental theorem of algebra and what was just shown.

It has been shown that every zero of $p(\lambda)$ is an eigenvalue which has an eigenvector in V. Now suppose μ is an eigenvalue which has an eigenvector in V so that $Av = \mu v$ for some $v \in V, v \neq 0$. Does it follow μ is a zero of $p(\lambda)$?

$$\mathbf{0} = p(A) v = p(\mu) v$$

and so μ is indeed a zero of $p(\lambda)$. This proves the theorem.

In summary, the theorem says that the eigenvalues which have eigenvectors in V are exactly the zeros of the minimal polynomial which are in the field of scalars, $\mathbb{F}$.

9.5 Exercises

1. If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible?
2. ♠Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by doing the computation $ABC\left(C^{-1}B^{-1}A^{-1}\right)$.
3. ♠If A is invertible, show $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
4. If A is invertible, show $\left(A^2\right)^{-1} = \left(A^{-1}\right)^2$.
5. If A is invertible, show $\left(A^{-1}\right)^{-1} = A$.
6. ♠Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.
7. Explain why $A\mathbf{x} = \mathbf{0}$ always has a solution whenever A is a linear transformation.
8. ♠Review problem: Suppose $\det(A - \lambda I) = 0$. Show using Theorem 3.1.17 there exists $\mathbf{x} \neq \mathbf{0}$ such that $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
9. Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one. **Hint:** Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n - m) \times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$.

10. ♠How does the minimal polynomial of an algebraic number relate to the minimal polynomial of a linear transformation? Can an algebraic number be thought of as a linear transformation? How?

11. ♠Recall the fact from algebra that if $p(\lambda)$ and $q(\lambda)$ are polynomials, then there exists $l(\lambda)$, a polynomial such that

$$q(\lambda) = p(\lambda) l(\lambda) + r(\lambda)$$

where the degree of $r(\lambda)$ is less than the degree of $p(\lambda)$ or else $r(\lambda) = 0$. With this in mind, why must the minimal polynomial always divide the characteristic polynomial? That is why does there always exist a polynomial $l(\lambda)$ such that $p(\lambda) l(\lambda) = q(\lambda)$? Can you give conditions which imply the minimal polynomial equals the characteristic polynomial? Go ahead and use the Cayley Hamilton theorem.

12. ♠In the following examples, a linear transformation, T is given by specifying its action on a basis β. Find its matrix with respect to this basis.

(a) $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}, T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(b) $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}, T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

13. ♠Let $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for $\mathbb{F}^n$ and let $T : \mathbb{F}^n \to \mathbb{F}^n$ be defined as follows.

$$T\left(\sum_{k=1}^{n} a_k \mathbf{u}_k\right) = \sum_{k=1}^{n} a_k b_k \mathbf{u}_k$$

First show that T is a linear transformation. Next show that the matrix of T with respect to this basis is $[T]_\beta =$

$$\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$$

Show that the above definition is equivalent to simply specifying T on the basis vectors of β by

$$T(\mathbf{u}_k) = b_k \mathbf{u}_k.$$

14. ♠ ↑In the situation of the above problem, let $\gamma = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$ where $\mathbf{e}_k$ is the vector which has 1 in the k^{th} entry and zeros elsewhere. Show that $[T]_\gamma =$

$$\begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} [T]_\beta \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}^{-1} \tag{9.9}$$

15. ♠ ↑Generalize the above problem to the situation where T is given by specifying its action on the vectors of a basis $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ as follows.

$$T\mathbf{u}_k = \sum_{j=1}^{n} a_{jk} \mathbf{u}_j.$$

Letting $A = (a_{ij})$, verify that for $\gamma = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, 9.9 still holds and that $[T]_\beta = A$.

16. ♠Let P_3 denote the set of real polynomials of degree no more than 3, defined on an interval $[a,b]$. Show that P_3 is a subspace of the vector space of all functions defined on this interval. Show that a basis for P_3 is $\left\{1,x,x^2,x^3\right\}$. Now let D denote the differentiation operator which sends a function to its derivative. Show D is a linear transformation which sends P_3 to P_3. Find the matrix of this linear transformation with respect to the given basis.

17. ♠Generalize the above problem to P_n, the space of polynomials of degree no more than n with basis $\{1,x,\cdots,x^n\}$.

18. ♠In the situation of the above problem, let the linear transformation be $T=D^2+1$, defined as $Tf=f''+f$. Find the matrix of this linear transformation with respect to the given basis $\{1,x,\cdots,x^n\}$.

19. ♠In calculus, the following situation is encountered. There exists a vector valued function $\mathbf{f}:U\to\mathbb{R}^m$ where U is an open subset of $\mathbb{R}^n$. Such a function is said to have a derivative or to be differentiable at $\mathbf{x}\in U$ if there exists a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\lim_{\mathbf{v}\to\mathbf{0}}\frac{|\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})-T\mathbf{v}|}{|\mathbf{v}|}=0.$$

First show that this linear transformation, if it exists, must be unique. Next show that for $\beta=\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$, the k^{th} column of $[T]_\beta$ is

$$\frac{\partial\mathbf{f}}{\partial x_k}(\mathbf{x}).$$

Actually, the result of this problem is a well kept secret. People typically don't see this in calculus. It is seen for the first time in advanced calculus if then.

20. ♠Recall that A is similar to B if there exists a matrix P such that $A=P^{-1}BP$. Show that if A and B are similar, then they have the same determinant. Give an example of two matrices which are not similar but have the same determinant.

21. ♠Suppose $A\in\mathcal{L}(V,W)$ where $\dim(V)>\dim(W)$. Show $\ker(A)\neq\{\mathbf{0}\}$. That is, show there exist nonzero vectors $\mathbf{v}\in V$ such that $A\mathbf{v}=\mathbf{0}$.

22. ♠ A vector $\mathbf{v}$ is in the convex hull of a nonempty set S if there are finitely many vectors of $S,\{\mathbf{v}_1,\cdots,\mathbf{v}_m\}$ and nonnegative scalars $\{t_1,\cdots,t_m\}$ such that

$$\mathbf{v}=\sum_{k=1}^m t_k\mathbf{v}_k,\ \sum_{k=1}^m t_k=1.$$

Such a linear combination is called a convex combination. Suppose now that $S\subseteq V$, a vector space of dimension n. Show that if $\mathbf{v}=\sum_{k=1}^m t_k\mathbf{v}_k$ is a vector in the convex hull for $m>n+1$, then there exist other scalars $\{t_k'\}$ such that

$$\mathbf{v}=\sum_{k=1}^{m-1}t_k'\mathbf{v}_k.$$

Thus every vector in the convex hull of S can be obtained as a convex combination of at most $n+1$ points of S. This incredible result is in Rudin [17]. **Hint:** Consider $L:\mathbb{R}^m\to V\times\mathbb{R}$ defined by

$$L(\mathbf{a})\equiv\left(\sum_{k=1}^m a_k\mathbf{v}_k,\sum_{k=1}^m a_k\right)$$

Explain why $\ker(L) \neq \{\mathbf{0}\}$. You might use Problem 9 to get this far. Next letting $\mathbf{a} \in \ker(L) \setminus \{\mathbf{0}\}$ and $\lambda \in \mathbb{R}$, note that $\lambda\mathbf{a} \in \ker(L)$. Thus for all $\lambda \in \mathbb{R}$,

$$\mathbf{v} = \sum_{k=1}^{m} (t_k + \lambda a_k)\,\mathbf{v}_k.$$

Now vary λ till some $t_k + \lambda a_k = 0$ for some $a_k \neq 0$.

23. For those who know about compactness, use Problem 22 to show that if $S \subseteq \mathbb{R}^n$ and S is compact, then so is its convex hull.

24. ♠Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution.

25. Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace.

26. Verify the linear transformation determined by the matrix of 2.27 maps $\mathbb{R}^3$ onto $\mathbb{R}^2$ but the linear transformation determined by this matrix is not one to one.

27. People like to consider the solutions of first order linear systems of equations which are of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where here A is an $n \times n$ matrix. From Schur's theorem, there exist S and S^{-1} such that $A = SJS^{-1}$ where J is an upper triangular matrix. Define $\mathbf{y}(t) \equiv S^{-1}\mathbf{x}(t)$. Show $\mathbf{y}' = J\mathbf{y}$. Now suppose $\Psi(t)$ is an $n \times n$ matrix whose columns are solutions of the above differential equation. Thus

$$\Psi' = A\Psi$$

Now let Φ be defined by $S\Phi S^{-1} = \Psi$. Show

$$\Phi' = J\Phi.$$

28. ♠Suppose V is an n dimensional vector space, $n \geq 1$, over a field of scalars $\mathbb{F}$ and let $A \in \mathcal{L}(V, V)$. Show that there exists a unique monic polynomial $p_A(\lambda)$ of smallest possible degree such that $p_A(A) = 0$. This is called the minimal polynomial of A. Show that if $q(\lambda)$ is any polynomial for which $q(A) = 0$, it follows $p_A(\lambda)$ must divide $q(\lambda)$. Also show that if the degree of $p_A(\lambda)$ is m, then $\left\{I, A, \cdots, A^{m-1}\right\}$ is linearly independent in $\mathcal{L}(V, V)$.

29. ♠ ↑ In the situation of the above problem, let A be an $n \times n$ matrix of elements of $\mathbb{F}$. There are two cases. In the first case, $\mathbb{F}$ contains the splitting field of $p_A(\lambda)$ so that $p(\lambda)$ factors into a product of linear polynomials having coefficients in $\mathbb{F}$. It is the second case which of interest here where $p_A(\lambda)$ does not factor into linear factors having coefficients in $\mathbb{F}$. Let $\mathbb{G}$ be the splitting field of $p_A(\lambda)$ and let $q_A(\lambda)$ be the minimal polynomial of A with respect to the field $\mathbb{G}$. Explain why $q_A(\lambda)$ must divide $p_A(\lambda)$. Now why must $q_A(\lambda)$ factor completely into linear factors?

Linear Transformations Canonical Forms

10.1 A Theorem Of Sylvester, Direct Sums

The notation is defined as follows.

Definition 10.1.1 *Let* $L \in \mathcal{L}(V,W)$. *Then* $\ker(L) \equiv \{v \in V : Lv = 0\}$.

Lemma 10.1.2 *Whenever* $L \in \mathcal{L}(V,W)$, $\ker(L)$ *is a subspace.*

Proof: If a, b are scalars and v,w are in $\ker(L)$, then

$$L(av + bw) = aL(v) + bL(w) = 0 + 0 = 0$$

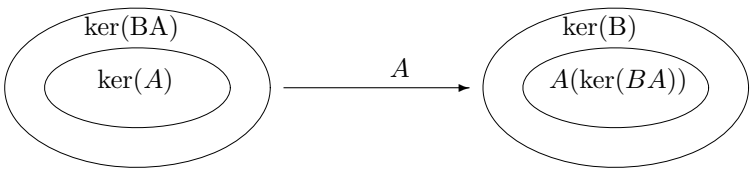
This proves the lemma.

Suppose now that $A \in \mathcal{L}(V,W)$ and $B \in \mathcal{L}(W,U)$ where V, W, U are all finite dimensional vector spaces. Then it is interesting to consider $\ker(BA)$. The following theorem of Sylvester is a very useful and important result.

Theorem 10.1.3 *Let* $A \in \mathcal{L}(V,W)$ *and* $B \in \mathcal{L}(W,U)$ *where* V, W, U *are all vector spaces over a field* $\mathbb{F}$. *Suppose also that* $\ker(A)$ *and* $A(\ker(BA))$ *are finite dimensional subspaces. Then*

$$\dim(\ker(BA)) \leq \dim(\ker(B)) + \dim(\ker(A)).$$

Proof: If $\mathbf{x} \in \ker(BA)$, then $A\mathbf{x} \in \ker(B)$ and so $A(\ker(BA)) \subseteq \ker(B)$. The following picture may help.



Now let $\{x_1, \cdots, x_n\}$ be a basis of $\ker(A)$ and let $\{Ay_1, \cdots, Ay_m\}$ be a basis for $A(\ker(BA))$. Take any $z \in \ker(BA)$. Then $Az = \sum_{i=1}^{m} a_i Ay_i$ and so

$$A\left(z - \sum_{i=1}^{m} a_i y_i\right) = \mathbf{0}$$

which means $z - \sum_{i=1}^{m} a_i y_i \in \ker(A)$ and so there are scalars b_i such that

$$z - \sum_{i=1}^{m} a_i y_i = \sum_{j=1}^{n} b_i x_i.$$

It follows $\text{span}(x_1, \cdots, x_n, y_1, \cdots, y_m) \supseteq \ker(BA)$ and so by the first part, (See the picture.)

$$\dim(\ker(BA)) \leq n + m \leq \dim(\ker(A)) + \dim(\ker(B))$$

This proves the theorem.

Of course this result holds for any finite product of commuting linear transformations by induction. One way this is quite useful is in the case where you have a finite product of linear transformations $\prod_{i=1}^{l} L_i$ all in $\mathcal{L}(V, V)$. Then

$$\dim\left(\ker \prod_{i=1}^{l} L_i\right) \leq \sum_{i=1}^{l} \dim(\ker L_i)$$

and so if you can find a linearly independent set of vectors in $\ker\left(\prod_{i=1}^{l} L_i\right)$ of size

$$\sum_{i=1}^{l} \dim(\ker L_i),$$

then it must be a basis for $\ker\left(\prod_{i=1}^{l} L_i\right)$. This is discussed below.

Definition 10.1.4 *Let $\{V_i\}_{i=1}^{r}$ be subspaces of V. Then*

$$\sum_{i=1}^{r} V_i$$

denotes all sums of the form $\sum_{i=1}^{r} v_i$ where $v_i \in V_i$. If whenever

$$\sum_{i=1}^{r} v_i = 0, v_i \in V_i, \tag{10.1}$$

it follows that $v_i = 0$ for each i, then a special notation is used to denote $\sum_{i=1}^{r} V_i$. This notation is

$$V_1 \oplus \cdots \oplus V_r,$$

and it is called a direct sum of subspaces.

Lemma 10.1.5 *If $V = V_1 \oplus \cdots \oplus V_r$ and if $\beta_i = \left\{v_1^i, \cdots, v_{m_i}^i\right\}$ is a basis for V_i, then a basis for V is $\{\beta_1, \cdots, \beta_r\}$.*

Proof: Suppose $\sum_{i=1}^{r} \sum_{j=1}^{m_i} c_{ij} v_j^i = 0$. then since it is a direct sum, it follows for each i,

$$\sum_{j=1}^{m_i} c_{ij} v_j^i = 0$$

and now since $\left\{v_1^i, \cdots, v_{m_i}^i\right\}$ is a basis, each $c_{ij} = 0$. This proves the lemma.

Here is a useful lemma.

Lemma 10.1.6 *Let L_i be in $\mathcal{L}(V,V)$ and suppose for $i \neq j, L_iL_j = L_jL_i$ and also L_i is one to one on $\ker(L_j)$ whenever $i \neq j$. Then*

$$\ker\left(\prod_{i=1}^{p} L_i\right) = \ker(L_1) \oplus + \cdots + \oplus \ker(L_p)$$

Here $\prod_{i=1}^{p} L_i$ is the product of all the linear transformations. A symbol like $\prod_{j\neq i} L_j$ is the product of all of them but L_i.

Proof: Note that since the operators commute, $L_j : \ker(L_i) \to \ker(L_i)$. Here is why. If $L_iy = 0$ so that $y \in \ker(L_i)$, then

$$L_iL_jy = L_jL_iy = L_j0 = 0$$

and so $L_j : \ker(L_i) \to \ker(L_i)$. Suppose

$$\sum_{i=1}^{p} v_i = 0,\ v_i \in \ker(L_i),$$

but some $v_i \neq 0$. Then do $\prod_{j\neq i} L_j$ to both sides. Since the linear transformations commute, this results in

$$\prod_{j\neq i} L_jv_i = 0$$

which contradicts the assumption that these L_j are one to one and the observation that they map $\ker(L_i)$ to $\ker(L_i)$. Thus if

$$\sum_{i} v_i = 0, v_i \in \ker(L_i)$$

then each $v_i = 0$.

Suppose $\beta_i = \left\{v_1^i, \cdots, v_{m_i}^i\right\}$ is a basis for $\ker(L_i)$. Then from what was just shown and Lemma 10.1.5, $\left\{\beta_1, \cdots, \beta_p\right\}$ must be linearly independent and a basis for

$$\ker(L_1) \oplus + \cdots + \oplus \ker(L_p).$$

It is also clear that since these operators commute,

$$\ker(L_1) \oplus + \cdots + \oplus \ker(L_p) \subseteq \ker\left(\prod_{i=1}^{p} L_i\right)$$

Therefore, by Sylvester's theorem and the above,

$$\begin{aligned}
\dim\left(\ker\left(\prod_{i=1}^{p} L_i\right)\right) &\leq \sum_{j=1}^{p} \dim(\ker(L_j)) \\
&= \dim(\ker(L_1) \oplus + \cdots + \oplus \ker(L_p)) \\
&\leq \dim\left(\ker\left(\prod_{i=1}^{p} L_i\right)\right).
\end{aligned}$$

Now in general, if W is a subspace of V, a finite dimensional vector space and the two have the same dimension, then $W = V$. This is because W has a basis and if v is not in the span of this basis, then v adjoined to the basis of W would be a linearly independent set, yielding a linearly independent set which has more vectors in it than a basis, a contradiction.

It follows

$$\ker(L_1) \oplus + \cdots + \oplus \ker(L_p) = \ker\left(\prod_{i=1}^{p} L_i\right)$$

and this proves the lemma.

10.2 Direct Sums, Block Diagonal Matrices

Let V be a finite dimensional vector space with field of scalars $\mathbb{F}$. Here I will make no assumption on $\mathbb{F}$. Also suppose $A \in \mathcal{L}(V,V)$.

The following lemma gives the existence of the minimal polynomial for a linear transformation A. This is the monic polynomial p which has smallest possible degree such that $p(A) = 0$

Lemma 10.2.1 *Let $A \in \mathcal{L}(V,V)$ where V is a finite dimensional vector space of dimension n with field of scalars $\mathbb{F}$. Then there exists a unique monic polynomial of the form*

$$p(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$$

such that $p(A) = 0$ and m is as small as possible for this to occur.

Proof: Consider the linear transformations, $I, A, A^2, \cdots, A^{n^2}$. There are n^2+1 of these transformations and so by Theorem 9.2.3 the set is linearly dependent. Thus there exist constants, $c_i \in \mathbb{F}$ not all zero such that

$$c_0 I + \sum_{k=1}^{n^2} c_k A^k = 0.$$

This implies there exists a polynomial, $q(\lambda)$ which has the property that $q(A) = 0$. In fact, one example is the above $q(\lambda) \equiv c_0 + \sum_{k=1}^{n^2} c_k\lambda^k$. Dividing by the leading term, it can be assumed this polynomial is of the form $\lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$, a monic polynomial. Now consider all such monic polynomials, q such that $q(A) = 0$ and pick the one which has the smallest degree, m. This is called the minimal polynomial and will be denoted here by $p(\lambda)$. If there were two minimal polynomials, the one just found and another,

$$\lambda^m + d_{m-1}\lambda^{m-1} + \cdots + d_1\lambda + d_0.$$

Then subtracting these would give the following polynomial,

$$q'(\lambda) = (d_{m-1} - c_{m-1})\lambda^{m-1} + \cdots + (d_1 - c_1)\lambda + d_0 - c_0$$

Since $q'(A) = 0$, this requires each $d_k = c_k$ since otherwise you could divide by $d_k - c_k$ where k is the largest subscript which is nonzero. Thus the choice of m would be contradicted. This proves the lemma.

Now here is a useful lemma which will be used below.

Lemma 10.2.2 *Let $L \in \mathcal{L}(V,V)$ where V is an n dimensional vector space. Then if L is one to one, it follows that L is also onto. In fact, if $\{v_1, \cdots, v_n\}$ is a basis, then so is $\{Lv_1, \cdots, Lv_n\}$.*

Proof: Let $\{v_1, \cdots, v_n\}$ be a basis for V. Then I claim that $\{Lv_1, \cdots, Lv_n\}$ is also a basis for V. First of all, I show $\{Lv_1, \cdots, Lv_n\}$ is linearly independent. Suppose

$$\sum_{k=1}^{n} c_k Lv_k = 0.$$

Then

$$L\left(\sum_{k=1}^{n} c_k v_k\right) = 0$$

and since L is one to one, it follows

$$\sum_{k=1}^{n} c_k v_k = 0$$

which implies each $c_k = 0$. Therefore, $\{Lv_1, \cdots, Lv_n\}$ is linearly independent. If there exists w not in the span of these vectors, then by Lemma 8.2.6, $\{Lv_1, \cdots, Lv_n, w\}$ would be independent and this contradicts the exchange theorem, Theorem 8.2.3 because it would be a linearly independent set having more vectors than the spanning set $\{v_1, \cdots, v_n\}$. This proves the lemma.

Now it is time to consider the notion of a direct sum of subspaces. Recall you can always assert the existence of a factorizaton of the minimal polynomial into a product of irreducible polynomials. This fact will now be used to show how to obtain such a direct sum of subspaces.

Definition 10.2.3 *For $A \in \mathcal{L}(V,V)$ where $\dim(V) = n$, suppose the minimal polynomial is*

$$p(\lambda) = \prod_{k=1}^{q} \phi_k(\lambda)^{r_k}$$

where the polynomials ϕ_k have coefficients in $\mathbb{F}$ and are irreducible. Now define the generalized eigenspaces

$$V_k \equiv \ker\left(\phi_k(A)^{r_k}\right)$$

Theorem 10.2.4 *In the context of Definition 10.2.3,*

$$V = V_1 \oplus \cdots \oplus V_q \tag{10.2}$$

and each V_k is A invariant, meaning $A(V_k) \subseteq V_k$. $\phi_l(A)$ is one to one on each V_k for $k \neq l$. If $\beta_i = \left\{v_1^i, \cdots, v_{m_i}^i\right\}$ is a basis for V_i, then $\left\{\beta_1, \beta_2, \cdots, \beta_q\right\}$ is a basis for V.

Proof: It is clear V_k is a subspace which is A invariant because A commutes with $\phi_k(A)^{m_k}$. It is clear the operators $\phi_k(A)^{r_k}$ commute. Thus if $v \in V_k$,

$$\phi_k(A)^{r_k}\phi_l(A)^{r_l} v = \phi_l(A)^{r_l}\phi_k(A)^{r_k} v = \phi_l(A)^{r_l} 0 = 0$$

and so $\phi_l(A)^{r_l} : V_k \to V_k$.

I claim $\phi_l(A)$ is one to one on V_k whenever $k \neq l$. The two polynomials $\phi_l(\lambda)$ and $\phi_k(\lambda)^{r_k}$ are relatively prime so there exist polynomials $m(\lambda), n(\lambda)$ such that

$$m(\lambda)\phi_l(\lambda) + n(\lambda)\phi_k(\lambda)^{r_k} = 1$$

where this holds for all $\lambda \in \mathbb{F}$. It follows that the sum of all coefficients of λ raised to a positive power are zero and the constant term on the left is 1. Therefore, using the convention $A^0 = I$ it follows

$$m(A)\phi_l(A) + n(A)\phi_k(A)^{r_k} = I$$

If $v \in V_k$, then from the above,

$$m(A)\phi_l(A) v + n(A)\phi_k(A)^{r_k} v = v$$

Since v is in V_k, it follows by definition,

$$m(A)\phi_l(A) v = v$$

and so $\phi_l(A)v \neq 0$ unless $v = 0$. Thus $\phi_l(A)$ and hence $\phi_l(A)^{r_l}$ is one to one on V_k for every $k \neq l$. By Lemma 10.1.6 and the fact that $\ker\left(\prod_{k=1}^{q} \phi_k(\lambda)^{r_k}\right) = V$, 10.2 is obtained. The claim about the bases follows from Lemma 10.1.5. This proves the theorem.

How does this relate to matrices?

Theorem 10.2.5 *Suppose V is a vector space with field of scalars $\mathbb{F}$ and $A \in \mathcal{L}(V,V)$. Suppose also*

$$V = V_1 \oplus \cdots \oplus V_q$$

where each V_k is A invariant. ($AV_k \subseteq V_k$) Also let β_k be a basis for V_k and let A_k denote the restriction of A to V_k. Letting M^k denote the matrix of A_k with respect to this basis, it follows the matrix of A with respect to the basis $\{\beta_1, \cdots, \beta_q\}$ is

$$\begin{pmatrix} M^1 & & 0 \\ & \ddots & \\ 0 & & M^q \end{pmatrix}$$

Proof: Recall the matrix M of a linear transformation A is defined such that the following diagram commutes.

$$\begin{array}{ccccc} & & A & & \\ \{v_1, \cdots, v_n\} & V & \rightarrow & V & \{v_1, \cdots, v_n\} \\ & q \uparrow & \circ & \uparrow q & \\ & \mathbb{F}^n & \rightarrow & \mathbb{F}^n & \\ & & M & & \end{array}$$

where

$$q(\mathbf{x}) \equiv \sum_{i=1}^{n} x_i v_i$$

and $\{v_1, \cdots, v_n\}$ is a basis for V. Now when $V = V_1 \oplus \cdots \oplus V_q$ each V_k being invariant with respect to the linear transformation A, and β_k a basis for V_k, $\beta_k = \{v_1^k, \cdots, v_{m_k}^k\}$, one can consider the matrix M^k of A_k taken with respect to the basis β_k where A_k is the restriction of A to V_k. Then the claim of the theorem is true because if M is given as described it causes the diagram to commute. To see this, let $\mathbf{x} \in \mathbb{F}^{m_k}$.

$$q\begin{pmatrix} M^1 & & & & 0 \\ & \ddots & & & \\ & & M^k & & \\ & & & \ddots & \\ 0 & & & & M^q \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{x} \\ \vdots \\ \mathbf{0} \end{pmatrix} = q\begin{pmatrix} \mathbf{0} \\ \vdots \\ M^k\mathbf{x} \\ \vdots \\ \mathbf{0} \end{pmatrix} \equiv \sum_{ij} M_{ij}^k x_j v_i^k$$

while

$$Aq\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{x} \\ \vdots \\ \mathbf{0} \end{pmatrix} \equiv A\sum_j x_j v_j^k = \sum_j x_j A_k v_j^k = \sum_j x_j \sum_i M_{ij}^k v_i^k$$

because, as discussed earlier, $Av_j^k = \sum_i M_{ij}^k v_i^k$ because M^k is the matrix of A_k with respect to the basis β_k. This proves the theorem.

An examination of the proof of the above theorem yields the following corollary.

Corollary 10.2.6 *If any β_k in the above consists of eigenvectors, then M^k is a diagonal matrix having the corresponding eigenvalues down the diagonal.*

It follows that it would be interesting to consider special bases for the vector spaces in the direct sum. This leads to the Jordan form or more generally other canonical forms such as the rational canonical form.

10.3 The Jordan Canonical Form

The first case to consider is the case in which the minimal polynomial can be completely factored. (Sometimes people refer to this by saying the minimal polynomial "splits".) This is the condition which makes all of what follows valid but it is common to simply let the field of scalars be $\mathbb{C}$ because this will ensure that this happens, thanks to the fundamental theorem of algebra, proofs of which may be found in an appendix.

Definition 10.3.1 *$J_k(\alpha)$ is a Jordan block if it is a $k \times k$ matrix of the form*

$$J_k(\alpha) = \begin{pmatrix} \alpha & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \alpha \end{pmatrix}$$

In words, there is an unbroken string of ones down the super diagonal and the number, α filling every space on the main diagonal with zeros everywhere else.

The following is the fundamental result which will yield the Jordan form.

Proposition 10.3.2 *Let $N \in \mathcal{L}(W, W)$ be nilpotent,*

$$N^m = 0$$

for some $m \in \mathbb{N}$. Here W is an p dimensional vector space with field of scalars $\mathbb{F}$. Then there exists a basis for W such that the matrix of N with respect to this basis is of the form

$$J = \begin{pmatrix} J_{r_1}(0) & & & 0 \\ & J_{r_2}(0) & & \\ & & \ddots & \\ 0 & & & J_{r_s}(0) \end{pmatrix}$$

where $r_1 \geq r_2 \geq \cdots \geq r_s \geq 1$ and $\sum_{i=1}^{s} r_i = p$. In the above, the $J_{r_j}(0)$ is a Jordan block of size $r_j \times r_j$ with 0 down the main diagonal.

Proof: First note the only eigenvalue of N is 0. This is because if

$$Nv = \lambda v,\ v \neq 0,$$

and $\lambda \neq 0$, then

$$N^k v = \lambda^k N^{k-1} v.$$

Let $k < m$ be the largest natural number such that $N^{k-1}v \neq 0$. There is such a largest natural number because $N^m = 0$. Then

$$0 = \lambda^k N^{k-1} v,$$

a contradiction.

Let v_1 be an eigenvector. Then $\{v_1, v_2, \cdots, v_r\}$ is called a chain based on v_1 if $Nv_{k+1} = v_k$ for all $k = 1, 2, \cdots, r$ and v_1 is an eigenvector, $v_0 \equiv 0$. It will be called a maximal chain if there is no solution, v, to the equation, $Nv = v_r$.

Claim 1: The vectors in any chain are linearly independent and for $\{v_1, v_2, \cdots, v_r\}$ a chain based on v_1,

$$N : \text{span}(v_1, v_2, \cdots, v_r) \to \text{span}(v_1, v_2, \cdots, v_r). \tag{10.3}$$

Also if $\{v_1, v_2, \cdots, v_r\}$ is a chain, then $r \leq p$.

Proof: First note that 10.3 is obvious because

$$N \sum_{i=1}^{r} c_i v_i = \sum_{i=2}^{r} c_i v_{i-1}.$$

It only remains to verify the vectors of a chain are independent. Suppose then

$$\sum_{k=1}^{r} c_k v_k = 0.$$

Do N^{r-1} to it to conclude $c_r = 0$. Next do N^{r-2} to it to conclude $c_{r-1} = 0$ and continue this way. Now it is obvious $r \leq p$ because the chain is independent. This proves the claim.

Consider the set of all chains based on eigenvectors. Since all have total length no larger than p it follows there exists one which has maximal length, $\left\{v_1^1, \cdots, v_{r_1}^1\right\} \equiv \beta_1$. If $\text{span}(\beta_1)$ contains all eigenvectors of N, then stop. Otherwise, consider all chains based on eigenvectors not in $\text{span}(\beta_1)$ and pick one, $\beta_2 \equiv \left\{v_1^2, \cdots, v_{r_2}^2\right\}$ which is as long as possible. Thus $r_2 \leq r_1$. If $\text{span}(\beta_1, \beta_2)$ contains all eigenvectors of N, stop. Otherwise, consider all chains based on eigenvectors not in $\text{span}(\beta_1, \beta_2)$ and pick one, $\beta_3 \equiv \left\{v_1^3, \cdots, v_{r_3}^3\right\}$ such that r_3 is as large as possible. Continue this way. Thus $r_k \geq r_{k+1}$.

Claim 2: The above process terminates with a finite list of chains, $\{\beta_1, \cdots, \beta_s\}$ because for any k, $\{\beta_1, \cdots, \beta_k\}$ is linearly independent.

Proof of Claim 2: The claim is true if $k = 1$. This follows from **Claim** 1. Suppose it is true for $k-1, k \geq 2$. Then $\left\{\beta_1, \cdots, \beta_{k-1}\right\}$ is linearly independent. Now consider the induction step going from $k-1$ to k. Suppose you have a linear combination of vectors from $\left\{\beta_1, \cdots, \beta_{k-1}, \beta_k\right\}$.

$$\sum_{q=1}^{p} c_q w_q = 0,\ c_q \neq 0$$

By induction, some of these w_q must come from β_k. Let v_i^k be the one for which i is as large as possible. Then do N^{i-1} to both sides to obtain v_1^k, the eigenvector upon which the chain β_k is based, is a linear combination of $\left\{\beta_1, \cdots, \beta_{k-1}\right\}$ contrary to the construction. Since $\{\beta_1, \cdots, \beta_k\}$ is linearly independent, the process of picking maximal chains terminates. This proves the claim.

Claim 3: Suppose $Nw = 0$. (w is an eigenvector) Then there exist scalars, c_i such that

$$w = \sum_{i=1}^{s} c_i v_1^i.$$

Recall that v_1^i is the eigenvector in the i^{th} chain on which this chain is based.

Proof of Claim 3: From the construction, $w \in \text{span}(\beta_1, \cdots, \beta_s)$ since otherwise, it could serve as a base for another chain. Therefore,

$$w = \sum_{i=1}^{s} \sum_{k=1}^{r_i} c_i^k v_k^i.$$

Now apply N to both sides.

$$0 = \sum_{i=1}^{s} \sum_{k=2}^{r_i} c_i^k v_{k-1}^i$$

and so by **Claim 2,** $c_i^k = 0$ if $k \geq 2, i = 1, \cdots, s$. Therefore,

$$w = \sum_{i=1}^{s} c_i^1 v_1^i$$

and this proves the claim.

In fact, there is a generalization of this claim.

Claim 4: Suppose $N^k w = 0, k$ a positive integer. Then $w \in \text{span}(\beta_1, \cdots, \beta_s)$.

Proof of Claim 4: From **Claim 3,** this is true if $k = 1$. Suppose then it is true for $k - 1 \geq 1$ and $N^k w = 0$. Then if $N^{k-1} w = 0$ the result follows by induction so suppose $N^{k-1} w \neq 0$. Then $N^{k-1} w, \cdots, N w, w$ is a chain based on $N^{k-1} w$ having length k. It follows that each β_i has length at least as long as k because each time a new chain was chosen, it was chosen to be as long as possible. From **Claim 3** there exist scalars c_1^j such that

$$N^{k-1} w = \sum_{j=1}^{s} c_1^j v_1^j = \sum_{j=1}^{s} c_1^j N^{k-1} v_k^j$$

and so

$$0 = N^{k-1} \left(w - \sum_{j=1}^{s} c_1^j v_k^j \right)$$

Now by induction

$$w - \sum_{j=1}^{s} c_1^j v_k^j \in \text{span}(\beta_1, \cdots, \beta_s)$$

which shows $w \in \text{span}(\beta_1, \cdots, \beta_s)$.

From this claim, it is clear that $\text{span}(\beta_1, \cdots, \beta_s) = W$ because if $w \in W$ is arbitrary, then since N is nilpotent, $N^m = 0$, there exists some $k \leq m$ such that $N^k w = 0$ and so by **Claim 4,** $w \in \text{span}(\beta_1, \cdots, \beta_s)$. Since $\{\beta_1, \cdots, \beta_s\}$ is linearly independent, this shows it is a basis for W.

Now consider the matrix of N with respect to this basis. Since $\{\beta_1, \cdots, \beta_s\}$ is a basis, it follows

$$W = \text{span}(\beta_1) \oplus \cdots \oplus \text{span}(\beta_s).$$

Also each $\text{span}(\beta_k)$ is N invariant. Then it follows from Theorem 10.2.5 that the matrix of N with respect to this basis is block diagonal in which the k^{th} block is the matrix of N_k, the restriction of N to $\text{span}(\beta_k)$. Denote this matrix as J^k. Then by definition of the matrix of a linear transformation,

$$v_{j-1}^k = N v_j^k = \sum_i J_{ij}^k v_i^k$$

and so

$$J_{ij}^k = \begin{cases} 1 \text{ if } i = j - 1 \\ 0 \text{ otherwise} \end{cases}$$

Hence this block is of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \equiv J_{r_k}(0)$$

where there are r_k vectors in β_k. This proves the proposition.

In fact, the matrix of the above proposition is unique.

Corollary 10.3.3 *Let J, J' both be matrices of the nilpotent linear transformation $N \in \mathcal{L}(W,W)$ which are of the form described in Proposition 10.3.2. Then $J = J'$. In fact, if the rank of J^k equals the rank of J'^k for all nonnegative integers k, then $J = J'$.*

Proof: Since J and J' are similar, it follows that for each k an integer, J^k and J'^k are similar. Hence, for each k, these matrices have the same rank. Now suppose $J \neq J'$. Note first that

$$J_r(0)^r = 0,\ J_r(0)^{r-1} \neq 0.$$

Denote the blocks of J as $J_{r_k}(0)$ and the blocks of J' as $J_{r'_k}(0)$. Let k be the first such that $J_{r_k}(0) \neq J_{r'_k}(0)$. Suppose that $r_k > r'_k$. By block multiplication and the above observation, it follows that the two matrices J^{r_k-1} and J'^{r_k-1} are respectively of the forms

$$\begin{pmatrix} M_{r_1} & & & & & & 0 \\ & \ddots & & & & & \\ & & M_{r_k} & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ 0 & & & & & & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} M_{r_1} & & & & & & 0 \\ & \ddots & & & & & \\ & & M_{r'_k} & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ 0 & & & & & & 0 \end{pmatrix}$$

where $M_{r_j} = M_{r'_j}$ for $j \leq k-1$ but $M_{r'_k}$ is a zero $r'_k \times r'_k$ matrix while M_{r_k} is a larger matrix which is of the form

$$M_{r_k} = \begin{pmatrix} 0 & \cdots & 1 \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}$$

Thus there is one more pivot column in M_{r_k} than in $M_{r'_k}$, contradicting the requirement that J^k and J'^k have the same rank. This proves the corollary.

Now let $A \in \mathcal{L}(V,V)$ where V is an n dimensional vector space and the minimal polynomial splits. That is, the minimal polynomial can be written in the form

$$p(\lambda) = \prod_{k=1}^{r} (\lambda - \lambda_k)^{m_k}$$

The following proposition is very interesting.

Proposition 10.3.4 *Let the minimal polynomial of $A \in \mathcal{L}(V,V)$ be given by*

$$p(\lambda) = \prod_{k=1}^{r} (\lambda - \lambda_k)^{m_k}$$

Then the eigenvalues of A are $\{\lambda_1, \cdots, \lambda_r\}$.

Proof: The scalars λ_k are each eigenvalues. To see this, note there exists a vector v such that

$$(A-\lambda_k I)^{m_k-1}\prod_{j\neq k}(A-\lambda_j I)^{m_j}\,v\neq 0$$

since otherwise $p(\lambda)$ given above is not really the minimal polynomial. Therefore, the vector

$$(A-\lambda_k I)^{m_k-1}\prod_{j\neq k}(A-\lambda_j I)^{m_j}\,v$$

is an eigenvector because it is nonzero and $A-\lambda_k I$ sends it to 0. Are there any other eigenvalues? Suppose $(A-\mu I)\,x=0$ for some $x\neq 0$. Then since $p(A)\,x=0$ and the polynomial $\lambda\to\lambda-\mu$ is irreducible, it must be the case that $\lambda-\mu$ divides $p(\lambda)$. This is because

$$p(\lambda)=g(\lambda)(\lambda-\mu)+\eta(\lambda)$$

where the degree of $\eta(\lambda)$ is less than the degree of $(\lambda-\mu)$ or is equal to 0. Thus $\eta(\lambda)$ is a constant or else 0. However, it can't be a nonzero constant η because then

$$\eta x=p(A)\,x-g(A)(A-\mu I)\,x=0$$

and $x\neq 0$. Thus $\eta=0$. By Corollary 8.3.9, this would require $\lambda-\mu$ to be one of the factors of $p(\lambda)$ so $\mu=\lambda_k$ for some k after all. Thus the λ_k are the only eigenvalues. This proves the proposition.

It follows from Corollary 10.2.4

$$\begin{aligned}V &= \ker(A-\lambda_1 I)^{m_1}\oplus\cdots\oplus\ker(A-\lambda_r I)^{m_r}\\ &\equiv V_1\oplus\cdots\oplus V_r\end{aligned}$$

where I denotes the identity linear transformation. Without loss of generality, let the dimensions of the V_k be decreasing from left to right.

It follows from the definition of V_k that $(A-\lambda_k I)$ is nilpotent on V_k and clearly each V_k is A invariant. Therefore from Proposition 10.3.2 and letting A_k denote the restriction of A to V_k, there exists an ordered basis for V_k,β_k such that with respect to this basis, the matrix of $(A_k-\lambda_k I)$ with respect to this basis is of the form given in that proposition, denoted here by J^k. What is the matrix of A_k with respect to β_k? Letting $\{b_1,\cdots,b_r\}=\beta_k$,

$$\begin{aligned}A_k b_j &= (A_k-\lambda_k I)\,b_j+\lambda_k I b_j\\ &\equiv \sum_s J^k_{sj}b_s+\sum_s\lambda_k\delta_{sj}b_s\\ &= \sum_s\left(J^k_{sj}+\lambda_k\delta_{sj}\right)b_s\end{aligned}$$

and so the matrix of A_k with respect to this basis is

$$J^k+\lambda_k I$$

where I is the identity matrix. Therefore, with respect to the ordered basis $\{\beta_1,\cdots,\beta_r\}$ the matrix of A is in Jordan canonical form. This means the matrix is of the form

$$\begin{pmatrix}J(\lambda_1) & & 0\\ & \ddots & \\ 0 & & J(\lambda_r)\end{pmatrix}\tag{10.4}$$

where $J(\lambda_k)$ is an $m_k \times m_k$ matrix of the form

$$\begin{pmatrix} J_{k_1}(\lambda_k) & & & 0 \\ & J_{k_2}(\lambda_k) & & \\ & & \ddots & \\ 0 & & & J_{k_r}(\lambda_k) \end{pmatrix} \tag{10.5}$$

where $k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$ and $\sum_{i=1}^{r} k_i = m_k$. Here $J_k(\lambda)$ is a $k \times k$ Jordan block of the form

$$\begin{pmatrix} \lambda & 1 & & 0 \\ 0 & \lambda & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 0 & \lambda \end{pmatrix} \tag{10.6}$$

This proves the existence part of the following fundamental theorem.

Note that if any of the β_k consists of eigenvectors, then the corresponding Jordan block will consist of a diagonal matrix having λ_k down the main diagonal.

To illustrate the main idea used in proving uniqueness in this theorem, consider the following two matrices.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first has one 3×3 block and the second has two 2×2 blocks. Initially both matrices have rank 2. Now lets raise them to a power 2.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 1 and

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has no rank. You see, discrepancies occur in the rank upon raising to higher powers if the blocks are not the same. Now with this preparation, here is the main theorem.

Theorem 10.3.5 *Let V be an n dimensional vector space with field of scalars $\mathbb{C}$ or some other field such that the minimal polynomial of $A \in \mathcal{L}(V,V)$ completely factors into powers of linear factors. Then there exists a unique Jordan canonical form for A as described in 10.4 - 10.6, where uniqueness is in the sense that any two have the same number and size of Jordan blocks.*

Proof: It only remains to verify uniqueness. Suppose there are two, J and J'. Then these are matrices of A with respect to possibly different bases and so they are similar. Therefore, they have the same minimal polynomials and the generalized eigenspaces have

the same dimension. Thus the size of the matrices $J(\lambda_k)$ and $J'(\lambda_k)$, corresponding to the algebraic multiplicity of λ_k, must be the same. Therefore, they comprise the same set of positive integers. Thus listing the eigenvalues in the same order, corresponding blocks $J(\lambda_k), J'(\lambda_k)$ are the same size.

It remains to show that $J(\lambda_k)$ and $J'(\lambda_k)$ are not just the same size but also are the same up to order of the Jordan blocks running down their respective diagonals. It is only necessary to worry about the number and size of the Jordan blocks making up $J(\lambda_k)$ and $J'(\lambda_k)$. Since J, J' are similar, so are $J-\lambda_k I$ and $J'-\lambda_k I$. Letting I be the identity of the right size, it is also the case that both $J(\lambda_j)-\lambda_k I$ and $J'(\lambda_j)-\lambda_k I$ are one to one for every $\lambda_j \neq \lambda_k$. The following two matrices are similar

$$A \equiv \begin{pmatrix} J(\lambda_1)-\lambda_k I & & & 0 \\ & \ddots & & \\ & & J(\lambda_k)-\lambda_k I & \\ & & & \ddots \\ 0 & & & J(\lambda_r)-\lambda_k I \end{pmatrix}$$

$$B \equiv \begin{pmatrix} J'(\lambda_1)-\lambda_k I & & & 0 \\ & \ddots & & \\ & & J'(\lambda_k)-\lambda_k I & \\ & & & \ddots \\ 0 & & & J'(\lambda_r)-\lambda_k I \end{pmatrix}$$

and consequently, $\operatorname{rank}\left(A^k\right) = \operatorname{rank}\left(B^k\right)$ for all $k \in \mathbb{N}$. Since all the blocks in both of these matrices are one to one except the blocks $J'(\lambda_k)-\lambda_k I, J(\lambda_k)-\lambda_k I$, it follows that this requires the two sequences of numbers $\{\operatorname{rank}\left((J(\lambda_k)-\lambda_k I)^m\right)\}_{m=1}^{\infty}$ and $\left\{\operatorname{rank}\left(\left(J'(\lambda_k)-\lambda_k I\right)^m\right)\right\}_{m=1}^{\infty}$ must be the same.

Then

$$J(\lambda_k)-\lambda_k I \equiv \begin{pmatrix} J_{k_1}(0) & & & 0 \\ & J_{k_2}(0) & & \\ & & \ddots & \\ 0 & & & J_{k_r}(0) \end{pmatrix}$$

and a similar formula holds for $J'(\lambda_k)$

$$J'(\lambda_k)-\lambda_k I \equiv \begin{pmatrix} J_{l_1}(0) & & & 0 \\ & J_{l_2}(0) & & \\ & & \ddots & \\ 0 & & & J_{l_p}(0) \end{pmatrix}$$

and it is required to verify that $p = r$ and that the same blocks occur in both. Without loss of generality, let the blocks be arranged according to size with the largest on upper left corner falling to smallest in lower right. Now the desired conclusion follows from Corollary 10.3.3. This proves the theorem.

Note that if any of the generalized eigenspaces $\ker(A-\lambda_k I)^{m_k}$ has a basis of eigenvectors, then it would be possible to use this basis and obtain a diagonal matrix in the block corresponding to λ_k. By uniqueness, this is **the** block corresponding to the eigenvalue λ_k. Thus when this happens, the block in the Jordan canonical form corresponding to λ_k is just the diagonal matrix having λ_k down the diagonal and there are **no generalized eigenvectors**.

The Jordan canonical form is very significant when you try to understand powers of a matrix. There exists an $n \times n$ matrix, S[1] such that

$$A = S^{-1}JS.$$

Therefore, $A^2 = S^{-1}JSS^{-1}JS = S^{-1}J^2S$ and continuing this way, it follows

$$A^k = S^{-1}J^kS.$$

where J is given in the above corollary. Consider J^k. By block multiplication,

$$J^k = \begin{pmatrix} J_1^k & & 0 \\ & \ddots & \\ 0 & & J_r^k \end{pmatrix}.$$

The matrix, J_s is an $m_s \times m_s$ matrix which is of the form

$$J_s = \begin{pmatrix} \alpha & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha \end{pmatrix} \tag{10.7}$$

which can be written in the form

$$J_s = D + N$$

for D a multiple of the identity and N an upper triangular matrix with zeros down the main diagonal. Therefore, by the Cayley Hamilton theorem, $N^{m_s} = 0$ because the characteristic equation for N is just $\lambda^{m_s} = 0$. (You could also verify this directly.) Now since D is just a multiple of the identity, it follows that $DN = ND$. Therefore, the usual binomial theorem may be applied and this yields the following equations for $k \geq m_s$.

$$\begin{aligned} J_s^k &= (D+N)^k = \sum_{j=0}^{k} \binom{k}{j} D^{k-j}N^j \\ &= \sum_{j=0}^{m_s} \binom{k}{j} D^{k-j}N^j, \end{aligned} \tag{10.8}$$

the third equation holding because $N^{m_s} = 0$. Thus J_s^k is of the form

$$J_s^k = \begin{pmatrix} \alpha^k & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha^k \end{pmatrix}.$$

Lemma 10.3.6 *Suppose J is of the form J_s described above in 10.7 where the constant, α, on the main diagonal is less than one in absolute value. Then*

$$\lim_{k\to\infty} \left(J^k\right)_{ij} = 0.$$

Proof: From 10.8, it follows that for large k, and $j \leq m_s$,

$$\binom{k}{j} \leq \frac{k\,(k-1)\cdots(k-m_s+1)}{m_s!}.$$

[1]The S here is written as S^{-1} in the corollary.

Therefore, letting C be the largest value of $\left|\left(N^j\right)_{pq}\right|$ for $0 \leq j \leq m_s$,

$$\left|\left(J^k\right)_{pq}\right| \leq m_s C \left(\frac{k\left(k-1\right)\cdots\left(k-m_s+1\right)}{m_s!}\right)\left|\alpha\right|^{k-m_s}$$

which converges to zero as $k \to \infty$. This is most easily seen by applying the ratio test to the series

$$\sum_{k=m_s}^{\infty}\left(\frac{k\left(k-1\right)\cdots\left(k-m_s+1\right)}{m_s!}\right)\left|\alpha\right|^{k-m_s}$$

and then noting that if a series converges, then the k^{th} term converges to zero.

10.4 Exercises

1. ♠When we say a polynomial equals zero, we mean that all the coefficients equal 0. If we assign a different meaning to it which says that a polynomial

$$p\left(\lambda\right) = \sum_{k=0}^{n} a_k \lambda^k = 0,$$

when the value of the polynomial equals zero whenever a particular value of $\lambda \in \mathbb{F}$ is placed in the formula for $p\left(\lambda\right)$, can the same conclusion be drawn? Is there any difference in the two definitions for ordinary fields like $\mathbb{Q}$? **Hint:** Consider $\mathbb{Z}_2$, the integers mod 2.

2. ♠Let $A \in \mathcal{L}\left(V,V\right)$ where V is a finite dimensional vector space with field of scalars $\mathbb{F}$. Let $p\left(\lambda\right)$ be the minimal polynomial and suppose $\phi\left(\lambda\right)$ is any nonzero polynomial such that $\phi\left(A\right)$ is not one to one and $\phi\left(\lambda\right)$ has smallest possible degree such that $\phi\left(A\right)$ is nonzero and not one to one. Show $\phi\left(\lambda\right)$ must divide $p\left(\lambda\right)$.

3. ♠Let $A \in \mathcal{L}\left(V,V\right)$ where V is a finite dimensional vector space with field of scalars $\mathbb{F}$. Let $p\left(\lambda\right)$ be the minimal polynomial and suppose $\phi\left(\lambda\right)$ is an irreducible polynomial with the property that $\phi\left(A\right)x = 0$ for some specific $x \neq 0$. Show that $\phi\left(\lambda\right)$ must divide $p\left(\lambda\right)$. **Hint:** First write

$$p\left(\lambda\right) = \phi\left(\lambda\right)g\left(\lambda\right) + r\left(\lambda\right)$$

where $r\left(\lambda\right)$ is either 0 or has degree smaller than the degree of $\phi\left(\lambda\right)$. If $r\left(\lambda\right) = 0$ you are done. Suppose it is not 0. Let $\eta\left(\lambda\right)$ be the monic polynomial of smallest degree with the property that $\eta\left(A\right)x = 0$. Now use the Euclidean algorithm to divide $\phi\left(\lambda\right)$ by $\eta\left(\lambda\right)$. Contradict the irreducibility of $\phi\left(\lambda\right)$.

4. ♠Suppose A is a linear transformation and let the characteristic polynomial be

$$\det\left(\lambda I - A\right) = \prod_{j=1}^{q}\phi_j\left(\lambda\right)^{n_j}$$

where the $\phi_j\left(\lambda\right)$ are irreducible. Explain using Corollary 8.3.9 why the irreducible factors of the minimal polynomial are $\phi_j\left(\lambda\right)$ and why the minimal polynomial is of the form

$$\prod_{j=1}^{q}\phi_j\left(\lambda\right)^{r_j}$$

where $r_j \leq n_j$. You can use the Cayley Hamilton theorem if you like.

5. ♠Let
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
Find the minimal polynomial for A.

6. ♠ Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be a vector. Consider the A cyclic set of vectors $\left\{\mathbf{v}, A\mathbf{v}, \cdots, A^{m-1}\mathbf{v}\right\}$ where this is an independent set of vectors but $A^m\mathbf{v}$ is a linear combination of the preceding vectors in the list. Show how to obtain a monic polynomial of smallest degree, m, $\phi_{\mathbf{v}}(\lambda)$ such that
$$\phi_{\mathbf{v}}(A)\,\mathbf{v} = \mathbf{0}$$
Now let $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ be a basis and let $\phi(\lambda)$ be the least common multiple of the $\phi_{\mathbf{w}_k}(\lambda)$. Explain why this must be the minimal polynomial of A. Give a reasonably easy algorithm for computing $\phi_{\mathbf{v}}(A)$.

7. ♠Here is a matrix.
$$\begin{pmatrix} -7 & -1 & -1 \\ -21 & -3 & -3 \\ 70 & 10 & 10 \end{pmatrix}$$
Using the process of Problem 6 find the minimal polynomial of this matrix. It turns out the characteristic polynomial is λ^3.

8. ♠Find the minimal polynomial for
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 2 & 1 \end{pmatrix}$$
by the above technique. Is what you found also the characteristic polynomial?

9. ♠Let A be an $n \times n$ matrix with field of scalars $\mathbb{C}$. Letting λ be an eigenvalue, show the dimension of the eigenspace equals the number of Jordan blocks in the Jordan canonical form which are associated with λ. Recall the eigenspace is $\ker(\lambda I - A)$.

10. ♠For any $n \times n$ matrix, why is the dimension of the eigenspace always less than or equal to the algebraic multiplicity of the eigenvalue as a root of the characteristic equation? **Hint:** Note the algebraic multiplicity is the size of the appropriate block in the Jordan form.

11. ♠Give an example of two nilpotent matrices which are not similar but have the same minimal polynomial if possible.

12. ♠Use the existence of the Jordan canonical form for a linear transformation whose minimal polynomial factors completely to give a proof of the Cayley Hamilton theorem which is valid for any field of scalars. **Hint:** First assume the minimal polynomial factors completely into linear factors. If this does not happen, consider the splitting field of the minimal polynomial. Then consider the minimal polynomial with respect to this larger field. How will the two minimal polynomials be related?

13. ♠Here is a matrix. Find its Jordan canonical form by directly finding the eigenvectors and generalized eigenvectors based on these to find a basis which will yield the Jordan form. The eigenvalues are 1 and 2.

$$\begin{pmatrix} -3 & -2 & 5 & 3 \\ -1 & 0 & 1 & 2 \\ -4 & -3 & 6 & 4 \\ -1 & -1 & 1 & 3 \end{pmatrix}$$

Why is it typically impossible to find the Jordan canonical form?

14. ♠People like to consider the solutions of first order linear systems of equations which are of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where here A is an $n \times n$ matrix. From the theorem on the Jordan canonical form, there exist S and S^{-1} such that $A = SJS^{-1}$ where J is a Jordan form. Define $\mathbf{y}(t) \equiv S^{-1}\mathbf{x}(t)$. Show $\mathbf{y}' = J\mathbf{y}$. Now suppose $\Psi(t)$ is an $n \times n$ matrix whose columns are solutions of the above differential equation. Thus

$$\Psi' = A\Psi$$

Now let Φ be defined by $S\Phi S^{-1} = \Psi$. Show

$$\Phi' = J\Phi.$$

15. ♠In the above Problem show that

$$\det(\Psi)' = \operatorname{trace}(A)\det(\Psi)$$

and so

$$\det(\Psi(t)) = Ce^{\operatorname{trace}(A)t}$$

This is called Abel's formula and $\det(\Psi(t))$ is called the Wronskian. **Hint:** Show it suffices to consider

$$\Phi' = J\Phi$$

and establish the formula for Φ. Next let

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

where the ϕ_j are the rows of Φ. Then explain why

$$\det(\Phi)' = \sum_{i=1}^{n} \det(\Phi_i) \tag{10.9}$$

where Φ_i is the same as Φ except the i^{th} row is replaced with ϕ_i' instead of the row ϕ_i. Now from the form of J,

$$\Phi' = D\Phi + N\Phi$$

where N has all nonzero entries above the main diagonal. Explain why

$$\phi_i'(t) = \lambda_i\phi_i(t) + a_i\phi_{i+1}$$

Now use this in the formula for the derivative of the Wronskian given in 10.9 and use properties of determinants to obtain

$$\det(\Phi)' = \sum_{i=1}^{n} \lambda_i \det(\Phi).$$

Obtain Abel's formula

$$\det(\Phi) = Ce^{\mathrm{trace}(A)t}$$

and so the Wronskian $\det \Phi$ either vanishes identically or never.

16. ♠Let A be an $n \times n$ matrix and let J be its Jordan canonical form. Recall J is a block diagonal matrix having blocks $J_k(\lambda)$ down the diagonal. Each of these blocks is of the form

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Now for $\varepsilon > 0$ given, let the diagonal matrix D_ε be given by

$$D_\varepsilon = \begin{pmatrix} 1 & & & 0 \\ & \varepsilon & & \\ & & \ddots & \\ 0 & & & \varepsilon^{k-1} \end{pmatrix}$$

Show that $D_\varepsilon^{-1} J_k(\lambda) D_\varepsilon$ has the same form as $J_k(\lambda)$ but instead of ones down the super diagonal, there is ε down the super diagonal. That is $J_k(\lambda)$ is replaced with

$$\begin{pmatrix} \lambda & \varepsilon & & 0 \\ & \lambda & \ddots & \\ & & \ddots & \varepsilon \\ 0 & & & \lambda \end{pmatrix}$$

Now show that for A an $n \times n$ matrix, it is similar to one which is just like the Jordan canonical form except instead of the blocks having 1 down the super diagonal, it has ε.

17. ♠Let A be in $\mathcal{L}(V, V)$ and suppose that $A^p x \neq 0$ for some $x \neq 0$. Show that $A^p e_k \neq 0$ for some $e_k \in \{e_1, \cdots, e_n\}$, a basis for V. If you have a matrix which is nilpotent, ($A^m = 0$ for some m) will it always be possible to find its Jordan form? Describe how to do it if this is the case. **Hint:** First explain why all the eigenvalues are 0. Then consider the way the Jordan form for nilpotent transformations was constructed in the above.

18. ♠Suppose A is an $n \times n$ matrix and that it has n distinct eigenvalues. How do the minimal polynomial and characteristic polynomials compare? Determine other conditions based on the Jordan Canonical form which will cause the minimal and characteristic polynomials to be different.

19. ♠Suppose A is a 3×3 matrix and it has at least two distinct eigenvalues. Is it possible that the minimal polynomial is different than the characteristic polynomial?

20. If A is an $n \times n$ matrix of entries from a field of scalars and if the minimal polynomial of A splits over this field of scalars, does it follow that the characteristic polynomial of A also splits? Explain why or why not.

21. In proving the uniqueness of the Jordan canonical form, it was asserted that if two $n \times n$ matrices A, B are similar, then they have the same minimal polynomial and also that if this minimal polynomial is of the form

$$p(\lambda) = \prod_{i=1}^{s} \phi_i(\lambda)^{r_i}$$

where the $\phi_i(\lambda)$ are irreducible, then $\ker(\phi_i(A)^{r_i})$ has the same dimension as $\ker(\phi_i(B)^{r_i})$. Why is this so? This was what was responsible for the blocks corresponding to an eigenvalue being of the same size.

10.5 The Rational Canonical Form

It was shown above that even in the case where nothing is known about whether the minimal polynomial factors, you can still decompose the vector space V into a direct sum,

$$V = V_1 \oplus \cdots \oplus V_q$$

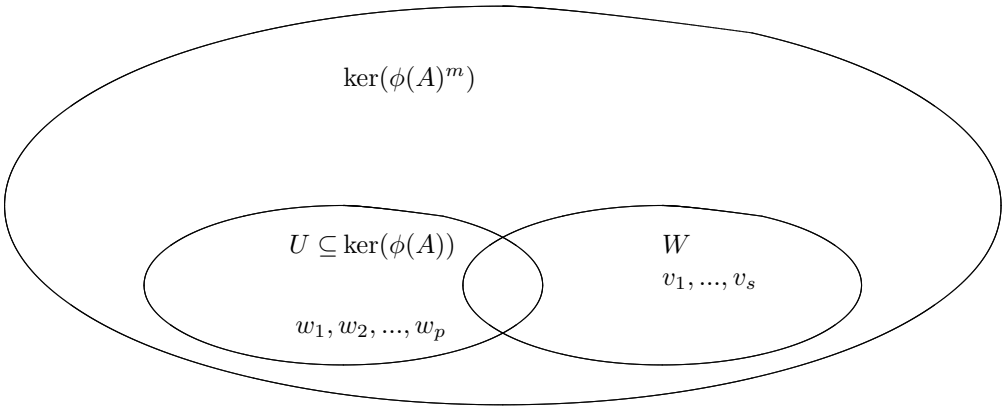
where

$$V_k = \ker(\phi(A)^{m_k})$$

and $\phi(\lambda)$ is an irreducible polynomial. Next I will consider the problem of finding bases for each V_k such that the matrix of A assumes a certain form. I am following the presentation given in [8]. This rational canonical form is due to Frobenius.

Definition 10.5.1 *Letting $x \neq 0$ denote by β_x the vectors $\{x, Ax, A^2x, \cdots, A^{m-1}x\}$ where m is the smallest such that $A^m x \in \operatorname{span}(x, \cdots, A^{m-1}x)$. This is called an A cyclic set.*

The following is the main idea. To help organize the ideas in this lemma, here is a diagram.



Lemma 10.5.2 *Let W be an A invariant subspace of $\ker(\phi(A)^m)$ for m a positive integer where $\phi(\lambda)$ is an irreducible monic polynomial of degree d. Then if $\eta(\lambda)$ is a monic polynomial of smallest degree such that for*

$$x \in \ker(\phi(A)^m) \setminus \{0\},$$

$$\eta(A)\,x = 0,$$

then

$$\eta(\lambda) = \phi(\lambda)^k$$

for some positive integer k. *If*

$$x \neq 0$$

then

$$\left\{x, Ax, \cdots, A^{d-1}x\right\}$$

is linearly independent.

Now let U *be an* A *invariant subspace of* $\ker(\phi(A))$.

If $\{v_1, \cdots, v_s\}$ *is a basis for* W *then if* $x \in U \setminus W$,

$$\left\{v_1, \cdots, v_s, x, Ax, \cdots, A^{d-1}x\right\}$$

is linearly independent.

Denoting by β *the vectors* $\{v_1, \cdots, v_s\}$, *there exist vectors* $w_1, \cdots, w_p$ *such that*

$$\left\{\beta, \beta_{w_1}, \cdots, \beta_{w_p}\right\}$$

is a basis for a subspace of $\ker(\phi(A)^m)$ *which contains*

$$U + W.$$

Proof: Consider the first claim. If $\eta(A)\,x = 0$, then writing

$$\phi(\lambda)^m = \eta(\lambda)\,g(\lambda) + r(\lambda)$$

where either $r(\lambda) = 0$ or the degree of $r(\lambda)$ is less than that of $\eta(\lambda)$, the latter possibility cannot occur because if it did, $r(A)\,x = 0$ and this would contradict the definition of $\eta(\lambda)$. Therefore $r(\lambda) = 0$ and so $\eta(\lambda)$ divides $\phi(\lambda)^m$. From Corollary 8.3.9,

$$\eta(\lambda) = \phi(\lambda)^k$$

for some integer, $k \leq m$. Since $x \neq 0$, it follows $k > 0$. In particular, the degree of $\eta(\lambda)$ is at least as large as d.

Now suppose

$$\sum_{j=1}^{d} d_j A^{j-1}x = 0,\ x \neq 0. \tag{10.10}$$

If not all the d_j are zero then $\sum_{j=1}^{d} d_j \lambda^{j-1}$ is a polynomial having smaller degree than that of $\phi(\lambda)$ in contradiction to what was just shown.

Suppose now $x \in U \setminus W$ and

$$\sum_{i=1}^{s} a_i v_i + \sum_{j=1}^{d} d_j A^{j-1}x = 0.$$

If $z \equiv \sum_{j=1}^{d} d_j A^{j-1}x$, then $z \in W \cap \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)$. Then also each

$$A^r z \in W \cap \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right) \subseteq W \cap \ker(\phi(A))$$

because W is A invariant and so

$$\begin{aligned} \operatorname{span}\left(z, Az, \cdots, A^{d-1}z\right) &\subseteq W \cap \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right) \\ &\subseteq \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right) \end{aligned} \tag{10.11}$$

Suppose $z \neq 0$. Then $\left\{z, Az, \cdots, A^{d-1}z\right\}$ must be linearly independent because if not, there would exist a monic polynomial $\eta(\lambda)$ having smaller degree than d such that $\eta(A)z = 0$ which is impossible by the first part of the argument. Therefore,

$$\begin{aligned} d &= \dim\left(\operatorname{span}\left(z, Az, \cdots, A^{d-1}z\right)\right) \\ &\leq \dim\left(W \cap \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)\right) \\ &\leq \dim\left(\operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)\right) = d \end{aligned}$$

Thus

$$\operatorname{span}\left(z, Az, \cdots, A^{d-1}z\right) \subseteq \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)$$

and both have the same dimension and so the two sets are equal. It follows from 10.11

$$W \cap \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right) = \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)$$

which would require $x \in W$ but this is assumed not to take place. Hence $z = 0$ and so the linearly independence of the $\{v_1, \cdots, v_s\}$ implies each $a_i = 0$. Then the linear independence of $\left\{x, Ax, \cdots, A^{d-1}x\right\}$ which follows from the first part of the argument shows each $d_j = 0$. Thus $\left\{v_1, \cdots, v_s, x, Ax, \cdots, A^{d-1}x\right\}$ is linearly independent as claimed.

Since $x \in U \subseteq \ker(\phi(A))$,

$$\beta_x = \left\{x, Ax, \cdots, A^{d-1}x\right\}.$$

This is because $\phi(\lambda)$ is irreducible. It follows that if

$$\sum_{i=1}^{d} c_i A^{i-1} x = 0$$

then all the $c_i = 0$ because otherwise, you could consider the polynomial

$$\eta(\lambda) = \sum_{i=1}^{d} c_i \lambda^{i-1} x$$

and it would have smaller degree than $\phi(\lambda)$ contrary to the first claim of the lemma proved above. Also $\phi(A)x = 0$ so this implies $A^m x \in \operatorname{span}\left(x, Ax, \cdots, A^{d-1}x\right)$.

Now pick $w_1 \in U \setminus W$ and consider the linearly independent vectors $\left\{\beta, \beta_{w_1}\right\}$ if the span of these contains $U + W$, stop. If not, let $W_1 \equiv W + \operatorname{span}\left(\beta_{w_1}\right)$ play the role of W and pick $w_2 \in U \setminus W_1$ and then consider $\left\{\beta, \beta_{w_1}, \beta_{w_2}\right\}$. If $\operatorname{span}\left(\beta, \beta_{w_1}, \beta_{w_2}\right)$ contains $U + W$, stop. Otherwise continue in this way. The spaces are all finite dimensional so the process must eventually stop and yield the desired basis. This proves the lemma.

Lemma 10.5.3 *Let V be a vector space and let $B \in \mathcal{L}(V, V)$. Then*

$$V = B(V) + \ker(B)$$

Proof: Let $\{Bv_1, \cdots, Bv_r\}$ be a basis for $B(V)$. Now let $\{w_1, \cdots, w_s\}$ be a basis for $\ker(B)$. Then if $v \in V$, there exist unique scalars c_i such that

$$Bv = \sum_{i=1}^{r} c_i Bv_i$$

and so $B\left(v - \sum_{i=1}^{r} c_i v_i\right) = 0$ and so there exist unique scalars d_i such that

$$v - \sum_{i=1}^{r} c_i v_i = \sum_{j=1}^{s} d_j w_j.$$

This proves the lemma.

Proposition 10.5.4 *Suppose $V = \ker(\phi(A)^m)$ where $\phi(\lambda)$ is irreducible with degree d and suppose $x \in V, x \neq 0$ and $\phi_x(\lambda)$ is the monic polynomial of smallest degree r such that $\phi_x(A)\, x = 0$. Then $\beta_x \equiv \left\{x, Ax, \cdots, A^{r-1}x\right\}$ is linearly independent and if the degree of $\phi(\lambda)$ is d, then $r = dk$ for some positive integer $k \leq m$.*

Proof: I need to consider the degree of $\phi_x(\lambda)$. This polynomial must divide $\phi(\lambda)^m$. Here is why. If not, then

$$\phi(\lambda)^m = g(\lambda)\,\phi_x(\lambda) + r(\lambda)$$

where $r(\lambda)$ has smaller degree than $\phi_x(\lambda)$. But then

$$r(A)\, x = \phi(A)^m x - g(A)\,\phi_x(A)\, x = 0$$

which contradicts the definition of $\phi_x(\lambda)$ as having the smallest degree. By Corollary 8.3.9 $\phi_x(\lambda) = \phi(\lambda)^k$ for some $k \leq m$. This proves the proposition.

With this preparation, here is the main result about a basis for $\ker(\phi(A)^m)$ for $\phi(\lambda)$ irreducible.

Theorem 10.5.5 *Let V be a vector space with field of scalars $\mathbb{F}$ and let $A \in \mathcal{L}(V, V)$. Suppose the minimal polynomial of A is $\phi(\lambda)^m$ where $\phi(\lambda)$ is irreducible and m is a positive integer. Then there exist vectors $\{v_1, \cdots, v_s\}$ and A cyclic bases β_{v_j} such that $\left\{\beta_{v_1}, \cdots, \beta_{v_s}\right\}$ is a basis for V.*

Proof 1: First suppose $m = 1$. Then letting $W = \{0\}, U = \ker(\phi(A))$, Lemma 10.5.2 immediately gives the desired basis consisting of A cyclic sets.

Suppose the theorem is true for $m-1$ and let $W \equiv \phi(A)(V)$. Then W is A invariant and the minimal polynomial for the restriction of A to W is $\phi(\lambda)^{m-1}$. Here is why this claim is true. If $\eta(\lambda)$ is the minimial polynomial then $\eta(\lambda)$ must divide $\phi(\lambda)^m$ and so by Corollary 8.3.9, $\eta(\lambda) = \phi(\lambda)^k$ where $k \leq m$. In fact $k \leq m-1$ because $\phi(\lambda)^{m-1}$ sends everything in $\phi(A)(V)$ to zero. Could $k < m-1$? No, this can't happen because if so then for an arbitrary $v \in V$,

$$0 = \phi(A)^k \left(\overbrace{\phi(A)\, v}^{\text{In } W} \right) = \phi(A)^{k+1} v,\ k+1 < m$$

and so $\phi(\lambda)^m$ wouldn't be the minimal polynomial for V after all.

By induction, there exist $\{w_1, \cdots, w_l\}$ such that $\left\{\beta_{w_1}, \cdots, \beta_{w_l}\right\}$ is a basis for W. Then by Lemma 10.5.2 again there exist $\{z_1, \cdots, z_p\} \in \ker(\phi(A))$ such that

$$\left\{\beta_{z_1}, \cdots, \beta_{z_p}, \beta_{w_1}, \cdots, \beta_{w_l}\right\}$$

is a basis for $\ker(\phi(A)) + W = V$ by Lemma 10.5.3.

Proof 2: This proof involves an algorithm for finding the β_{v_i}. This is one of the important aspects of the rational canonical form. It can be computed because it does not require a knowledge of the eigenvalues, in contrast to the Jordan form. First suppose $m = 1$. Then letting $W = \{0\}$ Lemma 10.5.2 immediately gives the desired basis consisting of A cyclic sets. To obtain it, just follow the procedure described in the proof of the lemma for finding $\beta_{w_1}, \beta_{w_2}, \cdots$.

What do you do in general where $m > 1$? By Lemma 10.5.3,

$$V = \phi(A)^{m-1}(V) + \ker\left(\phi(A)^{m-1}\right).$$

Let

$$W \equiv \ker\left(\phi(A)^{m-1}\right), U \equiv \phi(A)^{m-1}(V).$$

Then the minimal polynomial for $\phi(A)^{m-1}(V)$ is $\phi(\lambda)$. This is because if $\eta(\lambda)$ is the minimal polynomial, the Euclidean algorithm implies

$$\phi(\lambda) = g(\lambda)\eta(\lambda) + r(\lambda)$$

where either the degree of $r(\lambda)$ is less than the degree of $\eta(\lambda)$ or it equals zero. Its degree cannot be less than the degree of $\eta(\lambda)$ because the above equation would then yield $r(A) = 0$. Therefore, $r(\lambda) = 0$ and so $\eta(\lambda)$ divides $\phi(\lambda)$. But $\phi(\lambda)$ is irreducible and so $\eta(\lambda) = \phi(\lambda)$. Now obviously $\phi(A)^{m-1}(V) \subseteq \ker(\phi(A))$ because $V = \ker(\phi(A)^m)$. Also $\phi(A)^{m-1}(V)$ is A invariant. Therefore, from Lemma 10.5.2, one can let

$$U \equiv \phi(A)^{m-1}(V), W \equiv \ker\left(\phi(A)^{m-1}\right)$$

and use the process of that lemma to obtain $\{\beta_{v_1}, \cdots, \beta_{v_m}\}$ where $v_k \in U$ and a basis for V is $\{\beta_{v_1}, \cdots, \beta_{v_m}, z_1, \cdots, z_q\}$ where $\{z_1, \cdots, z_q\}$ is any basis for $W \equiv \ker\left(\phi(A)^{m-1}\right)$. Now repeat the above argument for $V_1 \equiv \ker\left(\phi(A)^{m-1}\right)$. The new U will be $\phi(A)^{m-2}(V_1)$ and the new W will be

$$\left\{x \in V_1 : \phi(A)^{m-2} x = 0\right\}.$$

Continue this way to obtain a basis for V which is of the form $\{\beta_{v_1}, \cdots, \beta_{v_s}\}$. This proves the theorem.

With this theorem it is time to establish the existence of the rational canonical form. It will be described in the following theorem. First here is a definition of the concept of a companion matrix.

Definition 10.5.6 *Let*

$$q(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

be a monic polynomial. The companion matrix of $q(\lambda)$, denoted as $C(q(\lambda))$ is the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & -a_1 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$$

Proposition 10.5.7 *Let $q(\lambda)$ be a polynomial and let $C(q(\lambda))$ be its companion matrix. Then $q(C(q(\lambda))) = 0$.*

Proof: Write C instead of $C(q(\lambda))$ for short. Note that

$$C\mathbf{e}_1 = \mathbf{e}_2, C\mathbf{e}_2 = \mathbf{e}_3, \cdots, C\mathbf{e}_{n-1} = \mathbf{e}_n$$

Thus

$$\mathbf{e}_k = C^{k-1}\mathbf{e}_1,\ k = 1, \cdots, n \tag{10.12}$$

and so it follows

$$\left\{\mathbf{e}_1, C\mathbf{e}_1, C^2\mathbf{e}_1, \cdots, C^{n-1}\mathbf{e}_1\right\} \tag{10.13}$$

are linearly independent. Hence these form a basis for $\mathbb{F}^n$. Now note that $C\mathbf{e}_n$ is given by

$$C\mathbf{e}_n = -a_0\mathbf{e}_1 - a_1\mathbf{e}_2 - \cdots - \mathbf{a}_{n-1}\mathbf{e}_n$$

and from 10.12 this implies

$$C^n\mathbf{e}_1 = -a_0\mathbf{e}_1 - a_1C\mathbf{e}_1 - \cdots - \mathbf{a}_{n-1}C^{n-1}\mathbf{e}_1$$

and so

$$q(C)\mathbf{e}_1 = \mathbf{0}.$$

Now since 10.13 is a basis, every vector of $\mathbb{F}^n$ is of the form

$$k(C)\mathbf{e}_1$$

for some polynomial $k(\lambda)$. Therefore, if $\mathbf{v} \in \mathbb{F}^n$,

$$q(C)\mathbf{v} = q(C)k(C)\mathbf{e}_1 = k(C)q(C)\mathbf{e}_1 = \mathbf{0}$$

which shows $q(C) = 0$. This proves the proposition.

The next lemma says the minimal polynomial of A restricted to

$$V_k \equiv \ker\left(\phi(A)^{m_k}\right)$$

is just $\phi(\lambda)^{m_k}$.

Lemma 10.5.8 *Let the minimal polynomial of A be $p(\lambda) = \prod_{k=1}^{q}\phi_k(\lambda)^{m_k}$ where each ϕ_k is irreducible. Let $V_k = \ker\left(\phi(A)^{m_k}\right)$. Then*

$$V_1 \oplus \cdots \oplus V_q = V$$

and letting A_k denote the restriction of A to V_k, it follows the minimal polynomial of A_k is $\phi_k(\lambda)^{m_k}$.

Proof: Recall the direct sum, $V_1 \oplus \cdots \oplus V_q = V$ where $V_k = \ker\left(\phi_k(A)^{m_k}\right)$ for $p(\lambda) = \prod_{k=1}^{q}\phi_k(\lambda)^{m_k}$ the minimal polynomial for A where the ϕ_k are all irreducible. Thus each V_k is invariant with respect to A. What is the minimal polynomial of A_k, the restriction of A to V_k? First note that $\phi_k(A_k)^{m_k}(V_k) = \{0\}$. Thus if $\eta(\lambda)$ is the minimal polynomial for A_k then it must divide $\phi_k(\lambda)^{m_k}$ and so by Corollary 8.3.9 $\eta(\lambda) = \phi_k(\lambda)^{r_k}$ where $r_k \leq m_k$. Could $r_k < m_k$? No, this is not possible because then $p(\lambda)$ would fail to be the minimal polynomial for A. You could substitute for the term $\phi_k(\lambda)^{m_k}$ in the factorization of $p(\lambda)$ with $\phi_k(\lambda)^{r_k}$ and the resulting polynomial p' would satisfy $p'(A) = 0$. Here is why. From Theorem 10.2.4, a typical $x \in V$ is of the form

$$\sum_{i=1}^{q} v_i,\ v_i \in V_i$$

Then since all the factors commute,

$$p'(A)\left(\sum_{i=1}^{q} v_i\right) = \prod_{i\neq k}^{q} \phi_i(A)^{m_i}\,\phi_k(A)^{r_k}\left(\sum_{i=1}^{q} v_i\right)$$

For $j \neq k$

$$\prod_{i\neq k}^{q} \phi_i(A)^{m_i}\,\phi_k(A)^{r_k} v_j = \prod_{i\neq k,j}^{q} \phi_i(A)^{m_i}\,\phi_k(A)^{r_k}\,\phi_j(A)^{m_j} v_j = 0$$

If $j = k$,

$$\prod_{i\neq k}^{q} \phi_i(A)^{m_i}\,\phi_k(A)^{r_k} v_k = 0$$

which shows $p'(\lambda)$ is a monic polynomial having smaller degree than $p(\lambda)$ such that $p'(A) = 0$. Thus the minimal polynomial for A_k is $\phi_k(\lambda)^{m_k}$ as claimed. This proves the lemma.

Note how elegant this is. The things you would want to have are actually valid. Now here is another little lemma.

Theorem 10.5.9 *Let $A \in \mathcal{L}(V,V)$ where V is a vector space with field of scalars $\mathbb{F}$ and minimal polynomial*

$$\prod_{i=1}^{q} \phi_i(\lambda)^{m_i}$$

where each $\phi_i(\lambda)$ is irreducible. Letting $V_k \equiv \ker\left(\phi_k(\lambda)^{m_k}\right)$, it follows

$$V = V_1 \oplus \cdots \oplus V_q$$

where each V_k is A invariant. Letting B_k denote a basis for V_k and M^k the matrix of the restriction of A to V_k, it follows that the matrix of A with respect to the basis $\{B_1, \cdots, B_q\}$ is the block diagonal matrix of the form

$$\begin{pmatrix} M^1 & & 0 \\ & \ddots & \\ 0 & & M^q \end{pmatrix} \tag{10.14}$$

If B_k is given as $\left\{\beta_{v_1}, \cdots, \beta_{v_s}\right\}$ as described in Theorem 10.5.5 where each β_{v_j} is an A cyclic set of vectors, then the matrix M^k is of the form

$$M^k = \begin{pmatrix} C\left(\phi_k(\lambda)^{r_1}\right) & & 0 \\ & \ddots & \\ 0 & & C\left(\phi_k(\lambda)^{r_s}\right) \end{pmatrix} \tag{10.15}$$

where the A cyclic sets of vectors may be arranged in order such that the positive integers r_j satisfy $r_1 \geq \cdots \geq r_s$ and $C\left(\phi_k(\lambda)^{r_j}\right)$ is the companion matrix of the polynomial $\phi_k(\lambda)^{r_j}$.

Proof: By Corollary 10.2.4 and Theorem 10.2.5 the matrix of A with respect to $\{B_1, \cdots, B_q\}$ is of the form given in 10.14. Now by Theorem 10.5.5 the basis B_k may be chosen in the form $\left\{\beta_{v_1}, \cdots, \beta_{v_s}\right\}$ where each β_{v_k} is an A cyclic set of vectors and also it can be assumed the lengths of these β_{v_k} are decreasing. Thus

$$V_k = \operatorname{span}\left(\beta_{v_1}\right) \oplus \cdots \oplus \operatorname{span}\left(\beta_{v_s}\right)$$

and it only remains to consider the matrix of A restricted to span $\left(\beta_{v_k}\right)$. Then you can apply Theorem 10.2.5 to get the result in 10.15. Say

$$\beta_{v_k} = v_k, Av_k, \cdots, A^{d-1}v_k$$

where $\eta(A)\, v_k = 0$ and the degree of $\eta(\lambda)$ is d, the smallest degree such that this is so, η being a monic polynomial. Then by Corollary 8.3.9, $\eta(\lambda) = \phi_k(\lambda)^{r_k}$ where $r_k \leq m_k$. Now

$$A\left(\text{span}\left(\beta_{v_k}\right)\right) \subseteq \text{span}\left(\beta_{v_k}\right)$$

and it remains to consider the matrix of A restricted to span $\left(\beta_{v_1}\right)$. If C is this matrix then, from the definition of the matrix of a linear transformation, if $j \leq d-1$

$$A\left(A^{j-1}v_k\right) = A^j v_k = \sum_{i=1}^{d} C_{ij} A^{i-1} v_k$$

and so $C_{ij} = 0$ unless $i = j+1$ and $C_{(j+1)j} = 1$. When $j = d$,

$$\begin{aligned} A\left(A^{d-1}v_k\right) &= A^d v_k = -a_0 v_k - a_1 A v_k - \cdots a_{d-1} A^{d-1} v_k \\ &= \sum_{i=1}^{d} C_{id} A^{i-1} v_k = \sum_{i=1}^{d} \left(-a_{i-1}\right) A^{i-1} v_k \end{aligned}$$

where

$$\phi_k(\lambda)^{r_k} \equiv a_0 + a_1\lambda + \cdots + a_{d-1}\lambda^{d-1} + \lambda^d$$

Thus the matrix C is a $d \times d$ matrix which is of the form

$$\begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & -a_1 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$$

In other words, $C = C\left(\phi_k(\lambda)^{r_k}\right)$ and so M^k has the form claimed in the theorem. This proves the theorem.

10.6 Uniqueness

Given $A \in \mathcal{L}(V, V)$ where V is a vector space having field of scalars $\mathbb{F}$, the above shows there exists a rational canonical form for A. Could A have more than one rational canonical form? Recall the definition of an A cyclic set. For convenience, here it is again.

Definition 10.6.1 *Letting $x \neq 0$ denote by β_x the vectors $\left\{x, Ax, A^2x, \cdots, A^{m-1}x\right\}$ where m is the smallest such that $A^m x \in \text{span}\left(x, \cdots, A^{m-1}x\right)$. This is called an A cyclic set, denoted by β_x.*

The following proposition ties these A cyclic sets to polynomials. It is really a condensation of ideas used above to prove existence.

Proposition 10.6.2 *Let $x \neq 0$ and consider $\left\{x, Ax, A^2x, \cdots, A^{m-1}x\right\}$. Then this is an A cyclic set if and only if there exists a monic polynomial $\eta(\lambda)$ such that $\eta(A)\, x = 0$ and among all such polynomials $\psi(\lambda)$ satisfying $\psi(A)\, x = 0$, $\eta(\lambda)$ has the smallest degree. If $V = \ker\left(\phi(\lambda)^m\right)$ where $\phi(\lambda)$ is irreducible, then for some positive integer $p \leq m, \eta(\lambda) = \phi(\lambda)^p$*

Proof: If the given set of vectors is an A cyclic set, then

$$A^m x = \sum_{k=1}^{m} -c_k A^{k-1} x$$

and so you can let

$$\eta(\lambda) \equiv \sum_{k=1}^{m} c_k \lambda^{k-1} x + \lambda^m.$$

Why is $\eta(\lambda)$ of smallest possible degree? This is because the vectors in the set are given to be linearly independent.

Now suppose there exists such a monic polynomial $\eta(\lambda)$ which satisfies $\eta(A)x = 0$ and $\eta(\lambda)$ has smallest possible degree out of all such polynomials. Then if

$$\sum_{k=1}^{m} c_k A^{k-1} x = 0$$

and not all $c_k = 0$, you could obtain a polynomial $\psi(\lambda)$ of smaller degree having the property $\psi(A)x = 0$. Just let $\psi(\lambda) = \sum_{k=1}^{m} c_k \lambda^{k-1}$.

To establish the last claim, note that $\phi(A)^m x = 0$ and you can write

$$\phi(A)^m = g(\lambda)\eta(\lambda) + r(\lambda)$$

where either $r(\lambda) = 0$ or it has smaller degree than $\eta(\lambda)$. The latter alternative does not occur because it would contradict the choice of $\eta(\lambda)$ since the above shows $r(A)x = 0$. Therefore, $\eta(\lambda)$ divides $\phi(\lambda)^m$ and so by Corollary 8.3.9, $\eta(\lambda) = \phi(\lambda)^p$ for some $p \leq m$. Since $x \neq 0, p > 0$. This proves the proposition.

Lemma 10.6.3 *Let V be a vector space and $A \in \mathcal{L}(V,V)$ has minimal polynomial $\phi(\lambda)^m$ where $\phi(\lambda)$ is irreducible and has degree d. Let the basis for V consist of $\{\beta_{v_1}, \cdots, \beta_{v_s}\}$ where β_{v_k} is A cyclic as described above and the rational canonical form for A is the matrix taken with respect to this basis. Then letting $|\beta_{v_k}|$ denote the number of vectors in β_{v_k}, it follows there is only one possible set of numbers $|\beta_{v_k}|$.*

Proof: Say β_{v_j} is associated with the polynomial $\phi(\lambda)^{p_j}$. Thus, as described above $|\beta_{v_j}|$ equals $p_j d$. Consider the following table which comes from the A cyclic set

$$\{v_j, Av_j, \cdots, A^{d-1}v_j, \cdots, A^{p_j d-1}v_j\}$$

α_0^j	α_1^j	α_2^j	$\cdots$	α_{d-1}^j
v_j	Av_j	$A^2 v_j$	$\cdots$	$A^{d-1}v_j$
$\phi(A)v_j$	$\phi(A)Av_j$	$\phi(A)A^2 v_j$	$\cdots$	$\phi(A)A^{d-1}v_j$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\phi(A)^{p_j-1}v_j$	$\phi(A)^{p_j-1}Av_j$	$\phi(A)^{p_j-1}A^2 v_j$	$\cdots$	$\phi(A)^{p_j-1}A^{d-1}v_j$

In the above, α_k^j signifies the vectors below it in the k^{th} column. None of these vectors below the top row are equal to 0 because the degree of $\phi(\lambda)^{p_j-1}\lambda^{d-1}$ is less than $p_j d$ and the smallest degree of a nonzero polynomial sending v_j to 0 is $p_j d$. Also, each of these vectors is in the span of β_{v_j} and there are dp_j of them, just as there are dp_j vectors in β_{v_j}.

Claim: The vectors $\left\{\alpha_0^j, \cdots, \alpha_{d-1}^j\right\}$ are linearly independent.

Proof of claim: Suppose

$$\sum_{i=0}^{d-1}\sum_{k=0}^{p_j-1} c_{ik}\phi(A)^k A^i v_j = 0$$

Then multiply both sides by $\phi(A)^{p_j-1}$ this yields

$$\sum_{i=0}^{d-1} c_{i0}\phi(A)^{p_j-1} A^i v_j = 0$$

Now if any of the c_{i0} is nonzero this would imply there exists a polynomial having degree smaller than $p_j d$ which sends v_j to 0. Since this does not happen, it follows each $c_{i0} = 0$. Thus

$$\sum_{i=0}^{d-1}\sum_{k=1}^{p_j-1} c_{ik}\phi(A)^k A^i v_j = 0$$

Now multiply both sides by $\phi(A)^{p_j-2}$ and do a similar argument to assert that $c_{i1} = 0$ for each i. Continuing this way, all the $c_{ik} = 0$ and this proves the claim.

Thus the vectors $\left\{\alpha_0^j, \cdots, \alpha_{d-1}^j\right\}$ are linearly independent and there are $p_j d = \left|\beta_{v_j}\right|$ of them. Therefore, they form a basis for span $\left(\beta_{v_j}\right)$. Also note that if you list the columns in reverse order starting from the bottom and going toward the top, the vectors $\left\{\alpha_0^j, \cdots, \alpha_{d-1}^j\right\}$ yield Jordan blocks in the matrix of $\phi(A)$. Hence, considering all these vectors $\left\{\alpha_0^j, \cdots, \alpha_{d-1}^j\right\}_{j=1}^{s}$ listed in the reverse order, the matrix of $\phi(A)$ with respect to this basis of V is in Jordan canonical form. See Proposition 10.3.2 and Theorem 10.3.5 on existence and uniqueness for the Jordan form. This Jordan form is unique up to order of the blocks. For a given j $\left\{\alpha_0^j, \cdots, \alpha_{d-1}^j\right\}$ yields d Jordan blocks of size p_j for $\phi(A)$. The size and number of Jordan blocks of $\phi(A)$ depends only on $\phi(A)$, hence only on A. Once A is determined, $\phi(A)$ is determined and hence the number and size of Jordan blocks is determined so the exponents p_j are determined and this shows the lengths of the $\beta_{v_j}, p_j d$ are also determined. This proves the lemma.

Note that if the p_j are known, then so is the rational canonical form because it comes from blocks which are companion matrices of the polynomials $\phi(\lambda)^{p_j}$. Now here is the main result.

Theorem 10.6.4 *Let V be a vector space having field of scalars $\mathbb{F}$ and let $A \in \mathcal{L}(V,V)$. Then the rational canonical form of A is unique up to order of the blocks.*

Proof: Let the minimal polynomial of A be $\prod_{k=1}^{q}\phi_k(\lambda)^{m_k}$. Then recall from Corollary 10.2.4

$$V = V_1 \oplus \cdots \oplus V_q$$

where $V_k = \ker\left(\phi_k(A)^{m_k}\right)$. Also recall from Lemma 10.5.8 that the mimimal polynomial of the restriction of A to V_k is $\phi_k(\lambda)^{m_k}$. Now apply Lemma 10.6.3 to A restricted to V_k. This proves the theorem.

In the case where two $n \times n$ matrices M, N are similar, recall this is equivalent to the two being matrices of the same linear transformation taken with respect to two different bases. Hence each are similar to the same rational canonical form.

Example 10.6.5 *Here is a matrix.*

$$A = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 10 & -2 \\ 9 & 0 & 9 \end{pmatrix}$$

Find a similarity transformation which will produce the rational canonical form for A.

The characteristic polynomial is $\lambda^3 - 24\lambda^2 + 180\lambda - 432$. This factors as

$$(\lambda - 6)^2 (\lambda - 12)$$

It turns out this is also the minimal polynomial. You can see this by plugging in A where you see λ and observing things don't work if you delete one of the $\lambda - 6$ factors. There is more on this in the exercises. It turns out you can compute the minimal polynomial pretty easily. Thus $\mathbb{C}^3$ is the direct sum of $\ker\left((A - 6I)^2\right)$ and $\ker(A - 12I)$. Consider the first of these. You see easily that this is

$$y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, y, z \in \mathbb{C}.$$

What about the length of A cyclic sets? It turns out it doesn't matter much. You can start with either of these and get a cycle of length 2. Lets pick the second one. This leads to the cycle

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} = A \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
$$, \begin{pmatrix} -12 \\ -48 \\ -36 \end{pmatrix} = A^2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

where the last of the three is a linear combination of the first two. Take the first two as the first two columns of S. To get the third, you need a cycle of length 1 corresponding to $\ker(A - 12I)$. This yields the eigenvector

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Thus

$$S = \begin{pmatrix} -1 & -4 & 1 \\ 0 & -4 & -2 \\ 1 & 0 & 3 \end{pmatrix}$$

Now using Proposition 9.3.8, the Rational canonical form for A should be

$$\begin{pmatrix} -1 & -4 & 1 \\ 0 & -4 & -2 \\ 1 & 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 5 & -2 & 1 \\ 2 & 10 & -2 \\ 9 & 0 & 9 \end{pmatrix} \begin{pmatrix} -1 & -4 & 1 \\ 0 & -4 & -2 \\ 1 & 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -36 & 0 \\ 1 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Example 10.6.6 *Here is a matrix.*

$$A = \begin{pmatrix} 12 & -3 & -19 & -14 & 8 \\ -4 & 1 & 1 & 6 & -4 \\ 4 & 5 & 5 & -2 & 4 \\ 0 & -5 & -5 & 2 & 0 \\ -4 & 3 & 11 & 6 & 0 \end{pmatrix}$$

Find a basis such that if S is the matrix which has these vectors as columns $S^{-1}AS$ is in rational canonical form assuming the field of scalars is $\mathbb{Q}$.

First it is necessary to find the minimal polynomial. Of course you can find the characteristic polynomial and then take away factors till you find the minimal polynomial. However, there is a much better way which is described in the exercises. Leaving out this detail, the minimal polynomial is

$$\lambda^3 - 12\lambda^2 + 64\lambda - 128$$

This polynomial factors as

$$(\lambda - 4)\left(\lambda^2 - 8\lambda + 32\right) \equiv \phi_1(\lambda)\,\phi_2(\lambda)$$

where the second factor is irreducible over $\mathbb{Q}$. Consider $\phi_2(\lambda)$ first. Messy computations yield

$$\phi_2(A) = \begin{pmatrix} -16 & -16 & -16 & -16 & -32 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 16 & 16 & 16 & 16 & 32 \end{pmatrix}$$

and so

$$\begin{aligned} \ker(\phi_2(A)) &= a\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &\quad + c\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Now start with one of these basis vectors and look for an A cycle. Picking the first one,

$$\begin{pmatrix} 12 & -3 & -19 & -14 & 8 \\ -4 & 1 & 1 & 6 & -4 \\ 4 & 5 & 5 & -2 & 4 \\ 0 & -5 & -5 & 2 & 0 \\ -4 & 3 & 11 & 6 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -15 \\ 5 \\ 1 \\ -5 \\ 7 \end{pmatrix}$$

and now

$$\begin{pmatrix} 12 & -3 & -19 & -14 & 8 \\ -4 & 1 & 1 & 6 & -4 \\ 4 & 5 & 5 & -2 & 4 \\ 0 & -5 & -5 & 2 & 0 \\ -4 & 3 & 11 & 6 & 0 \end{pmatrix}\begin{pmatrix} -15 \\ 5 \\ 1 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} -88 \\ 8 \\ 8 \\ -40 \\ 56 \end{pmatrix}$$

This yields an A cycle which is of length 2 as can be seen by making these vectors the columns and obtaining the row reduced echelon form. The span of these two cannot yield all of $\ker(\phi_2(A))$ and so look for another cycle. Begin with a vector which is not in the span of these two. The last one works well. Thus another A cycle is

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 12 & -3 & -19 & -14 & 8 \\ -4 & 1 & 1 & 6 & -4 \\ 4 & 5 & 5 & -2 & 4 \\ 0 & -5 & -5 & 2 & 0 \\ -4 & 3 & 11 & 6 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -16 \\ 4 \\ -4 \\ 0 \\ 8 \end{pmatrix}$$

It follows a basis for $\ker(\phi_2(A))$ is

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -16 \\ 4 \\ -4 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -15 \\ 5 \\ 1 \\ -5 \\ 7 \end{pmatrix}$$

From the above theory, these vectors are linearly independent. Finally consider a cycle coming from $\ker(\phi_1(A))$. This amounts to nothing more than finding an eigenvector for A corresponding to the eigenvalue 4. Such a vector is

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now the desired matrix is

$$S \equiv \begin{pmatrix} -2 & -16 & -1 & -15 & -1 \\ 0 & 4 & 1 & 5 & 0 \\ 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 1 & 8 & 0 & 7 & 1 \end{pmatrix}$$

Then

$$S^{-1}AS = \begin{pmatrix} 0 & -32 & 0 & 0 & 0 \\ 1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -32 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

and you see this is in rational canonical form, the two 2×2 blocks being companion matrices for the polynomial $\lambda^2 - 8\lambda + 32$ and the 1×1 block being a companion matrix for $\lambda - 4$.

Obviously there is a lot more which could be considered about rational canonical forms. Just begin with a strange field and start investigating what can be said. It is as far as I

feel like going on this subject at this time. One can also derive more systematic methods for finding the rational canonical form. The advantage of this is you don't need to find the eigenvalues in order to compute the rational canonical form and it can really be computed for this reason, unlike the Jordan form. This can be used to determine whether two matrices consisting of entries in some field are similar.

10.7 Exercises

1. In the argument for Theorem 10.2.4 it was shown that $m(A)\phi_l(A)v = v$ whenever $v \in \ker(\phi_k(A)^{r_k})$. Show that $m(A)$ restricted to $\ker(\phi_k(A)^{r_k})$ is the inverse of the linear transformation $\phi_l(A)$ on $\ker(\phi_k(A)^{r_k})$.

2. Suppose A is a linear transformation and let the characteristic polynomial be

$$\det(\lambda I - A) = \prod_{j=1}^{q} \phi_j(\lambda)^{n_j}$$

where the $\phi_j(\lambda)$ are irreducible. Explain using Corollary 8.3.9 why the irreducible factors of the minimal polynomial are $\phi_j(\lambda)$ and why the minimal polynomial is of the form

$$\prod_{j=1}^{q} \phi_j(\lambda)^{r_j}$$

where $r_j \leq n_j$. You can use the Cayley Hamilton theorem if you like.

3. Find the minimal polynomial for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ -3 & 2 & 1 \end{pmatrix}$$

by the above technique. Is what you found also the characteristic polynomial?

4. Show, using the rational root theorem, the minimal polynomial for A in the above problem is irreducible with respect to $\mathbb{Q}$. Letting the field of scalars be $\mathbb{Q}$ find the rational canonical form and a similarity transformation which will produce it.

5. Let $A : \mathbb{Q}^3 \to \mathbb{Q}^3$ be linear. Suppose the minimal polynomial is $(\lambda - 2)\left(\lambda^2 + 2\lambda + 7\right)$. Find the rational canonical form. Can you give generalizations of this rather simple problem to other situations?

6. Use the existence of the rational canonical form to give a proof of the Cayley Hamilton theorem valid for any field, even fields like the integers mod p for p a prime. The earlier proof was fine for fields like $\mathbb{Q}$ or $\mathbb{R}$ where you could let $\lambda \to \infty$ but it is not clear the same result holds in general. **Hint:** You might want to show by induction that the characteristic polynomial of the companion matrix of a polynomial is that polynomial.

7. Let $q(\lambda)$ be a polynomial and C its companion matrix. Show the characteristic and minimal polynomial of C are the same and both equal $q(\lambda)$. Consider the above problem again.

8. What is the form of the companion matrix of a nilpotent transformation? Recall a nilpotent transformation N satisfies $N^k = 0$ for k large enough. The companion matrix of a linear transformation A involved looking at A cyclic sets of the form

$x, Ax, \cdots, A^{k-1}x$ and these vectors are considered in that order. Turn the order around in the basis β_x writing $A^{k-1}x, A^{k-2}x, \cdots, x$. What would the companion matrix be if the A cyclic set was written in this order? Using this convention, what would you obtain for the rational canonical form? Can you use this to obtain a proof of the existence of the Jordan form starting with the existence of the rational canonical form? **Hint:** Turning the list around puts the ones on the side we like them to be on for the Jordan form.

Markov Chains And Migration Processes

11.1 Regular Markov Matrices

The existence of the Jordan form is the basis for the proof of limit theorems for certain kinds of matrices called Markov matrices.

Definition 11.1.1 *An $n \times n$ matrix, $A = (a_{ij})$, is a Markov matrix if $a_{ij} \geq 0$ for all i, j and*

$$\sum_i a_{ij} = 1.$$

It may also be called a stochastic matrix. A matrix which has nonnegative entries such that

$$\sum_j a_{ij} = 1$$

will also be called a stochastic matrix. A Markov or stochastic matrix is called regular if some power of A has all entries strictly positive. A vector, $\mathbf{v} \in \mathbb{R}^n$, is a steady state if $A\mathbf{v} = \mathbf{v}$.

Lemma 11.1.2 *The property of being a stochastic matrix is preserved by taking products.*

Proof: Suppose the sum over a row equals 1 for A and B. Then letting the entries be denoted by (a_{ij}) and (b_{ij}) respectively,

$$\sum_i \sum_k a_{ik} b_{kj} = \sum_k \left(\sum_i a_{ik} \right) b_{kj} = \sum_k b_{kj} = 1.$$

A similar argument yields the same result in the case where it is the sum over a column which is equal to 1. It is obvious that when the product is taken, if each $a_{ij}, b_{ij} \geq 0$, then the same will be true of sums of products of these numbers.

The following theorem is convenient for showing the existence of limits.

Theorem 11.1.3 *Let A be a real $p \times p$ matrix having the properties*

1. *$a_{ij} \geq 0$*
2. *Either $\sum_{i=1}^p a_{ij} = 1$ or $\sum_{j=1}^p a_{ij} = 1$.*
3. *The distinct eigenvalues of A are $\{1, \lambda_2, \ldots, \lambda_m\}$ where each $|\lambda_j| < 1$.*

Then $\lim_{n\to\infty} A^n = A_\infty$ *exists in the sense that* $\lim_{n\to\infty} a_{ij}^n = a_{ij}^\infty$, *the* ij^{th} *entry* A_∞. ***Here*** a_{ij}^n ***denotes the*** ij^{th} ***entry of*** A^n. *Also, if* $\lambda = 1$ *has algebraic multiplicity* r, *then the Jordan block corresponding to* $\lambda = 1$ *is just the* $r \times r$ *identity.*

Proof. By the existence of the Jordan form for A, it follows that there exists an invertible matrix P such that

$$P^{-1}AP = \begin{pmatrix} I+N & & & \\ & J_{r_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{r_m}(\lambda_m) \end{pmatrix} = J$$

where I is $r \times r$ for r the multiplicity of the eigenvalue 1 and N is a nilpotent matrix for which $N^r = 0$. We will show that because of Condition 2, $N = 0$.

First of all,

$$J_{r_i}(\lambda_i) = \lambda_i I + N_i$$

where N_i satisfies $N_i^{r_i} = 0$ for some $r_i > 0$. It is clear that $N_i(\lambda_i I) = (\lambda_i I) N$ and so

$$(J_{r_i}(\lambda_i))^n = \sum_{k=0}^{n} \binom{n}{k} N^k \lambda_i^{n-k} = \sum_{k=0}^{r} \binom{n}{k} N^k \lambda_i^{n-k}$$

which converges to 0 due to the assumption that $|\lambda_i| < 1$. There are finitely many terms and a typical one is a matrix whose entries are no larger than an expression of the form

$$|\lambda_i|^{n-k} C_k n(n-1)\cdots(n-k+1) \le C_k |\lambda_i|^{n-k} n^k$$

which converges to 0 because, by the root test, the series $\sum_{n=1}^{\infty} |\lambda_i|^{n-k} n^k$ converges. Thus for each $i = 2, \ldots, p$,

$$\lim_{n\to\infty} (J_{r_i}(\lambda_i))^n = 0.$$

By Condition 2, if a_{ij}^n denotes the ij^{th} entry of A^n, then either

$$\sum_{i=1}^{p} a_{ij}^n = 1 \text{ or } \sum_{j=1}^{p} a_{ij}^n = 1,\ a_{ij}^n \ge 0.$$

This follows from Lemma 11.1.2. It is obvious each $a_{ij}^n \ge 0$, and so the entries of A^n must be bounded independent of n.

It follows easily from

$$\overbrace{P^{-1}APP^{-1}APP^{-1}AP\cdots P^{-1}AP}^{n \text{ times}} = P^{-1}A^nP$$

that

$$P^{-1}A^nP = J^n \tag{11.1}$$

Hence J^n must also have bounded entries as $n \to \infty$. However, this requirement is incompatible with an assumption that $N \ne 0$.

If $N \ne 0$, then $N^s \ne 0$ but $N^{s+1} = 0$ for some $1 \le s \le r$. Then

$$(I+N)^n = I + \sum_{k=1}^{s} \binom{n}{k} N^k$$

One of the entries of N^s is nonzero by the definition of s. Let this entry be n_{ij}^s. Then this implies that one of the entries of $(I+N)^n$ is of the form $\binom{n}{s} n_{ij}^s$. This entry dominates the ij^{th} entries of $\binom{n}{k} N^k$ for all $k < s$ because

$$\lim_{n\to\infty} \binom{n}{s} / \binom{n}{k} = \infty$$

Therefore, the entries of $(I+N)^n$ cannot all be bounded. From block multiplication,

$$P^{-1}A^nP = \begin{pmatrix} (I+N)^n & & & \\ & (J_{r_2}(\lambda_2))^n & & \\ & & \ddots & \\ & & & (J_{r_m}(\lambda_m))^n \end{pmatrix}$$

and this is a contradiction because entries are bounded on the left and unbounded on the right.

Since $N = 0$, the above equation implies $\lim_{n\to\infty} A^n$ exists and equals

$$P \begin{pmatrix} I & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} P^{-1}$$

This proves the theorem.

Are there examples which will cause the eigenvalue condition of this theorem to hold? The following lemma gives such a condition. It turns out that if $a_{ij} > 0$, not just ≥ 0, then the eigenvalue condition of the above theorem is valid.

Lemma 11.1.4 *Suppose $A = (a_{ij})$ is a stochastic matrix. Then $\lambda = 1$ is an eigenvalue. If $a_{ij} > 0$ for all i, j, then if μ is an eigenvalue of A, either $|\mu| < 1$ or $\mu = 1$. In addition to this, if $A\mathbf{v} = \mathbf{v}$ for a nonzero vector, $\mathbf{v} \in \mathbb{R}^n$, then $v_j v_i \geq 0$ for all i, j so the components of $\mathbf{v}$ have the same sign.*

Proof: Suppose the matrix satisfies

$$\sum_j a_{ij} = 1.$$

Then if $\mathbf{v} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$, it is obvious that $A\mathbf{v} = \mathbf{v}$. Therefore, this matrix has $\lambda = 1$ as an eigenvalue. Suppose then that μ is an eigenvalue. Is $|\mu| < 1$ or $\mu = 1$? Let $\mathbf{v}$ be an eigenvector and let $|v_i|$ be the largest of the $|v_j|$.

$$\mu v_i = \sum_j a_{ij} v_j$$

and now multiply both sides by $\overline{\mu v_i}$ to obtain

$$\begin{aligned} |\mu|^2 |v_i|^2 &= \sum_j a_{ij} v_j \overline{v_i \mu} = \sum_j a_{ij} \operatorname{Re}(v_j \overline{v_i \mu}) \\ &\leq \sum_j a_{ij} |\mu| |v_i|^2 = |\mu| |v_i|^2 \end{aligned}$$

Therefore, $|\mu| \leq 1$. If $|\mu| = 1$, then equality must hold in the above, and so $v_j\overline{v_i\mu}$ must be real and nonnegative for each j. In particular, this holds for $j = 1$ which shows $\overline{\mu}$ and hence μ are real. Thus, in this case, $\mu = 1$. The only other case is where $|\mu| < 1$.

If instead, $\sum_i a_{ij} = 1$, consider A^T. Both A and A^T have the same characteristic polynomial and so their eigenvalues are exactly the same. This proves the lemma.

Lemma 11.1.5 *Let A be any Markov matrix and let $\mathbf{v}$ be a vector having all its components non negative with $\sum_i v_i = c$. Then if $\mathbf{w} = A\mathbf{v}$, it follows that $w_i \geq 0$ for all i and $\sum_i w_i = c$.*

Proof: From the definition of $\mathbf{w}$,

$$w_i \equiv \sum_j a_{ij}v_j \geq 0.$$

Also

$$\sum_i w_i = \sum_i \sum_j a_{ij}v_j = \sum_j \sum_i a_{ij}v_j = \sum_j v_j = c.$$

The following theorem about limits is now easy to obtain.

Theorem 11.1.6 *Suppose A is a Markov matrix (The sum over a column equals 1) in which $a_{ij} > 0$ for all i, j and suppose $\mathbf{w}$ is a vector. Then for each i,*

$$\lim_{k\to\infty} \left(A^k\mathbf{w}\right)_i = v_i$$

where $A\mathbf{v} = \mathbf{v}$. In words, $A^k\mathbf{w}$ always converges to a steady state. In addition to this, if the vector, $\mathbf{w}$ satisfies $w_i \geq 0$ for all i and $\sum_i w_i = c$, then the vector $\mathbf{v}$ will also satisfy the conditions, $v_i \geq 0$, $\sum_i v_i = c$.

Proof: By Lemma 11.1.4, since each $a_{ij} > 0$, the eigenvalues are either 1 or have absolute value less than 1. Therefore, the claimed limit exists by Theorem 11.1.3. The assertion that the components are nonnegative and sum to c follows from Lemma 11.1.5. That $A\mathbf{v} = \mathbf{v}$ follows from

$$\mathbf{v} = \lim_{n\to\infty} A^n\mathbf{w} = \lim_{n\to\infty} A^{n+1}\mathbf{w} = A\lim_{n\to\infty} A^n\mathbf{w} = A\mathbf{v}.$$

This proves the theorem.

It is not hard to generalize the conclusion of this theorem to regular Markov processes.

Corollary 11.1.7 *Suppose A is a regular Markov matrix and suppose $\mathbf{w}$ is a vector. Then for each i,*

$$\lim_{n\to\infty} \left(A^n\mathbf{w}\right)_i = v_i$$

where $A\mathbf{v} = \mathbf{v}$. In words, $A^n\mathbf{w}$ always converges to a steady state. In addition to this, if the vector, $\mathbf{w}$ satisfies $w_i \geq 0$ for all i and $\sum_i w_i = c$, Then the vector, $\mathbf{v}$ will also satisfy the conditions, $v_i \geq 0$, $\sum_i v_i = c$.

Proof: Let the entries of A^k be all positive. Now suppose that $a_{ij} \geq 0$ for all i, j and $A = (a_{ij})$ is a transition matrix. Then if $B = (b_{ij})$ is a transition matrix with $b_{ij} > 0$ for all ij, it follows that BA is a transition matrix which has strictly positive entries. The ij^{th} entry of BA is

$$\sum_k b_{ik}a_{kj} > 0,$$

Thus, from Lemma 11.1.4, A^k has an eigenvalue equal to 1 for all k sufficiently large, and all the other eigenvalues have absolute value strictly less than 1. The same must be true of A, for if λ is an eigenvalue of A with $|\lambda| = 1$, then λ^k is an eigenvalue for A^k and so, for all k large enough, $\lambda^k = 1$ which is absurd unless $\lambda = 1$. By Theorem 11.1.3, $\lim_{n\to\infty} A^n\mathbf{w}$ exists. The rest follows as in Theorem 11.1.6. This proves the corollary.

11.2 Migration Matrices

Definition 11.2.1 *Let n locations be denoted by the numbers $1, 2, \cdots, n$. Also suppose it is the case that each year a_{ij} denotes the proportion of residents in location j which move to location i. Also suppose no one escapes or emigrates from without these n locations. This last assumption requires $\sum_i a_{ij} = 1$. Thus (a_{ij}) is a Markov matrix referred to as a migration matrix.*

If $\mathbf{v} = (x_1, \cdots, x_n)^T$ where x_i is the population of location i at a given instant, you obtain the population of location i one year later by computing $\sum_j a_{ij}x_j = (A\mathbf{v})_i$. Therefore, the population of location i after k years is $\left(A^k\mathbf{v}\right)_i$. Furthermore, Corollary 11.1.7 can be used to predict in the case where A is regular what the long time population will be for the given locations.

As an example of the above, consider the case where $n = 3$ and the migration matrix is of the form

$$\begin{pmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{pmatrix}.$$

Now

$$\begin{pmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{pmatrix}^2 = \begin{pmatrix} .38 & .02 & .15 \\ .28 & .64 & .02 \\ .34 & .34 & .83 \end{pmatrix}$$

and so the Markov matrix is regular. Therefore, $\left(A^k\mathbf{v}\right)_i$ will converge to the i^{th} component of a steady state. It follows the steady state can be obtained from solving the system

$$\begin{gathered} .6x + .1z = x \\ .2x + .8y = y \\ .2x + .2y + .9z = z \end{gathered}$$

along with the stipulation that the sum of x, y, and z must equal the constant value present at the beginning of the process. The solution to this system is

$$\{y = x, z = 4x, x = x\}.$$

If the total population at the beginning is 150,000, then you solve the following system

$$\begin{gathered} y = x \\ z = 4x \\ x + y + z = 150000 \end{gathered}$$

whose solution is easily seen to be $\{x = 25\,000, z = 100\,000, y = 25\,000\}$. Thus, after a long time there would be about four times as many people in the third location as in either of the other two.

11.3 Markov Chains

A random variable is just a function which can have certain values which have probabilities associated with them. Thus it makes sense to consider the probability that the random variable has a certain value or is in some set. The idea of a Markov chain is a sequence of random variables, $\{X_n\}$ which can be in any of a collection of states which can be labeled with nonnegative integers. Thus you can speak of the probability the random variable, X_n

is in state i. The probability that X_{n+1} is in state j given that X_n is in state i is called a one step transition probability. When this probability does not depend on n it is called stationary and this is the case of interest here. Since this probability does not depend on n it can be denoted by p_{ij}. Here is a simple example called a random walk.

Example 11.3.1 *Let there be n points, x_i, and consider a process of something moving randomly from one point to another. Suppose X_n is a sequence of random variables which has values $\{1, 2, \cdots, n\}$ where $X_n = i$ indicates the process has arrived at the i^{th} point. Let p_{ij} be the probability that X_{n+1} has the value j given that X_n has the value i. Since X_{n+1} must have some value, it must be the case that $\sum_j a_{ij} = 1$. Note this says that the sum over a row equals 1 and so the situation is a little different than the above in which the sum was over a column.*

As an example, let x_1, x_2, x_3, x_4 be four points taken in order on $\mathbb{R}$ and suppose x_1 and x_4 are absorbing. This means that $p_{4k} = 0$ for all $k \neq 4$ and $p_{1k} = 0$ for all $k \neq 1$. Otherwise, you can move either to the left or to the right with probability $\frac{1}{2}$. The Markov matrix associated with this situation is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 11.3.2 *Let the stationary transition probabilities, p_{ij} be defined above. The resulting matrix having p_{ij} as its ij^{th} entry is called the matrix of transition probabilities. The sequence of random variables for which these p_{ij} are the transition probabilities is called a Markov chain. The matrix of transition probabilities is called a stochastic matrix.*

The next proposition is fundamental and shows the significance of the powers of the matrix of transition probabilities.

Proposition 11.3.3 *Let p_{ij}^n denote the probability that X_n is in state j given that X_0 was in state i. Then p_{ij}^n is the ij^{th} entry of the matrix, P^n where $P = (p_{ij})$.*

Proof: This is clearly true if $n = 1$ and follows from the definition of the p_{ij}. Suppose true for n. Then the probability that X_{n+1} is at j given that X_0 was at i equals $\sum_k p_{ik}^n p_{kj}$ because X_n must have some value, k, and so this represents all possible ways to go from i to j. You can go from i to 1 in n steps with probability p_{i1} and then from 1 to j in one step with probability p_{1j} and so the probability of this is $p_{i1}^n p_{1j}$ but you can also go from i to 2 and then from 2 to j and from i to 3 and then from 3 to j etc. Thus the sum of these is just what is given and represents the probability of X_{n+1} having the value j given X_0 has the value i.

In the above random walk example, lets take a power of the transition probability matrix to determine what happens. Rounding off to two decimal places,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{20} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .67 & 9.5 \times 10^{-7} & 0 & .33 \\ .33 & 0 & 9.5 \times 10^{-7} & .67 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus p_{21} is about 2/3 while p_{32} is about 1/3 and terms like p_{22} are very small. You see this seems to be converging to the matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

After many iterations of the process, if you start at 2 you will end up at 1 with probability 2/3 and at 4 with probability 1/3. This makes good intuitive sense because it is twice as far from 2 to 4 as it is from 2 to 1.

Theorem 11.3.4 *The eigenvalues of*

$$\begin{pmatrix} 0 & p & 0 & \cdots & 0 \\ q & 0 & p & \cdots & 0 \\ 0 & q & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & p \\ 0 & \vdots & 0 & q & 0 \end{pmatrix}$$

have absolute value less than 1. Here $p+q=1$ *and both* $p,q>0$.

Proof: By Gerschgorin's theorem, if λ is an eigenvalue, then $|\lambda| \leq 1$. Now suppose $\mathbf{v}$ is an eigenvector for λ. Then

$$A\mathbf{v} = \begin{pmatrix} pv_2 \\ qv_1 + pv_3 \\ \vdots \\ qv_{n-2} + pv_n \\ qv_{n-1} \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$$

Suppose $|\lambda| = 1$. Then the top row shows $p|v_2| = |v_1|$ so $|v_1| < |v_2|$. Suppose $|v_1| < |v_2| < \cdots < |v_k|$ for some $k < n$. Then

$$|\lambda v_k| = |v_k| \leq q|v_{k-1}| + p|v_{k+1}| < q|v_k| + p|v_{k+1}|$$

and so subtracting $q|v_k|$ from both sides,

$$p|v_k| < p|v_{k+1}|$$

showing $\{|v_k|\}_{k=1}^n$ is an increasing sequence. Now a contradiction results on the last line which requires $|v_{n-1}| > |v_n|$. Therefore, $|\lambda| < 1$ for any eigenvalue of the above matrix and this proves the theorem.

Corollary 11.3.5 *Let* p,q *be positive numbers and let* $p+q=1$. *The eigenvalues of*

$$\begin{pmatrix} a & p & 0 & \cdots & 0 \\ q & a & p & \cdots & 0 \\ 0 & q & a & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & p \\ 0 & \vdots & 0 & q & a \end{pmatrix}$$

are all strictly closer than 1 to a. *That is, whenever* λ *is an eigenvalue,*

$$|\lambda - a| < 1$$

have absolute value less than 1.

Proof: Let A be the above matrix and suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then letting A' denote

$$\begin{pmatrix} 0 & p & 0 & \cdots & 0 \\ q & & p & \cdots & 0 \\ 0 & q & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & p \\ 0 & \vdots & 0 & q & 0 \end{pmatrix},$$

it follows

$$A'\mathbf{x} = (\lambda - a)\,\mathbf{x}$$

and so from the above theorem,

$$|\lambda - a| < 1.$$

This proves the corollary.

Example 11.3.6 *In the gambler's ruin problem a gambler plays a game with someone, say a casino, until he either wins all the other's money or loses all of his own. A simple version of this is as follows. Let X_k denote the amount of money the gambler has. Each time the game is played he wins with probability $p \in (0,1)$ or loses with probability $(1-p) \equiv q$. In case he wins, his money increases to $X_k + 1$ and if he loses, his money decreases to $X_k - 1$.*

The transition probability matrix, P, describing this situation is as follows.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & \vdots \\ 0 & 0 & q & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & p & 0 \\ 0 & 0 & \vdots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.2}$$

Here the matrix is $b+1 \times b+1$ because the possible values of X_k are all integers from 0 up to b. The 1 in the upper left corner corresponds to the gampler's ruin. It involves $X_k = 0$ so he has no money left. Once this state has been reached, it is not possible to ever leave it. This is indicated by the row of zeros to the right of this entry the k^{th} of which gives the probability of going from state 1 corresponding to no money to state k[1].

In this case 1 is a repeated root of the characteristic equation of multiplicity 2 and all the other eigenvalues have absolute value less than 1. To see that this is the case, note that the characteristic polynomial is of the form

$$(1-\lambda)^2 \det \begin{pmatrix} -\lambda & p & 0 & \cdots & 0 \\ q & -\lambda & p & \cdots & 0 \\ 0 & q & -\lambda & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & p \\ 0 & \vdots & 0 & q & -\lambda \end{pmatrix}$$

[1]No one will give the gambler money. This is why the only reasonable number for entries in this row to the right of 1 is 0.

and the factor after $(1-\lambda)^2$ has zeros which are in absolute value less than 1. Its zeros are the eigenvalues of the matrix

$$A \equiv \begin{pmatrix} 0 & p & 0 & \cdots & 0 \\ q & 0 & p & \cdots & 0 \\ 0 & q & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & p \\ 0 & \vdots & 0 & q & 0 \end{pmatrix}$$

and by Corollary 11.3.5 these all have absolute value less than 1.

Therefore, by Theorem 11.1.3 $\lim_{n\to\infty} P^n$ exists. The case of $\lim_{n\to\infty} p^n_{j0}$ is particularly interesting because it gives the probability that, starting with an amount j, the gambler eventually ends up at 0 and is **ruined**. From the matrix, it follows

$$\begin{aligned} p^n_{j0} &= qp^{n-1}_{(j-1)0} + pp^{n-1}_{(j+1)0} \text{ for } j \in [1, b-1], \\ p^n_{00} &= 1, \text{ and } p^n_{b0} = 0. \end{aligned}$$

To simplify the notation, define $P_j \equiv \lim_{n\to\infty} p^n_{j0}$ as the probability of ruin given the initial fortune of the gambler equals j. Then the above simplifies to

$$\begin{aligned} P_j &= qP_{j-1} + pP_{j+1} \text{ for } j \in [1, b-1], \\ P_0 &= 1, \text{ and } P_b = 0. \end{aligned} \tag{11.3}$$

Now, knowing that P_j exists, it is not too hard to find it from 11.3. This equation is called a difference equation and there is a standard procedure for finding solutions of these. You try a solution of the form $P_j = x^j$ and then try to find x such that things work out. Therefore, substitute this in to the first equation of 11.3 and obtain

$$x^j = qx^{j-1} + px^{j+1}.$$

Therefore,

$$px^2 - x + q = 0$$

and so in case $p \neq q$, you can use the fact that $p + q = 1$ to obtain

$$\begin{aligned} x &= \frac{1}{2p}\left(1 + \sqrt{(1-4pq)}\right) \text{ or } \frac{1}{2p}\left(1 - \sqrt{(1-4pq)}\right) \\ &= \frac{1}{2p}\left(1 + \sqrt{(1-4p(1-p))}\right) \text{ or } \frac{1}{2p}\left(1 - \sqrt{(1-4p(1-p))}\right) \\ &= 1 \text{ or } \frac{q}{p}. \end{aligned}$$

Now it follows that both $P_j = 1$ and $P_j = \left(\frac{q}{p}\right)^j$ satisfy the difference equation 11.3. Therefore, anything of the form

$$\alpha + \beta\left(\frac{q}{p}\right)^j \tag{11.4}$$

will satisfy this equation. Find a, b such that this also satisfies the second equation of 11.3. Thus it is required that

$$\alpha + \beta = 1, \ \alpha + \beta\left(\frac{q}{p}\right)^b = 0$$

and so

$$\begin{aligned}\alpha+\beta&=1\\ \alpha+\beta\left(\tfrac{q}{p}\right)^b&=0\end{aligned}$$

Solution is : $\left\{\beta=-\frac{1}{-1+\left(\frac{q}{p}\right)^b},\alpha=\frac{\left(\frac{q}{p}\right)^b}{-1+\left(\frac{q}{p}\right)^b}\right\}$. Substituting this in to 11.4 and simplifying, yields the following in the case that $p\neq q$.

$$P_j=\frac{p^{b-j}q^j-q^b}{p^b-q^b} \tag{11.5}$$

Note that

$$\lim_{p\to q}\frac{p^{b-j}q^j-q^b}{p^b-q^b}=\frac{b-j}{b}.$$

Thus as the game becomes more fair in the sense the probabilities of winning become closer to 1/2, the probability of ruin given an initial amount j is $\frac{b-j}{b}$.

Alternatively, you could consider the difference equation directly in the case where $p=q=1/2$. In this case, you can see that two solutions to the difference equation

$$\begin{aligned}P_j&=\frac{1}{2}P_{j-1}+\frac{1}{2}P_{j+1}\text{ for }j\in[1,b-1],\\ P_0&=1,\text{ and }P_b=0.\end{aligned} \tag{11.6}$$

are $P_j=1$ and $P_j=j$. This leads to a solution to the above of

$$P_j=\frac{b-j}{b}. \tag{11.7}$$

This last case is pretty interesting because it shows, for example that if the gambler starts with a fortune of 1 so that he starts at state $j=1$, then his probability of losing all is $\frac{b-1}{b}$ which might be quite large, especially if the other player has a lot of money to begin with. As the gambler starts with more and more money, his probability of losing everything does decrease.

11.4 Exercises

1. ♠Suppose the migration matrix for three locations is

$$\begin{pmatrix}.5&0&.3\\.3&.8&0\\.2&.2&.7\end{pmatrix}.$$

   Find a comparison for the populations in the three locations after a long time.

2. ♠Show that if $\sum_i a_{ij}=1$, then if $A=(a_{ij})$, then the sum of the entries of $A\mathbf{v}$ equals the sum of the entries of $\mathbf{v}$. Thus it does not matter whether $a_{ij}\geq 0$ for this to be so.

3. ♠If A satisfies the conditions of the above problem, can it be concluded that $\lim_{n\to\infty}A^n$ exists?

4. ♠Give an example of a non regular Markov matrix which has an eigenvalue equal to -1.

5. ♠Show that when a Markov matrix is non defective, all of the above theory can be proved very easily. In particular, prove the theorem about the existence of $\lim_{n\to\infty} A^n$ if the eigenvalues are either 1 or have absolute value less than 1.

6. ♠Find a formula for A^n where

$$A = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} & 0 & -1 \\ 5 & 0 & 0 & -4 \\ \frac{7}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ \frac{7}{2} & -\frac{1}{2} & 0 & -2 \end{pmatrix}$$

Does $\lim_{n\to\infty} A^n$ exist? Note that all the rows sum to 1. **Hint:** This matrix is similar to a diagonal matrix. The eigenvalues are $1, -1, \frac{1}{2}, \frac{1}{2}$.

7. ♠Find a formula for A^n where

$$A = \begin{pmatrix} 2 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 4 & 0 & 1 & -4 \\ \frac{5}{2} & -\frac{1}{2} & 1 & -2 \\ 3 & -\frac{1}{2} & \frac{1}{2} & -2 \end{pmatrix}$$

Note that the rows sum to 1 in this matrix also. **Hint:** This matrix is not similar to a diagonal matrix but you can find the Jordan form and consider this in order to obtain a formula for this product. The eigenvalues are $1, -1, \frac{1}{2}, \frac{1}{2}$.

8. ♠Find $\lim_{n\to\infty} A^n$ if it exists for the matrix

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}$$

The eigenvalues are $\frac{1}{2}, 1, 1, 1$.

9. ♠Given an example of a matrix A which has eigenvalues which are either equal to 1 or have absolute value strictly less than 1 but which has the property that $\lim_{n\to\infty} A^n$ does not exist.

10. ♠If A is an $n \times n$ matrix such that all the eigenvalues have absolute value less than 1, show $\lim_{n\to\infty} A^n = 0$.

11. ♠Find an example of a 3×3 matrix A such that $\lim_{n\to\infty} A^n$ does not exist but $\lim_{r\to\infty} A^{5r}$ does exist.

12. ♠If A is a Markov matrix and B is similar to A, does it follow that B is also a Markov matrix?

13. ♠In Theorem 11.1.3 suppose everything is unchanged except that you assume either $\sum_j a_{ij} \leq 1$ or $\sum_i a_{ij} \leq 1$. Would the same conclusion be valid? What if you don't insist that each $a_{ij} \geq 0$? Would the conclusion hold in this case?

Inner Product Spaces

12.1 General Theory

The usual example of an inner product space is $\mathbb{C}^n$ or $\mathbb{R}^n$ with the dot product. However, there are many other inner product spaces and the topic is of such importance that it seems appropriate to discuss the general theory of these spaces.

Definition 12.1.1 *A vector space X is said to be a normed linear space if there exists a function, denoted by $|\cdot| : X \to [0,\infty)$ which satisfies the following axioms.*

1. $|x| \geq 0$ *for all* $x \in X$, *and* $|x| = 0$ *if and only if* $x = 0$.
2. $|ax| = |a|\,|x|$ *for all* $a \in \mathbb{F}$.
3. $|x+y| \leq |x| + |y|$.

This function $|\cdot|$ is called a norm.

The notation $||x||$ is also often used. Not all norms are created equal. There are many geometric properties which they may or may not possess. There is also a concept called an inner product which is discussed next. It turns out that the best norms come from an inner product.

Definition 12.1.2 *A mapping $(\cdot,\cdot) : V \times V \to \mathbb{F}$ is called an inner product, also denoted by the usual dot product notation if it satisfies the following axioms.*

1. $(x,y) = \overline{(y,x)}$.
2. $(x,x) \geq 0$ *for all* $x \in V$ *and equals zero if and only if* $x = 0$.
3. $(ax+by,z) = a\,(x,z) + b\,(y,z)$ *whenever* $a, b \in \mathbb{F}$.

Note that 2 and 3 imply $(x, ay+bz) = \overline{a}(x,y) + \overline{b}(x,z)$.
Then a norm is given by

$$(x,x)^{1/2} \equiv |x|.$$

It remains to verify this really is a norm.

Definition 12.1.3 *A normed linear space in which the norm comes from an inner product as just described is called an inner product space.*

Example 12.1.4 *Let $V = \mathbb{C}^n$ with the inner product given by*

$$(\mathbf{x}, \mathbf{y}) \equiv \sum_{k=1}^{n} x_k \overline{y}_k.$$

This is an example of a complex inner product space already discussed.

Example 12.1.5 *Let $V = \mathbb{R}^n$,*

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \equiv \sum_{j=1}^{n} x_j y_j.$$

This is an example of a real inner product space.

Example 12.1.6 *Let V be any finite dimensional vector space and let $\{v_1, \cdots, v_n\}$ be a basis. Decree that*

$$(v_i, v_j) \equiv \delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

and define the inner product by

$$(x, y) \equiv \sum_{i=1}^{n} x^i \overline{y^i}$$

where

$$x = \sum_{i=1}^{n} x^i v_i, \ y = \sum_{i=1}^{n} y^i v_i.$$

The above is well defined because $\{v_1, \cdots, v_n\}$ is a basis. Thus the components, x_i associated with any given $x \in V$ are uniquely determined.

This example shows there is no loss of generality when studying finite dimensional vector spaces in assuming the vector space is actually an inner product space. The following theorem was presented earlier with slightly different notation.

Theorem 12.1.7 *(Cauchy Schwarz) In any inner product space*

$$|(x, y)| \leq |x||y|.$$

where $|x| \equiv (x, x)^{1/2}$.

Proof: Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and $\overline{\omega}(x, y) = |(x, y)| = \mathrm{Re}(x, y\omega)$. Let

$$F(t) = (x + ty\omega, x + t\omega y).$$

If $y = 0$ there is nothing to prove because

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0)$$

and so $(x, 0) = 0$. Thus, there is no loss of generality in assuming $y \neq 0$. Then from the axioms of the inner product,

$$F(t) = |x|^2 + 2t\,\mathrm{Re}(x, \omega y) + t^2|y|^2 \geq 0.$$

This yields

$$|x|^2 + 2t|(x, y)| + t^2|y|^2 \geq 0.$$

Since this inequality holds for all $t \in \mathbb{R}$, it follows from the quadratic formula that

$$4|(x,y)|^2 - 4|x|^2|y|^2 \leq 0.$$

This yields the conclusion and proves the theorem.

Earlier it was claimed that the inner product defines a norm. In this next proposition this claim is proved.

Proposition 12.1.8 *For an inner product space, $|x| \equiv (x,x)^{1/2}$ does specify a norm.*

Proof: All the axioms are obvious except the triangle inequality. To verify this,

$$\begin{aligned} |x+y|^2 &\equiv (x+y, x+y) \equiv |x|^2 + |y|^2 + 2\,\mathrm{Re}\,(x,y) \\ &\leq |x|^2 + |y|^2 + 2\,|(x,y)| \\ &\leq |x|^2 + |y|^2 + 2\,|x|\,|y| = (|x| + |y|)^2. \end{aligned}$$

The best norms of all are those which come from an inner product because of the following identity which is known as the parallelogram identity.

Proposition 12.1.9 *If $(V, (\cdot,\cdot))$ is an inner product space then for $|x| \equiv (x,x)^{1/2}$, the following identity holds.*

$$|x+y|^2 + |x-y|^2 = 2\,|x|^2 + 2\,|y|^2\,.$$

It turns out that the validity of this identity is equivalent to the existence of an inner product which determines the norm as described above. These sorts of considerations are topics for more advanced courses on functional analysis.

Definition 12.1.10 *A basis for an inner product space, $\{u_1, \cdots, u_n\}$ is an orthonormal basis if*

$$(u_k, u_j) = \delta_{kj} \equiv \begin{cases} 1 \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases}.$$

Note that if a list of vectors satisfies the above condition for being an orthonormal set, then the list of vectors is automatically linearly independent. To see this, suppose

$$\sum_{j=1}^{n} c^j u_j = 0$$

Then taking the inner product of both sides with u_k,

$$0 = \sum_{j=1}^{n} c^j\,(u_j, u_k) = \sum_{j=1}^{n} c^j \delta_{jk} = c^k.$$

12.2 The Gramm Schmidt Process

Lemma 12.2.1 *Let X be a finite dimensional inner product space of dimension n whose basis is $\{x_1, \cdots, x_n\}$. Then there exists an orthonormal basis for X, $\{u_1, \cdots, u_n\}$ which has the property that for each $k \leq n$, span$(x_1, \cdots, x_k) = \mathrm{span}\,(u_1, \cdots, u_k)$.*

Proof: Let $\{x_1, \cdots, x_n\}$ be a basis for X. Let $u_1 \equiv x_1/|x_1|$. Thus for $k = 1$, $\text{span}(u_1) = \text{span}(x_1)$ and $\{u_1\}$ is an orthonormal set. Now suppose for some $k < n$, $u_1, \cdots, u_k$ have been chosen such that $(u_j, u_l) = \delta_{jl}$ and $\text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$. Then define

$$u_{k+1} \equiv \frac{x_{k+1} - \sum_{j=1}^{k} (x_{k+1}, u_j)\, u_j}{\left| x_{k+1} - \sum_{j=1}^{k} (x_{k+1}, u_j)\, u_j \right|}, \tag{12.1}$$

where the denominator is not equal to zero because the x_j form a basis and so

$$x_{k+1} \notin \text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$$

Thus by induction,

$$u_{k+1} \in \text{span}(u_1, \cdots, u_k, x_{k+1}) = \text{span}(x_1, \cdots, x_k, x_{k+1}).$$

Also, $x_{k+1} \in \text{span}(u_1, \cdots, u_k, u_{k+1})$ which is seen easily by solving 12.1 for x_{k+1} and it follows

$$\text{span}(x_1, \cdots, x_k, x_{k+1}) = \text{span}(u_1, \cdots, u_k, u_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
(u_{k+1}, u_l) &= C\left((x_{k+1}, u_l) - \sum_{j=1}^{k} (x_{k+1}, u_j)\, (u_j, u_l) \right) \\
&= C\left((x_{k+1}, u_l) - \sum_{j=1}^{k} (x_{k+1}, u_j)\, \delta_{lj} \right) \\
&= C\left((x_{k+1}, u_l) - (x_{k+1}, u_l) \right) = 0.
\end{aligned}$$

The vectors, $\{u_j\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process.

The following corollary is obtained from the above process.

Corollary 12.2.2 *Let X be a finite dimensional inner product space of dimension n whose basis is $\{u_1, \cdots, u_k, x_{k+1}, \cdots, x_n\}$. Then if $\{u_1, \cdots, u_k\}$ is orthonormal, then the Gram Schmidt process applied to the given list of vectors in order leaves $\{u_1, \cdots, u_k\}$ unchanged.*

Lemma 12.2.3 *Suppose $\{u_j\}_{j=1}^{n}$ is an orthonormal basis for an inner product space X. Then for all $x \in X$,*

$$x = \sum_{j=1}^{n} (x, u_j)\, u_j.$$

Proof: By assumption that this is an orthonormal basis,

$$\sum_{j=1}^{n} (x, u_j) \overbrace{(u_j, u_l)}^{\delta_{jl}} = (x, u_l).$$

Letting $y = \sum_{k=1}^{n} (x, u_k)\, u_k$, it follows

$$\begin{aligned}
(x - y, u_j) &= (x, u_j) - \sum_{k=1}^{n} (x, u_k)\, (u_k, u_j) \\
&= (x, u_j) - (x, u_j) = 0
\end{aligned}$$

for all j. Hence, for any choice of scalars, $c^1, \cdots, c^n$,

$$\left(x - y, \sum_{j=1}^{n} c^j u_j \right) = 0$$

and so $(x-y,z)=0$ for all $z \in X$. Thus this holds in particular for $z = x - y$. Therefore, $x = y$ and this proves the theorem.

The following theorem is of fundamental importance. First note that a subspace of an inner product space is also an inner product space because you can use the same inner product.

Theorem 12.2.4 *Let M be a subspace of X, a finite dimensional inner product space and let $\{x_i\}_{i=1}^m$ be an orthonormal basis for M. Then if $y \in X$ and $w \in M$,*

$$|y-w|^2 = \inf \left\{ |y-z|^2 : z \in M \right\} \tag{12.2}$$

if and only if

$$(y-w,z) = 0 \tag{12.3}$$

for all $z \in M$. Furthermore,

$$w = \sum_{i=1}^{m} (y, x_i)\, x_i \tag{12.4}$$

is the unique element of M which has this property.

Proof: Let $t \in \mathbb{R}$. Then from the properties of the inner product,

$$|y - (w + t(z-w))|^2 = |y-w|^2 + 2t \operatorname{Re}(y-w, w-z) + t^2 |z-w|^2 . \tag{12.5}$$

If $(y-w,z)=0$ for all $z \in M$, then letting $t=1$, the middle term in the above expression vanishes and so $|y-z|^2$ is minimized when $z=w$.

Conversely, if 12.2 holds, then the middle term of 12.5 must also vanish since otherwise, you could choose small real t such that

$$|y-w|^2 > |y - (w + t(z-w))|^2 .$$

Here is why. If $\operatorname{Re}(y-w, w-z) < 0$, then let t be very small and positive. The middle term in 12.5 will then be more negative than the last term is positive and the right side of this formula will then be less than $|y-w|^2$. If $\operatorname{Re}(y-w, w-z) > 0$ then choose t small and negative to achieve the same result.

It follows, letting $z_1 = w - z$ that

$$\operatorname{Re}(y-w, z_1) = 0$$

for all $z_1 \in M$. Now letting $\omega \in \mathbb{C}$ be such that $\omega(y-w,z_1) = |(y-w,z_1)|$,

$$|(y-w,z_1)| = (y-w, \overline{\omega} z_1) = \operatorname{Re}(y-w, \overline{\omega} z_1) = 0,$$

which proves the first part of the theorem since z_1 is arbitrary.

It only remains to verify that w given in 12.4 satisfies 12.3 and is the only point of M which does so. To do this, note that if c_i, d_i are scalars, then the properties of the inner product and the fact the $\{x_i\}$ are orthonormal implies

$$\left(\sum_{i=1}^{m} c_i x_i, \sum_{j=1}^{m} d_j x_j \right) = \sum_i c_i \overline{d_i}.$$

By Lemma 12.2.3,

$$z = \sum_i (z, x_i)\, x_i$$

and so

$$\begin{aligned}\left(y - \sum_{i=1}^m (y, x_i)\, x_i, z\right) &= \left(y - \sum_{i=1}^m (y, x_i)\, x_i, \sum_{i=1}^m (z, x_i)\, x_i\right) \\ &= \sum_{i=1}^m \overline{(z, x_i)}\, (y, x_i) - \left(\sum_{i=1}^m (y, x_i)\, x_i, \sum_{j=1}^m (z, x_j)\, x_j\right) \\ &= \sum_{i=1}^m \overline{(z, x_i)}\, (y, x_i) - \sum_{i=1}^m (y, x_i)\, \overline{(z, x_i)} = 0.\end{aligned}$$

This shows w given in 12.4 does minimize the function, $z \to |y - z|^2$ for $z \in M$. It only remains to verify uniqueness. Suppose than that $w_i, i = 1, 2$ minimizes this function of z for $z \in M$. Then from what was shown above,

$$\begin{aligned}|y - w_1|^2 &= |y - w_2 + w_2 - w_1|^2 \\ &= |y - w_2|^2 + 2\,\mathrm{Re}\,(y - w_2, w_2 - w_1) + |w_2 - w_1|^2 \\ &= |y - w_2|^2 + |w_2 - w_1|^2 \leq |y - w_2|^2,\end{aligned}$$

the last equal sign holding because w_2 is a minimizer and the last inequality holding because w_1 minimizes.

12.3 Riesz Representation Theorem

The next theorem is one of the most important results in the theory of inner product spaces. It is called the Riesz representation theorem.

Theorem 12.3.1 *Let $f \in \mathcal{L}(X, \mathbb{F})$ where X is an inner product space of dimension n. Then there exists a unique $z \in X$ such that for all $x \in X$,*

$$f(x) = (x, z).$$

Proof: First I will verify uniqueness. Suppose z_j works for $j = 1, 2$. Then for all $x \in X$,

$$0 = f(x) - f(x) = (x, z_1 - z_2)$$

and so $z_1 = z_2$.

It remains to verify existence. By Lemma 12.2.1, there exists an orthonormal basis, $\{u_j\}_{j=1}^n$. Define

$$z \equiv \sum_{j=1}^n \overline{f(u_j)} u_j.$$

Then using Lemma 12.2.3,

$$\begin{aligned}(x, z) &= \left(x, \sum_{j=1}^n \overline{f(u_j)} u_j\right) = \sum_{j=1}^n f(u_j)\, (x, u_j) \\ &= f\left(\sum_{j=1}^n (x, u_j)\, u_j\right) = f(x).\end{aligned}$$

This proves the theorem.

Corollary 12.3.2 *Let $A \in \mathcal{L}(X,Y)$ where X and Y are two inner product spaces of finite dimension. Then there exists a unique $A^* \in \mathcal{L}(Y,X)$ such that*

$$(Ax,y)_Y = (x,A^*y)_X \tag{12.6}$$

for all $x \in X$ and $y \in Y$. The following formula holds

$$(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$$

Proof: Let $f_y \in \mathcal{L}(X,\mathbb{F})$ be defined as

$$f_y(x) \equiv (Ax,y)_Y .$$

Then by the Riesz representation theorem, there exists a unique element of X, $A^*(y)$ such that

$$(Ax,y)_Y = (x,A^*(y))_X .$$

It only remains to verify that A^* is linear. Let a and b be scalars. Then for all $x \in X$,

$$(x,A^*(ay_1+by_2))_X \equiv (Ax,(ay_1+by_2))_Y$$

$$\equiv \overline{a}(Ax,y_1) + \overline{b}(Ax,y_2) \equiv$$

$$\overline{a}(x,A^*(y_1)) + \overline{b}(x,A^*(y_2)) = (x,aA^*(y_1)+bA^*(y_2)) .$$

Since this holds for every x, it follows

$$A^*(ay_1+by_2) = aA^*(y_1) + bA^*(y_2)$$

which shows A^* is linear as claimed.

Consider the last assertion that * is conjugate linear.

$$\left(x,(\alpha A+\beta B)^* y\right) \equiv ((\alpha A+\beta B)x,y)$$

$$\begin{aligned} &= \alpha(Ax,y) + \beta(Bx,y) = \alpha(x,A^*y) + \beta(x,B^*y) \\ &= (x,\overline{\alpha}A^*y) + \left(x,\overline{\beta}A^*y\right) = \left(x,\left(\overline{\alpha}A^* + \overline{\beta}A^*\right)y\right) . \end{aligned}$$

Since x is arbitrary,

$$(\alpha A+\beta B)^* y = \left(\overline{\alpha}A^* + \overline{\beta}A^*\right)y$$

and since this is true for all y,

$$(\alpha A+\beta B)^* = \overline{\alpha}A^* + \overline{\beta}A^* .$$

This proves the corollary.

Definition 12.3.3 *The linear map, A^* is called the adjoint of A. In the case when $A : X \to X$ and $A = A^*$, A is called a self adjoint map.*

Theorem 12.3.4 *Let M be an $m \times n$ matrix. Then $M^* = \left(\overline{M}\right)^T$ in words, the transpose of the conjugate of M is equal to the adjoint.*

Proof: Using the definition of the inner product in $\mathbb{C}^n$,

$$(M\mathbf{x},\mathbf{y}) = (\mathbf{x},M^*\mathbf{y}) \equiv \sum_i x_i \overline{\sum_j (M^*)_{ij}\, y_j} = \sum_{i,j} \overline{(M^*)_{ij}}\, \overline{y_j} x_i.$$

Also

$$(M\mathbf{x},\mathbf{y}) = \sum_j \sum_i M_{ji} \overline{y_j} x_i.$$

Since $\mathbf{x},\mathbf{y}$ are arbitrary vectors, it follows that $M_{ji} = \overline{(M^*)_{ij}}$ and so, taking conjugates of both sides,

$$M^*_{ij} = \overline{M_{ji}}$$

which gives the conclusion of the theorem.

The next theorem is interesting. You have a p dimensional subspace of $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Of course this might be "slanted". However, there is a linear transformation Q which preserves distances which maps this subspace to $\mathbb{F}^p$.

Theorem 12.3.5 *Suppose V is a subspace of $\mathbb{F}^n$ having dimension $p \leq n$. Then there exists a $Q \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ such that*

$$QV \subseteq \operatorname{span}(\mathbf{e}_1, \cdots, \mathbf{e}_p)$$

and $|Q\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x}$. Also

$$Q^*Q = QQ^* = I.$$

Proof: By Lemma 12.2.1 there exists an orthonormal basis for $V, \{\mathbf{v}_i\}_{i=1}^p$. By using the Gram Schmidt process this may be extended to an orthonormal basis of the whole space, $\mathbb{F}^n$,

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_p, \mathbf{v}_{p+1}, \cdots, \mathbf{v}_n\}.$$

Now define $Q \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ by $Q(\mathbf{v}_i) \equiv \mathbf{e}_i$ and extend linearly. If $\sum_{i=1}^n x_i \mathbf{v}_i$ is an arbitrary element of $\mathbb{F}^n$,

$$\left| Q\left(\sum_{i=1}^n x_i \mathbf{v}_i \right) \right|^2 = \left| \sum_{i=1}^n x_i \mathbf{e}_i \right|^2 = \sum_{i=1}^n |x_i|^2 = \left| \sum_{i=1}^n x_i \mathbf{v}_i \right|^2.$$

It remains to verify that $Q^*Q = QQ^* = I$. To do so, let $\mathbf{x},\mathbf{y} \in \mathbb{F}^n$. Then

$$(Q(\mathbf{x}+\mathbf{y}), Q(\mathbf{x}+\mathbf{y})) = (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}).$$

Thus

$$|Q\mathbf{x}|^2 + |Q\mathbf{y}|^2 + 2\operatorname{Re}(Q\mathbf{x},Q\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\operatorname{Re}(\mathbf{x},\mathbf{y})$$

and since Q preserves norms, it follows that for all $\mathbf{x},\mathbf{y} \in \mathbb{F}^n$,

$$\operatorname{Re}(Q\mathbf{x},Q\mathbf{y}) = \operatorname{Re}(\mathbf{x},Q^*Q\mathbf{y}) = \operatorname{Re}(\mathbf{x},\mathbf{y}).$$

Thus

$$\operatorname{Re}(\mathbf{x},Q^*Q\mathbf{y} - \mathbf{y}) = 0 \tag{12.7}$$

for all $\mathbf{x},\mathbf{y}$. Let ω be a complex number such that $|\omega| = 1$ and

$$\omega(\mathbf{x},Q^*Q\mathbf{y} - \mathbf{y}) = |(\mathbf{x},Q^*Q\mathbf{y} - \mathbf{y})|.$$

Then from 12.7,

$$\begin{aligned} 0 &= \operatorname{Re}(\omega\mathbf{x}, Q^*Q\mathbf{y} - \mathbf{y}) = \operatorname{Re}\omega(\mathbf{x},Q^*Q\mathbf{y} - \mathbf{y}) \\ &= |(\mathbf{x},Q^*Q\mathbf{y} - \mathbf{y})| \end{aligned}$$

and since $\mathbf{x}$ is arbitrary, it follows that for all $\mathbf{y}$,

$$Q^*Q\mathbf{y} - \mathbf{y} = \mathbf{0}$$

Thus

$$I = Q^*Q.$$

Similarly $QQ^* = I$ and this proves the theorem.

12.4 The Tensor Product Of Two Vectors

Definition 12.4.1 *Let X and Y be inner product spaces and let $x \in X$ and $y \in Y$. Define the tensor product of these two vectors, $y \otimes x$, an element of $\mathcal{L}(X,Y)$ by*

$$y \otimes x\,(u) \equiv y\,(u,x)_X\,.$$

This is also called a rank one transformation because the image of this transformation is contained in the span of the vector, y.

The verification that this is a linear map is left to you. Be sure to verify this! The following lemma has some of the most important properties of this linear transformation.

Lemma 12.4.2 *Let X, Y, Z be inner product spaces. Then for α a scalar,*

$$(\alpha\,(y \otimes x))^* = \overline{\alpha}x \otimes y \tag{12.8}$$

$$(z \otimes y_1)\,(y_2 \otimes x) = (y_2, y_1)\,z \otimes x \tag{12.9}$$

Proof: Let $u \in X$ and $v \in Y$. Then

$$(\alpha\,(y \otimes x)\,u, v) = (\alpha\,(u,x)\,y, v) = \alpha\,(u,x)\,(y,v)$$

and

$$(u, \overline{\alpha}x \otimes y\,(v)) = (u, \overline{\alpha}\,(v,y)\,x) = \alpha\,(y,v)\,(u,x)\,.$$

Therefore, this verifies 12.8.

To verify 12.9, let $u \in X$.

$$(z \otimes y_1)\,(y_2 \otimes x)\,(u) = (u,x)\,(z \otimes y_1)\,(y_2) = (u,x)\,(y_2,y_1)\,z$$

and

$$(y_2,y_1)\,z \otimes x\,(u) = (y_2,y_1)\,(u,x)\,z.$$

Since the two linear transformations on both sides of 12.9 give the same answer for every $u \in X$, it follows the two transformations are the same. This proves the lemma.

Definition 12.4.3 *Let X, Y be two vector spaces. Then define for $A, B \in \mathcal{L}(X,Y)$ and $\alpha \in \mathbb{F}$, new elements of $\mathcal{L}(X,Y)$ denoted by $A+B$ and αA as follows.*

$$(A+B)\,(x) \equiv Ax + Bx,\ (\alpha A)\,x \equiv \alpha\,(Ax)\,.$$

Theorem 12.4.4 *Let X and Y be finite dimensional inner product spaces. Then $\mathcal{L}(X,Y)$ is a vector space with the above definition of what it means to multiply by a scalar and add. Let $\{v_1, \cdots, v_n\}$ be an orthonormal basis for X and $\{w_1, \cdots, w_m\}$ be an orthonormal basis for Y. Then a basis for $\mathcal{L}(X,Y)$ is*

$$\{w_j \otimes v_i : i = 1, \cdots, n,\ j = 1, \cdots, m\}\,.$$

Proof: It is obvious that $\mathcal{L}(X,Y)$ is a vector space. It remains to verify the given set is a basis. Consider the following:

$$\left(\left(A-\sum_{k,l}\left(Av_k,w_l\right)w_l\otimes v_k\right)v_p,w_r\right)=\left(Av_p,w_r\right)-$$

$$\sum_{k,l}\left(Av_k,w_l\right)\left(v_p,v_k\right)\left(w_l,w_r\right)$$

$$=\left(Av_p,w_r\right)-\sum_{k,l}\left(Av_k,w_l\right)\delta_{pk}\delta_{rl}$$

$$=\left(Av_p,w_r\right)-\left(Av_p,w_r\right)=0.$$

Letting $A-\sum_{k,l}\left(Av_k,w_l\right)w_l\otimes v_k=B$, this shows that $Bv_p=0$ since w_r is an arbitrary element of the basis for Y. Since v_p is an arbitrary element of the basis for X, it follows $B=0$ as hoped. This has shown $\{w_j\otimes v_i : i=1,\cdots,n,\ j=1,\cdots,m\}$ spans $\mathcal{L}(X,Y)$.

It only remains to verify the $w_j\otimes v_i$ are linearly independent. Suppose then that

$$\sum_{i,j}c_{ij}w_j\otimes v_i=0$$

Then do both sides to v_s. By definition this gives

$$0=\sum_{i,j}c_{ij}w_j\left(v_s,v_i\right)=\sum_{i,j}c_{ij}w_j\delta_{si}=\sum_j c_{sj}w_j$$

Now the vectors $\{w_1,\cdots,w_m\}$ are independent because it is an orthonormal set and so the above requires $c_{sj}=0$ for each j. Since s was arbitrary, this shows the linear transformations, $\{w_j\otimes v_i\}$ form a linearly independent set. This proves the theorem.

Note this shows the dimension of $\mathcal{L}(X,Y)=nm$. The theorem is also of enormous importance because it shows you can always consider an arbitrary linear transformation as a sum of rank one transformations whose properties are easily understood. The following theorem is also of great interest.

Theorem 12.4.5 *Let $A=\sum_{i,j}c_{ij}w_i\otimes v_j\in\mathcal{L}(X,Y)$ where as before, the vectors, $\{w_i\}$ are an orthonormal basis for Y and the vectors, $\{v_j\}$ are an orthonormal basis for X. Then if the matrix of A has entries M_{ij}, it follows that $M_{ij}=c_{ij}$.*

Proof: Recall

$$Av_i\equiv\sum_k M_{ki}w_k$$

Also

$$\begin{aligned}Av_i &= \sum_{k,j}c_{kj}w_k\otimes v_j\left(v_i\right)\\ &= \sum_{k,j}c_{kj}w_k\left(v_i,v_j\right)\\ &= \sum_{k,j}c_{kj}w_k\delta_{ij}=\sum_k c_{ki}w_k\end{aligned}$$

Therefore,

$$\sum_k M_{ki}w_k=\sum_k c_{ki}w_k$$

and so $M_{ki}=c_{ki}$ for all k. This happens for each i. Therefore, this proves the theorem.

12.5 Least Squares

A common problem in experimental work is to find a straight line which approximates as well as possible a collection of points in the plane $\{(x_i, y_i)\}_{i=1}^p$. The usual way of dealing with these problems is by the method of least squares and it turns out that all these sorts of approximation problems can be reduced to $A\mathbf{x} = \mathbf{b}$ where the problem is to find the best $\mathbf{x}$ for solving this equation even when there is no solution.

Lemma 12.5.1 *Let V and W be finite dimensional inner product spaces and let $A : V \to W$ be linear. For each $y \in W$ there exists $x \in V$ such that*

$$|Ax - y| \leq |Ax_1 - y|$$

for all $x_1 \in V$. Also, $x \in V$ is a solution to this minimization problem if and only if x is a solution to the equation, $A^*Ax = A^*y$.

Proof: By Theorem 12.2.4 on Page 299 there exists a point, Ax_0, in the finite dimensional subspace, $A(V)$, of W such that for all $x \in V, |Ax - y|^2 \geq |Ax_0 - y|^2$. Also, from this theorem, this happens if and only if $Ax_0 - y$ is perpendicular to every $Ax \in A(V)$. Therefore, the solution is characterized by $(Ax_0 - y, Ax) = 0$ for all $x \in V$ which is the same as saying $(A^*Ax_0 - A^*y, x) = 0$ for all $x \in V$. In other words the solution is obtained by solving $A^*Ax_0 = A^*y$ for x_0. This proves the lemma.

Consider the problem of finding the least squares regression line in statistics. Suppose you have given points in the plane, $\{(x_i, y_i)\}_{i=1}^n$ and you would like to find constants m and b such that the line $y = mx + b$ goes through all these points. Of course this will be impossible in general. Therefore, try to find m, b such that you do the best you can to solve the system

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix}$$

which is of the form $\mathbf{y} = A\mathbf{x}$. In other words try to make $\left| A \begin{pmatrix} m \\ b \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right|^2$ as small as possible. According to what was just shown, it is desired to solve the following for m and b.

$$A^*A \begin{pmatrix} m \\ b \end{pmatrix} = A^* \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Since $A^* = A^T$ in this case,

$$\begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

Solving this system of equations for m and b,

$$m = \frac{-\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) + \left(\sum_{i=1}^n x_i y_i\right) n}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}$$

and

$$b = \frac{-\left(\sum_{i=1}^n x_i\right)\sum_{i=1}^n x_i y_i + \left(\sum_{i=1}^n y_i\right)\sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}.$$

One could clearly do a least squares fit for curves of the form $y = ax^2 + bx + c$ in the same way. In this case you solve as well as possible for a, b, and c the system

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

using the same techniques.

12.6 Fredholm Alternative Again

The best context in which to study the Fredholm alternative is in inner product spaces. This is done here.

Definition 12.6.1 *Let S be a subset of an inner product space, X. Define*

$$S^{\perp} \equiv \{x \in X : (x, s) = 0 \text{ for all } s \in S\}.$$

The following theorem also follows from the above lemma. It is sometimes called the Fredholm alternative.

Theorem 12.6.2 *Let $A : V \to W$ where A is linear and V and W are inner product spaces. Then $A(V) = \ker(A^*)^{\perp}$.*

Proof: Let $y = Ax$ so $y \in A(V)$. Then if $A^*z = 0$,

$$(y, z) = (Ax, z) = (x, A^*z) = 0$$

showing that $y \in \ker(A^*)^{\perp}$. Thus $A(V) \subseteq \ker(A^*)^{\perp}$.

Now suppose $y \in \ker(A^*)^{\perp}$. Does there exists x such that $Ax = y$? Since this might not be immediately clear, take the least squares solution to the problem. Thus let x be a solution to $A^*Ax = A^*y$. It follows $A^*(y - Ax) = 0$ and so $y - Ax \in \ker(A^*)$ which implies from the assumption about y that $(y - Ax, y) = 0$. Also, since Ax is the closest point to y in $A(V)$, Theorem 12.2.4 on Page 299 implies that $(y - Ax, Ax_1) = 0$ for all $x_1 \in V$. In particular this is true for $x_1 = x$ and so $0 = (y - Ax, y) - \overbrace{(y - Ax, Ax)}^{=0} = |y - Ax|^2$, showing that $y = Ax$. Thus $A(V) \supseteq \ker(A^*)^{\perp}$ and this proves the Theorem.

Corollary 12.6.3 *Let A, V, and W be as described above. If the only solution to $A^*y = 0$ is $y = 0$, then A is onto W.*

Proof: If the only solution to $A^*y = 0$ is $y = 0$, then $\ker(A^*) = \{0\}$ and so every vector from W is contained in $\ker(A^*)^{\perp}$ and by the above theorem, this shows $A(V) = W$.

12.7 Exercises

1. ♠Find the best solution to the system

$$\begin{aligned} x + 2y &= 6 \\ 2x - y &= 5 \\ 3x + 2y &= 0 \end{aligned}$$

2. ♠Find an orthonormal basis for $\mathbb{R}^3$, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ given that $\mathbf{w}_1$ is a multiple of the vector $(1, 1, 2)$.

3. ♠Suppose $A = A^T$ is a symmetric real $n \times n$ matrix which has all positive eigenvalues. Define
$$(\mathbf{x}, \mathbf{y}) \equiv (A\mathbf{x}, \mathbf{y}) .$$
Show this is an inner product on $\mathbb{R}^n$. What does the Cauchy Schwarz inequality say in this case?

4. ♠Let
$$||\mathbf{x}||_\infty \equiv \max \{|x_j| : j = 1, 2, \cdots, n\} .$$
Show this is a norm on $\mathbb{C}^n$. Here $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$. Show
$$||\mathbf{x}||_\infty \leq |\mathbf{x}| \equiv (\mathbf{x}, \mathbf{x})^{1/2}$$
where the above is the usual inner product on $\mathbb{C}^n$.

5. ♠Let
$$||\mathbf{x}||_1 \equiv \sum_{j=1}^{n} |x_j| .$$
Show this is a norm on $\mathbb{C}^n$. Here $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$. Show
$$||\mathbf{x}||_1 \geq |\mathbf{x}| \equiv (\mathbf{x}, \mathbf{x})^{1/2}$$
where the above is the usual inner product on $\mathbb{C}^n$.

6. ♠Show that if $||\cdot||$ is any norm on any vector space, then
$$|||x|| - ||y||| \leq ||x - y|| .$$

7. ♠Relax the assumptions in the axioms for the inner product. Change the axiom about $(x, x) \geq 0$ and equals 0 if and only if $x = 0$ to simply read $(x, x) \geq 0$. Show the Cauchy Schwarz inequality still holds in the following form.
$$|(x, y)| \leq (x, x)^{1/2} (y, y)^{1/2} .$$

8. ♠Let H be an inner product space and let $\{u_k\}_{k=1}^n$ be an orthonormal basis for H. Show
$$(x, y) = \sum_{k=1}^{n} (x, u_k) \overline{(y, u_k)}.$$

9. ♠Let the vector space V consist of real polynomials of degree no larger than 3. Thus a typical vector is a polynomial of the form
$$a + bx + cx^2 + dx^3.$$
For $p, q \in V$ define the inner product,
$$(p, q) \equiv \int_0^1 p(x)\, q(x)\, dx.$$
Show this is indeed an inner product. Then state the Cauchy Schwarz inequality in terms of this inner product. Show $\left\{1, x, x^2, x^3\right\}$ is a basis for V. Finally, find an orthonormal basis for V. This is an example of some orthonormal polynomials.

10. ♠Let P_n denote the polynomials of degree no larger than $n-1$ which are defined on an interval $[a,b]$. Let $\{x_1, \cdots, x_n\}$ be n distinct points in $[a,b]$. Now define for $p, q \in P_n$,

$$(p,q) \equiv \sum_{j=1}^{n} p(x_j)\,\overline{q(x_j)}$$

Show this yields an inner product on P_n. **Hint:** Most of the axioms are obvious. The one which says $(p,p) = 0$ if and only if $p = 0$ is the only interesting one. To verify this one, note that a nonzero polynomial of degree no more than $n-1$ has at most $n-1$ zeros.

11. ♠Let $C([0,1])$ denote the vector space of continuous real valued functions defined on $[0,1]$. Let the inner product be given as

$$(f,g) \equiv \int_0^1 f(x)\,g(x)\,dx$$

Show this is an inner product. Also let V be the subspace described in Problem 9. Using the result of this problem, find the vector in V which is closest to x^4.

12. A **regular Sturm Liouville problem** involves the differential equation, for an unknown function of x which is denoted here by y,

$$(p(x)\,y')' + (\lambda q(x) + r(x))\,y = 0,\ x \in [a,b]$$

and it is assumed that $p(t), q(t) > 0$ for any $t \in [a,b]$ and also there are boundary conditions,

$$\begin{aligned} C_1 y(a) + C_2 y'(a) &= 0 \\ C_3 y(b) + C_4 y'(b) &= 0 \end{aligned}$$

where

$$C_1^2 + C_2^2 > 0, \text{ and } C_3^2 + C_4^2 > 0.$$

There is an immense theory connected to these important problems. The constant, λ is called an eigenvalue. Show that if y is a solution to the above problem corresponding to $\lambda = \lambda_1$ and if z is a solution corresponding to $\lambda = \lambda_2 \neq \lambda_1$, then

$$\int_a^b q(x)\,y(x)\,z(x)\,dx = 0. \tag{12.10}$$

and this defines an inner product. **Hint:** Do something like this:

$$\begin{aligned} (p(x)\,y')'\,z + (\lambda_1 q(x) + r(x))\,yz &= 0, \\ (p(x)\,z')'\,y + (\lambda_2 q(x) + r(x))\,zy &= 0. \end{aligned}$$

Now subtract and either use integration by parts or show

$$(p(x)\,y')'\,z - (p(x)\,z')'\,y = ((p(x)\,y')\,z - (p(x)\,z')\,y)'$$

and then integrate. Use the boundary conditions to show that $y'(a)\,z(a) - z'(a)\,y(a) = 0$ and $y'(b)\,z(b) - z'(b)\,y(b) = 0$. The formula, 12.10 is called an orthogonality relation. It turns out there are typically infinitely many eigenvalues and it is interesting to write given functions as an infinite series of these "eigenfunctions".

13. ♠Consider the continuous functions defined on $[0,\pi]$, $C([0,\pi])$. Show

$$(f,g) \equiv \int_0^\pi fg dx$$

is an inner product on this vector space. Show the functions $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ are an orthonormal set. What does this mean about the dimension of the vector space $C([0,\pi])$? Now let $V_N = \text{span}\left(\sqrt{\frac{2}{\pi}}\sin(x), \cdots, \sqrt{\frac{2}{\pi}}\sin(Nx)\right)$. For $f \in C([0,\pi])$ find a formula for the vector in V_N which is closest to f with respect to the norm determined from the above inner product. This is called the N^{th} partial sum of the Fourier series of f. An important problem is to determine whether and in what way this Fourier series converges to the function f. The norm which comes from this inner product is sometimes called the mean square norm.

14. Consider the subspace $V \equiv \ker(A)$ where

$$A = \begin{pmatrix} 1 & 4 & -1 & -1 \\ 2 & 1 & 2 & 3 \\ 4 & 9 & 0 & 1 \\ 5 & 6 & 3 & 4 \end{pmatrix}$$

Find an orthonormal basis for V. **Hint:** You might first find a basis and then use the Gram Schmidt procedure.

15. ♠The Gram Schmidt process starts with a basis for a subspace $\{v_1, \cdots, v_n\}$ and produces an orthonormal basis for the same subspace $\{u_1, \cdots, u_n\}$ such that

$$\text{span}(v_1, \cdots, v_k) = \text{span}(u_1, \cdots, u_k)$$

for each k. Show that in the case of $\mathbb{R}^m$ the QR factorization does the same thing. More specifically, if

$$A = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

and if

$$A = QR \equiv \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{pmatrix} R$$

then the vectors $\{\mathbf{q}_1, \cdots, \mathbf{q}_n\}$ is an orthonormal set of vectors and for each k,

$$\text{span}(\mathbf{q}_1, \cdots, \mathbf{q}_k) = \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_k)$$

16. ♠Let V be a subspace of a finite dimensional inner product space H. Also let P denote the map which send every vector of H to its closest point in V. Show that P is linear and also

$$|Px - Py| \leq |x - y|.$$

17. ♠Verify the parallelogram identify for any inner product space,

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

Why is it called the parallelogram identity?

18. ♠Let H be an inner product space and let $K \subseteq H$ be a nonempty convex subset. This means that if $k_1, k_2 \in K$, then the line segment consisting of points of the form

$$tk_1 + (1-t)k_2 \text{ for } t \in [0,1]$$

is also contained in K. Suppose for each $x \in H$, there exists Px defined to be a point of K closest to x. Show that Px is unique so that P actually is a map. **Hint:** Suppose z_1 and z_2 both work as closest points. Consider the midpoint, $(z_1 + z_2)/2$ and use the parallelogram identity of Problem 17 in an auspicious manner.

19. ♠In the situation of Problem 18 suppose K is a closed convex subset and that H is complete. This means every Cauchy sequence converges. Recall from calculus a sequence $\{k_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N_ε such that whenever $m, n > N_\varepsilon$, it follows $|k_m - k_n| < \varepsilon$. Let $\{k_n\}$ be a sequence of points of K such that
$$\lim_{n\to\infty} |x - k_n| = \inf\{|x - k| : k \in K\}$$
This is called a minimizing sequence. Show there exists a unique $k \in K$ such that $\lim_{n\to\infty} |k_n - k|$ and that $k = Px$. That is, there exists a well defined projection map onto the convex subset of H. **Hint:** Use the parallelogram identity in an auspicious manner to show $\{k_n\}$ is a Cauchy sequence which must therefore converge. Since K is closed it follows this will converge to something in K which is the desired vector.

20. ♠Let H be an inner product space which is also complete and let P denote the projection map onto a convex closed subset, K. Show this projection map is characterized by the inequality
$$\mathrm{Re}\,(k - Px, x - Px) \leq 0$$
for all $k \in K$. That is, a point $z \in K$ equals Px if and only if the above variational inequality holds. This is what that inequality is called. This is because k is allowed to vary and the inequality continues to hold for all $k \in K$.

21. ♠Using Problem 20 and Problems 18 - 19 show the projection map, P onto a closed convex subset is Lipschitz continuous with Lipschitz constant 1. That is
$$|Px - Py| \leq |x - y|$$

22. Give an example of two vectors in $\mathbb{R}^4$ $\mathbf{x}, \mathbf{y}$ and a subspace V such that $\mathbf{x} \cdot \mathbf{y} = 0$ but $P\mathbf{x} \cdot P\mathbf{y} \neq 0$ where P denotes the projection map which sends $\mathbf{x}$ to its closest point on V.

23. ♠Suppose you are given the data, $(1,2), (2,4), (3,8), (0,0)$. Find the linear regression line using the formulas derived above. Then graph the given data along with your regression line.

24. ♠Generalize the least squares procedure to the situation in which data is given and you desire to fit it with an expression of the form $y = af(x) + bg(x) + c$ where the problem would be to find a, b and c in order to minimize the error. Could this be generalized to higher dimensions? How about more functions?

25. Let $A \in \mathcal{L}(X, Y)$ where X and Y are finite dimensional vector spaces with the dimension of X equal to n. Define $\mathrm{rank}(A) \equiv \dim(A(X))$ and $\mathrm{nullity}(A) \equiv \dim(\ker(A))$. Show that $\mathrm{nullity}(A) + \mathrm{rank}(A) = \dim(X)$. **Hint:** Let $\{x_i\}_{i=1}^r$ be a basis for $\ker(A)$ and let $\{x_i\}_{i=1}^r \cup \{y_i\}_{i=1}^{n-r}$ be a basis for X. Then show that $\{Ay_i\}_{i=1}^{n-r}$ is linearly independent and spans AX.

26. ♠Let A be an $m \times n$ matrix. Show the column rank of A equals the column rank of A^*A. Next verify column rank of A^*A is no larger than column rank of A^*. Next

justify the following inequality to conclude the column rank of A equals the column rank of A^*.

$$\text{rank } (A) = \text{rank } (A^*A) \leq \text{rank } (A^*) \leq$$

$$= \text{rank } (AA^*) \leq \text{rank } (A).$$

Hint: Start with an orthonormal basis, $\{A\mathbf{x}_j\}_{j=1}^r$ of $A(\mathbb{F}^n)$ and verify $\{A^*A\mathbf{x}_j\}_{j=1}^r$ is a basis for $A^*A(\mathbb{F}^n)$.

27. ♠Let A be a real $m \times n$ matrix and let $A = QR$ be the QR factorization with Q orthogonal and R upper triangular. Show that there exists a solution $\mathbf{x}$ to the equation

$$R^T R\mathbf{x} = R^T Q^T \mathbf{b}$$

and that this solution is also a least squares solution defined above such that $A^T A\mathbf{x} = A^T\mathbf{b}$.

12.8 The Determinant And Volume

The determinant is the essential algebraic tool which provides a way to give a unified treatment of the concept of p dimensional volume of a parallelepiped in $\mathbb{R}^M$. Here is the definition of what is meant by such a thing.

Definition 12.8.1 *Let $\mathbf{u}_1, \cdots, \mathbf{u}_p$ be vectors in $\mathbb{R}^M, M \geq p$. The parallelepiped determined by these vectors will be denoted by $P(\mathbf{u}_1, \cdots, \mathbf{u}_p)$ and it is defined as*

$$P(\mathbf{u}_1, \cdots, \mathbf{u}_p) \equiv \left\{ \sum_{j=1}^p s_j \mathbf{u}_j : s_j \in [0,1] \right\}.$$

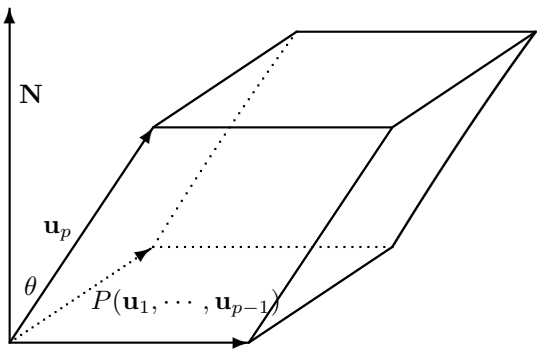
The volume of this parallelepiped is defined as

$$\textit{volume of } P(\mathbf{u}_1, \cdots, \mathbf{u}_p) \equiv v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p)) \equiv (\det(\mathbf{u}_i \cdot \mathbf{u}_j))^{1/2}.$$

If the vectors are dependent, this definition will give the volume to be 0.

First lets observe the last assertion is true. Say $\mathbf{u}_i = \sum_{j \neq i} \alpha_j \mathbf{u}_j$. Then the i^{th} row is a linear combination of the other rows and so from the properties of the determinant, the determinant of this matrix is indeed zero as it should be.

A parallelepiped is a sort of a squashed box. Here is a picture which shows the relationship between $P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})$ and $P(\mathbf{u}_1, \cdots, \mathbf{u}_p)$.



In a sense, we can define the volume any way we want but if it is to be reasonable, the following relationship must hold. The appropriate definition of the volume of $P(\mathbf{u}_1, \cdots, \mathbf{u}_p)$ in terms of $P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})$ is

$$v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p)) = |\mathbf{u}_p| \, |\cos(\theta)| \, v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})) \tag{12.11}$$

In the case where $p = 1$, the parallelepiped $P(\mathbf{v})$ consists of the single vector and the one dimensional volume should be $|\mathbf{v}| = \left(\mathbf{v}^T\mathbf{v}\right)^{1/2}$. Now having made this definition, I will show that this is the appropriate definition of p dimensional volume for every p.

Definition 12.8.2 *Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ be vectors. Then*

$$v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p)) \equiv$$

$$\equiv \det\left(\begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{pmatrix}\right)^{1/2}$$

As just pointed out, this is the only reasonable definition of volume in the case of one vector. The next theorem shows that it is the only reasonable definition of volume of a parallelepiped in the case of p vectors because 12.11 holds.

Theorem 12.8.3 *With the above definition of volume, 12.11 holds.*

Proof: To check whether this is so, it is necessary to find $|\cos(\theta)|$. This involves finding the vector perpendicular to $P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})$. Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_p\}$ be an orthonormal basis for $\text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_p)$ such that $\text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ for each $k \leq p$. Such an orthonormal basis exists because of the Gramm Schmidt procedure. First note that since $\{\mathbf{w}_k\}$ is an orthonormal basis for $\text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_p)$,

$$\mathbf{u}_j = \sum_{k=1}^{p} (\mathbf{u}_j \cdot \mathbf{w}_k) \, \mathbf{w}_k$$

and if $i, j \leq k$

$$\mathbf{u}_j \cdot \mathbf{u}_i = \sum_{k=1}^{k} (\mathbf{u}_j \cdot \mathbf{w}_k) \, (\mathbf{u}_i \cdot \mathbf{w}_k)$$

Therefore, for each $k \leq p$

$$\det\left(\begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_k^T \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}\right)$$

is the determinant of a matrix whose ij^{th} entry is

$$\mathbf{u}_i^T \mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j = \sum_{r=1}^{k} (\mathbf{u}_i \cdot \mathbf{w}_r) \, (\mathbf{w}_r \cdot \mathbf{u}_j)$$

Thus this matrix is the product of the two $k \times k$ matrices, one which is the transpose of the other.

$$\begin{pmatrix} (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_1 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_1 \cdot \mathbf{w}_k) \\ (\mathbf{u}_2 \cdot \mathbf{w}_1) & (\mathbf{u}_2 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_2 \cdot \mathbf{w}_k) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_k \cdot \mathbf{w}_1) & (\mathbf{u}_k \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_k) \end{pmatrix}.$$
$$\begin{pmatrix} (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_2 \cdot \mathbf{w}_1) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_1) \\ (\mathbf{u}_1 \cdot \mathbf{w}_2) & (\mathbf{u}_2 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_2) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_1 \cdot \mathbf{w}_k) & (\mathbf{u}_2 \cdot \mathbf{w}_k) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_k) \end{pmatrix}$$

It follows

$$\det\left(\begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_k^T \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}\right)$$
$$= \left(\det\begin{pmatrix} (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_1 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_1 \cdot \mathbf{w}_k) \\ (\mathbf{u}_2 \cdot \mathbf{w}_1) & (\mathbf{u}_2 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_2 \cdot \mathbf{w}_k) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_k \cdot \mathbf{w}_1) & (\mathbf{u}_k \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_k) \end{pmatrix}\right)^2$$

and so from the definition,

$$v\left(P\left(\mathbf{u}_1, \cdots, \mathbf{u}_k\right)\right) =$$
$$\left|\det\begin{pmatrix} (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_1 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_1 \cdot \mathbf{w}_k) \\ (\mathbf{u}_2 \cdot \mathbf{w}_1) & (\mathbf{u}_2 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_2 \cdot \mathbf{w}_k) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_k \cdot \mathbf{w}_1) & (\mathbf{u}_k \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_k \cdot \mathbf{w}_k) \end{pmatrix}\right|$$

Now consider the vector

$$\mathbf{N} \equiv \det\begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_p \\ (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_1 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_1 \cdot \mathbf{w}_p) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_{p-1} \cdot \mathbf{w}_1) & (\mathbf{u}_{p-1} \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_{p-1} \cdot \mathbf{w}_p) \end{pmatrix}$$

which results from formally expanding along the top row. Note that from what was just discussed,

$$v\left(P\left(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1}\right)\right) = \pm A_{1p}$$

Now it follows from the formula for expansion of a determinant along the top row that for each $j \leq p-1$

$$\mathbf{N} \cdot \mathbf{u}_j = \sum_{k=1}^{p} \left(\mathbf{u}_j \cdot \mathbf{w}_k\right) \left(\mathbf{N} \cdot \mathbf{w}_k\right) = \sum_{k=1}^{p} \left(\mathbf{u}_j \cdot \mathbf{w}_k\right) A_{1k}$$

where A_{1k} is the $1k^{th}$ cofactor of the above matrix. Thus if $j \leq p-1$

$$\mathbf{N} \cdot \mathbf{u}_j = \det\begin{pmatrix} (\mathbf{u}_j \cdot \mathbf{w}_1) & (\mathbf{u}_j \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_j \cdot \mathbf{w}_p) \\ (\mathbf{u}_1 \cdot \mathbf{w}_1) & (\mathbf{u}_1 \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_1 \cdot \mathbf{w}_p) \\ \vdots & \vdots & & \vdots \\ (\mathbf{u}_{p-1} \cdot \mathbf{w}_1) & (\mathbf{u}_{p-1} \cdot \mathbf{w}_2) & \cdots & (\mathbf{u}_{p-1} \cdot \mathbf{w}_p) \end{pmatrix} = 0$$

because the matrix has two equal rows while if $j = p$, the above discussion shows $\mathbf{N} \cdot \mathbf{u}_p$ equals $\pm v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p))$. Therefore, $\mathbf{N}$ points in the direction of the normal vector in the above picture or else it points in the opposite direction to this vector. From the geometric description of the dot product,

$$|\cos(\theta)| = \frac{|\mathbf{N} \cdot \mathbf{u}_p|}{|\mathbf{u}_p||\mathbf{N}|}$$

and it follows

$$|\mathbf{u}_p||\cos(\theta)| v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})) = |\mathbf{u}_p| \frac{|\mathbf{N} \cdot \mathbf{u}_p|}{|\mathbf{u}_p||\mathbf{N}|} v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1}))$$

$$= \frac{v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p))}{|\mathbf{N}|} v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1}))$$

Now at this point, note that from the construction, $\mathbf{w}_p \cdot \mathbf{u}_k = 0$ whenever $k \leq p-1$ because $\mathbf{u}_k \in \text{span}(\mathbf{w}_1, \cdots, \mathbf{w}_{p-1})$. Therefore, $|\mathbf{N}| = |A_{1p}| = v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1}))$ and so the above reduces to

$$|\mathbf{u}_p||\cos(\theta)| v(P(\mathbf{u}_1, \cdots, \mathbf{u}_{p-1})) = v(P(\mathbf{u}_1, \cdots, \mathbf{u}_p)).$$

This proves the theorem.

The theorem shows that the only reasonable definition of p dimensional volume of a parallelepiped is the one given in the above definition.

12.9 Exercises

1. ♠Here are three vectors in $\mathbb{R}^4 : (1,2,0,3)^T, (2,1,-3,2)^T, (0,0,1,2)^T$. Find the three dimensional volume of the parallelepiped determined by these three vectors.

2. Here are two vectors in $\mathbb{R}^4 : (1,2,0,3)^T, (2,1,-3,2)^T$. Find the volume of the parallelepiped determined by these two vectors.

3. ♠Here are three vectors in $\mathbb{R}^2 : (1,2)^T, (2,1)^T, (0,1)^T$. Find the three dimensional volume of the parallelepiped determined by these three vectors. Recall that from the above theorem, this should equal 0.

4. Find the equation of the plane through the three points $(1,2,3), (2,-3,1), (1,1,7)$.

Self Adjoint Operators

13.1 Simultaneous Diagonalization

Recall the following definition of what it means for a matrix to be diagonalizable.

Definition 13.1.1 *Let A be an $n \times n$ matrix. It is said to be diagonalizable if there exists an invertible matrix S such that*

$$S^{-1}AS = D$$

where D is a diagonal matrix.

Also, here is a useful observation.

Observation 13.1.2 *If A is an $n \times n$ matrix and $AS = SD$ for D a diagonal matrix, then each column of S is an eigenvector or else it is the zero vector. This follows from observing that for $\mathbf{s}_k$ the k^{th} column of S and from the way we multiply matrices,*

$$A\mathbf{s}_k = \lambda_k \mathbf{s}_k$$

It is sometimes interesting to consider the problem of finding a single similarity transformation which will diagonalize all the matrices in some set.

Lemma 13.1.3 *Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. Denote by C the matrix,*

$$C \equiv \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Then C is diagonalizable if and only if both A and B are diagonalizable.

Proof: Suppose $S_A^{-1}AS_A = D_A$ and $S_B^{-1}BS_B = D_B$ where D_A and D_B are diagonal matrices. You should use block multiplication to verify that $S \equiv \begin{pmatrix} S_A & 0 \\ 0 & S_B \end{pmatrix}$ is such that $S^{-1}CS = D_C$, a diagonal matrix.

Conversely, suppose C is diagonalized by $S = (\mathbf{s}_1, \cdots, \mathbf{s}_{n+m})$. Thus S has columns $\mathbf{s}_i$. For each of these columns, write in the form

$$\mathbf{s}_i = \begin{pmatrix} \mathbf{x}_i \\ \mathbf{y}_i \end{pmatrix}$$

where $\mathbf{x}_i \in \mathbb{F}^n$ and where $\mathbf{y}_i \in \mathbb{F}^m$. The result is

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

where S_{11} is an $n \times n$ matrix and S_{22} is an $m \times m$ matrix. Then there is a diagonal matrix

$$D = diag\,(\lambda_1, \cdots, \lambda_{n+m}) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

such that

$$\begin{aligned} & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \\ = & \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \end{aligned}$$

Hence by block multiplication

$$AS_{11} = S_{11}D_1,\ BS_{22} = S_{22}D_2$$

$$BS_{21} = S_{21}D_1,\ AS_{12} = S_{12}D_2$$

It follows each of the $\mathbf{x}_i$ is an eigenvector of A or else is the zero vector and that each of the $\mathbf{y}_i$ is an eigenvector of B or is the zero vector. If there are n linearly independent $\mathbf{x}_i$, then A is diagonalizable by Theorem 9.3.10 on Page 9.3.10.

The row rank of the matrix, $(\mathbf{x}_1, \cdots, \mathbf{x}_{n+m})$ must be n because if this is not so, the rank of S would be less than $n+m$ which would mean S^{-1} does not exist. Therefore, since the column rank equals the row rank, this matrix has column rank equal to n and this means there are n linearly independent eigenvectors of A implying that A is diagonalizable. Similar reasoning applies to B. This proves the lemma.

The following corollary follows from the same type of argument as the above.

Corollary 13.1.4 *Let A_k be an $n_k \times n_k$ matrix and let C denote the block diagonal*

$$\left(\sum_{k=1}^{r} n_k\right) \times \left(\sum_{k=1}^{r} n_k\right)$$

matrix given below.

$$C \equiv \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}.$$

Then C is diagonalizable if and only if each A_k is diagonalizable.

Definition 13.1.5 *A set, $\mathcal{F}$ of $n \times n$ matrices is simultaneously diagonalizable if and only if there exists a single invertible matrix, S such that for every $A \in \mathcal{F}$, $S^{-1}AS = D_A$ where D_A is a diagonal matrix.*

Lemma 13.1.6 *If $\mathcal{F}$ is a set of $n \times n$ matrices which is simultaneously diagonalizable, then $\mathcal{F}$ is a commuting family of matrices.*

Proof: Let $A, B \in \mathcal{F}$ and let S be a matrix which has the property that $S^{-1}AS$ is a diagonal matrix for all $A \in \mathcal{F}$. Then $S^{-1}AS = D_A$ and $S^{-1}BS = D_B$ where D_A and D_B are diagonal matrices. Since diagonal matrices commute,

$$\begin{aligned} AB &= SD_AS^{-1}SD_BS^{-1} = SD_AD_BS^{-1} \\ &= SD_BD_AS^{-1} = SD_BS^{-1}SD_AS^{-1} = BA. \end{aligned}$$

Lemma 13.1.7 *Let D be a diagonal matrix of the form*

$$D \equiv \begin{pmatrix} \lambda_1 I_{n_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_r I_{n_r} \end{pmatrix}, \tag{13.1}$$

where I_{n_i} denotes the $n_i \times n_i$ identity matrix and $\lambda_i \neq \lambda_j$ for $i \neq j$ and suppose B is a matrix which commutes with D. Then B is a block diagonal matrix of the form

$$B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_r \end{pmatrix} \tag{13.2}$$

where B_i is an $n_i \times n_i$ matrix.

Proof: Let $B = (B_{ij})$ where $B_{ii} = B_i$ a block matrix as above in 13.2.

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \ddots & B_{2r} \\ \vdots & \ddots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{pmatrix}$$

Then by block multiplication, since B is given to commute with D, Then using block multiplication,

$$\lambda_j B_{ij} = \lambda_i B_{ij}$$

Therefore, if $i \neq j, B_{ij} = 0$. This proves the lemma.

Lemma 13.1.8 *Let $\mathcal{F}$ denote a commuting family of $n \times n$ matrices such that each $A \in \mathcal{F}$ is diagonalizable. Then $\mathcal{F}$ is simultaneously diagonalizable.*

Proof: First note that if every matrix in $\mathcal{F}$ has only one eigenvalue, there is nothing to prove. This is because for A such a matrix,

$$S^{-1}AS = \lambda I$$

and so

$$A = \lambda I$$

Thus all the matrices in $\mathcal{F}$ are diagonal matrices and you could pick any S to diagonalize them all. Therefore, without loss of generality, assume some matrix in $\mathcal{F}$ has more than one eigenvalue.

The significant part of the lemma is proved by induction on n. If $n = 1$, there is nothing to prove because all the 1×1 matrices are already diagonal matrices. Suppose then that the theorem is true for all $k \leq n-1$ where $n \geq 2$ and let $\mathcal{F}$ be a commuting family of diagonalizable $n \times n$ matrices. Pick $A \in \mathcal{F}$ which has more than one eigenvalue and let S be an invertible matrix such that $S^{-1}AS = D$ where D is of the form given in 13.1. By permuting the columns of S there is no loss of generality in assuming D has this form. Now denote by $\widetilde{\mathcal{F}}$ the collection of matrices, $\left\{S^{-1}CS : C \in \mathcal{F}\right\}$. Note $\widetilde{\mathcal{F}}$ features the single matrix S.

It follows easily that $\widetilde{\mathcal{F}}$ is also a commuting family of diagonalizable matrices. By Lemma 13.1.7 every $B \in \widetilde{\mathcal{F}}$ is of the form given in 13.2 because each of these commutes with D described above as $S^{-1}AS$ and so by block multiplication, the diagonal blocks B_i corresponding to different $B \in \widetilde{\mathcal{F}}$ commute.

By Corollary 13.1.4 each of these blocks is diagonalizable. This is because B is known to be so. Therefore, by induction, since all the blocks are no larger than $n-1 \times n-1$ thanks to the assumption that A has more than one eigenvalue, there exist invertible $n_i \times n_i$ matrices, T_i such that $T_i^{-1}B_iT_i$ is a diagonal matrix whenever B_i is one of the matrices making up the block diagonal of any $B \in \mathcal{F}$. It follows that for T defined by

$$T \equiv \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_r \end{pmatrix},$$

then $T^{-1}BT =$ a diagonal matrix for every $B \in \widetilde{\mathcal{F}}$ including D. Consider ST. It follows that for all $C \in \mathcal{F}$,

$$T^{-1} \overbrace{S^{-1}CS}^{\text{something in } \widetilde{\mathcal{F}}} T = (ST)^{-1} C (ST) = \text{ a diagonal matrix.}$$

This proves the lemma.

Theorem 13.1.9 *Let $\mathcal{F}$ denote a family of matrices which are diagonalizable. Then $\mathcal{F}$ is simultaneously diagonalizable if and only if $\mathcal{F}$ is a commuting family.*

Proof: If $\mathcal{F}$ is a commuting family, it follows from Lemma 13.1.8 that it is simultaneously diagonalizable. If it is simultaneously diagonalizable, then it follows from Lemma 13.1.6 that it is a commuting family. This proves the theorem.

13.2 Schur's Theorem

Recall that for a linear transformation, $L \in \mathcal{L}(V,V)$ for V a finite dimensional inner product space, it could be represented in the form

$$L = \sum_{ij} l_{ij} \mathbf{v}_i \otimes \mathbf{v}_j$$

where $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal basis. Of course different bases will yield different matrices, (l_{ij}). Schur's theorem gives the existence of a basis in an inner product space such that (l_{ij}) is particularly simple.

Definition 13.2.1 *Let $L \in \mathcal{L}(V,V)$ where V is vector space. Then a subspace U of V is L invariant if $L(U) \subseteq U$.*

In what follows, $\mathbb{F}$ will be the field of scalars, usually $\mathbb{C}$ but maybe something else.

Theorem 13.2.2 *Let $L \in \mathcal{L}(H,H)$ for H a finite dimensional inner product space such that the restriction of L^* to every L invariant subspace has its eigenvalues in $\mathbb{F}$. Then there exist constants, c_{ij} for $i \leq j$ and an orthonormal basis, $\{\mathbf{w}_i\}_{i=1}^n$ such that*

$$L = \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij} \mathbf{w}_i \otimes \mathbf{w}_j$$

The constants, c_{ii} are the eigenvalues of L.

Proof: If $\dim(H) = 1$, let $H = \text{span}(\mathbf{w})$ where $|\mathbf{w}| = 1$. Then $L\mathbf{w} = k\mathbf{w}$ for some k. Then

$$L = k\mathbf{w} \otimes \mathbf{w}$$

because by definition, $\mathbf{w} \otimes \mathbf{w}(\mathbf{w}) = \mathbf{w}$. Therefore, the theorem holds if H is 1 dimensional.

Now suppose the theorem holds for $n-1 = \dim(H)$. Let $\mathbf{w}_n$ be an eigenvector for L^*. Dividing by its length, it can be assumed $|\mathbf{w}_n| = 1$. Say $L^*\mathbf{w}_n = \mu\mathbf{w}_n$. Using the Gram Schmidt process, there exists an orthonormal basis for H of the form $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{w}_n\}$. Then

$$(L\mathbf{v}_k, \mathbf{w}_n) = (\mathbf{v}_k, L^*\mathbf{w}_n) = (\mathbf{v}_k, \mu\mathbf{w}_n) = 0,$$

which shows

$$L : H_1 \equiv \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}) \to \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}).$$

Denote by L_1 the restriction of L to H_1. Since H_1 has dimension $n-1$, the induction hypothesis yields an orthonormal basis, $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$ for H_1 such that

$$L_1 = \sum_{j=1}^{n-1} \sum_{i=1}^{j} c_{ij}\mathbf{w}_i \otimes \mathbf{w}_j. \tag{13.3}$$

Then $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is an orthonormal basis for H because every vector in

$$\text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1})$$

has the property that its dot product with $\mathbf{w}_n$ is 0 so in particular, this is true for the vectors $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$. Now define c_{in} to be the scalars satisfying

$$L\mathbf{w}_n \equiv \sum_{i=1}^{n} c_{in}\mathbf{w}_i \tag{13.4}$$

and let

$$B \equiv \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij}\mathbf{w}_i \otimes \mathbf{w}_j.$$

Then by 13.4,

$$B\mathbf{w}_n = \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{nj} = \sum_{j=1}^{n} c_{in}\mathbf{w}_i = L\mathbf{w}_n.$$

If $1 \le k \le n-1$,

$$B\mathbf{w}_k = \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{kj} = \sum_{i=1}^{k} c_{ik}\mathbf{w}_i$$

while from 13.3,

$$L\mathbf{w}_k = L_1\mathbf{w}_k = \sum_{j=1}^{n-1} \sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{jk} = \sum_{i=1}^{k} c_{ik}\mathbf{w}_i.$$

Since $L = B$ on the basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$, it follows $L = B$.

It remains to verify the constants, c_{kk} are the eigenvalues of L, solutions of the equation, $\det(\lambda I - L) = 0$. However, the definition of $\det(\lambda I - L)$ is the same as

$$\det(\lambda I - C)$$

where C is the upper triangular matrix which has c_{ij} for $i \leq j$ and zeros elsewhere. This equals 0 if and only if λ is one of the diagonal entries, one of the c_{kk}. This proves the theorem.

Now with the above Schur's theorem, the following diagonalization theorem comes very easily. Recall the following definition.

Definition 13.2.3 *Let $L \in \mathcal{L}(H,H)$ where H is a finite dimensional inner product space. Then L is Hermitian if $L^* = L$.*

Theorem 13.2.4 *Let $L \in \mathcal{L}(H,H)$ where H is an n dimensional inner product space. If L is Hermitian, then all of its eigenvalues λ_k are real and there exists an orthonormal basis of eigenvectors $\{\mathbf{w}_k\}$ such that*

$$L = \sum_k \lambda_k \mathbf{w}_k \otimes \mathbf{w}_k.$$

Proof: By Schur's theorem, Theorem 13.2.2, there exist $l_{ij} \in \mathbb{F}$ such that

$$L = \sum_{j=1}^{n} \sum_{i=1}^{j} l_{ij} \mathbf{w}_i \otimes \mathbf{w}_j$$

Then by Lemma 12.4.2,

$$\begin{aligned} \sum_{j=1}^{n} \sum_{i=1}^{j} l_{ij} \mathbf{w}_i \otimes \mathbf{w}_j &= L = L^* = \sum_{j=1}^{n} \sum_{i=1}^{j} (l_{ij} \mathbf{w}_i \otimes \mathbf{w}_j)^* \\ &= \sum_{j=1}^{n} \sum_{i=1}^{j} \overline{l_{ij}} \mathbf{w}_j \otimes \mathbf{w}_i = \sum_{i=1}^{n} \sum_{j=1}^{i} \overline{l_{ji}} \mathbf{w}_i \otimes \mathbf{w}_j \end{aligned}$$

By independence, if $i = j$,

$$l_{ii} = \overline{l_{ii}}$$

and so these are all real. If $i < j$, it follows from independence again that

$$l_{ij} = 0$$

because the coefficients corresponding to $i < j$ are all 0 on the right side. Similarly if $i > j$, it follows $l_{ij} = 0$. Letting $\lambda_k = l_{kk}$, this shows

$$L = \sum_k \lambda_k \mathbf{w}_k \otimes \mathbf{w}_k$$

That each of these $\mathbf{w}_k$ is an eigenvector corresponding to λ_k is obvious from the definition of the tensor product. This proves the theorem.

13.3 Spectral Theory Of Self Adjoint Operators

The following theorem is about the eigenvectors and eigenvalues of a self adjoint operator. The proof given generalizes to the situation of a compact self adjoint operator on a Hilbert space and leads to many very useful results. It is also a very elementary proof because it does not use the fundamental theorem of algebra and it contains a way, very important in applications, of finding the eigenvalues. This proof depends more directly on the methods of analysis than the preceding material. The field of scalars will be $\mathbb{R}$ or $\mathbb{C}$. The following is useful notation.

Definition 13.3.1 *Let X be an inner product space and let $S \subseteq X$. Then*

$$S^{\perp} \equiv \{x \in X : (x,s) = 0 \text{ for all } s \in S\}.$$

Note that even if S is not a subspace, $S^{\perp}$ is.

Definition 13.3.2 *A Hilbert space is a complete inner product space. Recall this means that every Cauchy sequence,$\{x_n\}$, one which satisfies*

$$\lim_{n,m\to\infty} |x_n - x_m| = 0,$$

converges. It can be shown, although I will not do so here, that for the field of scalars either $\mathbb{R}$ or $\mathbb{C}$, any finite dimensional inner product space is automatically complete.

Theorem 13.3.3 *Let $A \in \mathcal{L}(X,X)$ be self adjoint where X is a finite dimensional Hilbert space. Thus $A = A^*$. Then there exists an orthonormal basis of eigenvectors, $\{u_j\}_{j=1}^n$.*

Proof: Consider (Ax,x). This quantity is always a real number because

$$\overline{(Ax,x)} = (x,Ax) = (x,A^*x) = (Ax,x)$$

thanks to the assumption that A is self adjoint. Now define

$$\lambda_1 \equiv \inf\{(Ax,x) : |x| = 1, x \in X_1 \equiv X\}.$$

Claim: λ_1 is finite and there exists $v_1 \in X$ with $|v_1| = 1$ such that $(Av_1, v_1) = \lambda_1$.

Proof of claim: Let $\{u_j\}_{j=1}^n$ be an orthonormal basis for X and for $x \in X$, let $(x_1, \cdots, x_n)$ be defined as the components of the vector x. Thus,

$$x = \sum_{j=1}^n x_j u_j.$$

Since this is an orthonormal basis, it follows from the axioms of the inner product that

$$|x|^2 = \sum_{j=1}^n |x_j|^2.$$

Thus

$$(Ax,x) = \left(\sum_{k=1}^n x_k Au_k, \sum_{j=1} x_j u_j\right) = \sum_{k,j} x_k \overline{x_j}\,(Au_k, u_j),$$

a real valued continuous function of $(x_1, \cdots, x_n)$ which is defined on the compact set

$$K \equiv \{(x_1, \cdots, x_n) \in \mathbb{F}^n : \sum_{j=1}^n |x_j|^2 = 1\}.$$

Therefore, it achieves its minimum from the extreme value theorem. Then define

$$v_1 \equiv \sum_{j=1}^n x_j u_j$$

where $(x_1, \cdots, x_n)$ is the point of K at which the above function achieves its minimum. This proves the claim.

Continuing with the proof of the theorem, let $X_2 \equiv \{v_1\}^\perp$. This is a closed subspace of X. Let

$$\lambda_2 \equiv \inf\{(Ax, x) : |x| = 1, x \in X_2\}$$

As before, there exists $v_2 \in X_2$ such that $(Av_2, v_2) = \lambda_2, \lambda_1 \leq \lambda_2$. Now let $X_3 \equiv \{v_1, v_2\}^\perp$ and continue in this way. This leads to an increasing sequence of real numbers, $\{\lambda_k\}_{k=1}^n$ and an orthonormal set of vectors, $\{v_1, \cdots, v_n\}$. It only remains to show these are eigenvectors and that the λ_j are eigenvalues.

Consider the first of these vectors. Letting $w \in X_1 \equiv X$, the function of the real variable, t, given by

$$f(t) \equiv \frac{(A(v_1 + tw), v_1 + tw)}{|v_1 + tw|^2}$$

$$= \frac{(Av_1, v_1) + 2t\,\mathrm{Re}(Av_1, w) + t^2(Aw, w)}{|v_1|^2 + 2t\,\mathrm{Re}(v_1, w) + t^2|w|^2}$$

achieves its minimum when $t = 0$. Therefore, the derivative of this function evaluated at $t = 0$ must equal zero. Using the quotient rule, this implies, since $|v_1| = 1$ that

$$2\,\mathrm{Re}(Av_1, w)|v_1|^2 - 2\,\mathrm{Re}(v_1, w)(Av_1, v_1)$$

$$= 2(\mathrm{Re}(Av_1, w) - \mathrm{Re}(v_1, w)\lambda_1) = 0.$$

Thus $\mathrm{Re}(Av_1 - \lambda_1 v_1, w) = 0$ for all $w \in X$. This implies $Av_1 = \lambda_1 v_1$. To see this, let $w \in X$ be arbitrary and let θ be a complex number with $|\theta| = 1$ and

$$|(Av_1 - \lambda_1 v_1, w)| = \theta(Av_1 - \lambda_1 v_1, w).$$

Then

$$|(Av_1 - \lambda_1 v_1, w)| = \mathrm{Re}\left(Av_1 - \lambda_1 v_1, \overline{\theta} w\right) = 0.$$

Since this holds for all w, $Av_1 = \lambda_1 v_1$.

Now suppose $Av_k = \lambda_k v_k$ for all $k < m$. Observe that $A : X_m \to X_m$ because if $y \in X_m$ and $k < m$,

$$(Ay, v_k) = (y, Av_k) = (y, \lambda_k v_k) = 0,$$

showing that $Ay \in \{v_1, \cdots, v_{m-1}\}^\perp \equiv X_m$. Thus the same argument just given shows that for all $w \in X_m$,

$$(Av_m - \lambda_m v_m, w) = 0. \tag{13.5}$$

Since $Av_m \in X_m$, I can let $w = Av_m - \lambda_m v_m$ in the above and thereby conclude $Av_m = \lambda_m v_m$. This proves the theorem.

Contained in the proof of this theorem is the following important corollary.

Corollary 13.3.4 *Let $A \in \mathcal{L}(X, X)$ be self adjoint where X is a finite dimensional Hilbert space. Then all the eigenvalues are real and for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of A, there exists an orthonormal set of vectors $\{u_1, \cdots, u_n\}$ for which*

$$Au_k = \lambda_k u_k.$$

Furthermore,

$$\lambda_k \equiv \inf\{(Ax, x) : |x| = 1, x \in X_k\}$$

where

$$X_k \equiv \{u_1, \cdots, u_{k-1}\}^\perp, X_1 \equiv X.$$

Corollary 13.3.5 *Let $A \in \mathcal{L}(X,X)$ be self adjoint where X is a finite dimensional Hilbert space. Then the largest eigenvalue of A is given by*

$$\max\{(A\mathbf{x},\mathbf{x}) : |\mathbf{x}| = 1\} \tag{13.6}$$

and the minimum eigenvalue of A is given by

$$\min\{(A\mathbf{x},\mathbf{x}) : |\mathbf{x}| = 1\}. \tag{13.7}$$

Proof: The proof of this is just like the proof of Theorem 13.3.3. Simply replace inf with sup and obtain a decreasing list of eigenvalues. This establishes 13.6. The claim 13.7 follows from Theorem 13.3.3.

Another important observation is found in the following corollary.

Corollary 13.3.6 *Let $A \in \mathcal{L}(X,X)$ where A is self adjoint. Then $A = \sum_i \lambda_i v_i \otimes v_i$ where $Av_i = \lambda_i v_i$ and $\{v_i\}_{i=1}^n$ is an orthonormal basis.*

Proof : If v_k is one of the orthonormal basis vectors, $Av_k = \lambda_k v_k$. Also,

$$\begin{aligned}\sum_i \lambda_i v_i \otimes v_i(v_k) &= \sum_i \lambda_i v_i (v_k, v_i) \\ &= \sum_i \lambda_i \delta_{ik} v_i = \lambda_k v_k.\end{aligned}$$

Since the two linear transformations agree on a basis, it follows they must coincide. This proves the corollary.

By Theorem 12.4.5 this says the matrix of A with respect to this basis $\{v_i\}_{i=1}^n$ is the diagonal matrix having the eigenvalues $\lambda_1, \cdots, \lambda_n$ down the main diagonal.

The result of Courant and Fischer which follows resembles Corollary 13.3.4 but is more useful because it does not depend on a knowledge of the eigenvectors.

Theorem 13.3.7 *Let $A \in \mathcal{L}(X,X)$ be self adjoint where X is a finite dimensional Hilbert space. Then for $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ the eigenvalues of A, there exist orthonormal vectors $\{u_1, \cdots, u_n\}$ for which*

$$Au_k = \lambda_k u_k.$$

Furthermore,

$$\lambda_k \equiv \max_{w_1,\cdots,w_{k-1}} \left\{ \min\left\{ (Ax,x) : |x| = 1, x \in \{w_1, \cdots, w_{k-1}\}^\perp \right\} \right\} \tag{13.8}$$

where if $k = 1$, $\{w_1, \cdots, w_{k-1}\}^\perp \equiv X$.

Proof: From Theorem 13.3.3, there exist eigenvalues and eigenvectors with $\{u_1, \cdots, u_n\}$ orthonormal and $\lambda_i \le \lambda_{i+1}$. Therefore, by Corollary 13.3.6

$$A = \sum_{j=1}^n \lambda_j u_j \otimes u_j$$

Fix $\{w_1, \cdots, w_{k-1}\}$.

$$\begin{aligned}(Ax,x) &= \sum_{j=1}^n \lambda_j (x,u_j)(u_j,x) \\ &= \sum_{j=1}^n \lambda_j |(x,u_j)|^2\end{aligned}$$

Then let $Y = \{w_1, \cdots, w_{k-1}\}^\perp$

$$\inf\{(Ax,x) : |x| = 1, x \in Y\}$$
$$= \inf\left\{\sum_{j=1}^{n} \lambda_j |(x,u_j)|^2 : |x| = 1, x \in Y\right\}$$
$$\leq \inf\left\{\sum_{j=1}^{k} \lambda_j |(x,u_j)|^2 : |x| = 1, (x,u_j) = 0 \text{ for } j > k, \text{ and } x \in Y\right\}. \tag{13.9}$$

The reason this is so is that the infimum is taken over a smaller set. Therefore, the infimum gets larger. Now 13.9 is no larger than

$$\inf\left\{\lambda_k \sum_{j=1}^{k} |(x,u_j)|^2 : |x| = 1, (x,u_j) = 0 \text{ for } j > k, \text{ and } x \in Y\right\} = \lambda_k$$

because since $\{u_1, \cdots, u_n\}$ is an orthonormal basis, $|x|^2 = \sum_{j=1}^{n} |(x,u_j)|^2$. It follows since $\{w_1, \cdots, w_{k-1}\}$ is arbitrary,

$$\sup_{w_1,\cdots,w_{k-1}} \left\{\inf\left\{(Ax,x) : |x| = 1, x \in \{w_1, \cdots, w_{k-1}\}^\perp\right\}\right\} \leq \lambda_k. \tag{13.10}$$

However, for each $w_1, \cdots, w_{k-1}$, the infimum is achieved so you can replace the inf in the above with min. In addition to this, it follows from Corollary 13.3.4 that there exists a set, $\{w_1, \cdots, w_{k-1}\}$ for which

$$\inf\left\{(Ax,x) : |x| = 1, x \in \{w_1, \cdots, w_{k-1}\}^\perp\right\} = \lambda_k.$$

Pick $\{w_1, \cdots, w_{k-1}\} = \{u_1, \cdots, u_{k-1}\}$. Therefore, the sup in 13.10 is achieved and equals λ_k and 13.8 follows. This proves the theorem.

The following corollary is immediate.

Corollary 13.3.8 *Let $A \in \mathcal{L}(X,X)$ be self adjoint where X is a finite dimensional Hilbert space. Then for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of A, there exist orthonormal vectors $\{u_1, \cdots, u_n\}$ for which*

$$Au_k = \lambda_k u_k.$$

Furthermore,

$$\lambda_k \equiv \max_{w_1,\cdots,w_{k-1}} \left\{\min\left\{\frac{(Ax,x)}{|x|^2} : x \neq 0, x \in \{w_1, \cdots, w_{k-1}\}^\perp\right\}\right\} \tag{13.11}$$

where if $k = 1$, $\{w_1, \cdots, w_{k-1}\}^\perp \equiv X$.

Here is a version of this for which the roles of max and min are reversed.

Corollary 13.3.9 *Let $A \in \mathcal{L}(X,X)$ be self adjoint where X is a finite dimensional Hilbert space. Then for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of A, there exist orthonormal vectors $\{u_1, \cdots, u_n\}$ for which*

$$Au_k = \lambda_k u_k.$$

Furthermore,

$$\lambda_k \equiv \min_{w_1,\cdots,w_{n-k}} \left\{\max\left\{\frac{(Ax,x)}{|x|^2} : x \neq 0, x \in \{w_1, \cdots, w_{n-k}\}^\perp\right\}\right\} \tag{13.12}$$

where if $k = n$, $\{w_1, \cdots, w_{n-k}\}^\perp \equiv X$.

13.4 Positive And Negative Linear Transformations

The notion of a positive definite or negative definite linear transformation is very important in many applications. In particular it is used in versions of the second derivative test for functions of many variables. Here the main interest is the case of a linear transformation which is an $n \times n$ matrix but the theorem is stated and proved using a more general notation because all these issues discussed here have interesting generalizations to functional analysis.

Lemma 13.4.1 *Let X be a finite dimensional Hilbert space and let $A \in \mathcal{L}(X,X)$. Then if $\{v_1, \cdots, v_n\}$ is an orthonormal basis for X and $M(A)$ denotes the matrix of the linear transformation A then $M(A^*) = (M(A))^*$. In particular, A is self adjoint, if and only if $M(A)$ is.*

Proof: Consider the following picture

$$\begin{array}{ccc} & A & \\ X & \rightarrow & X \\ q \uparrow & \circ & \uparrow q \\ \mathbb{F}^n & \rightarrow & \mathbb{F}^n \\ & M(A) & \end{array}$$

where q is the coordinate map which satisfies $q(\mathbf{x}) \equiv \sum_i x_i v_i$. Therefore, since $\{v_1, \cdots, v_n\}$ is orthonormal, it is clear that $|\mathbf{x}| = |q(\mathbf{x})|$. Therefore,

$$\begin{aligned} |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\operatorname{Re}(\mathbf{x},\mathbf{y}) &= |\mathbf{x}+\mathbf{y}|^2 = |q(\mathbf{x}+\mathbf{y})|^2 \\ &= |q(\mathbf{x})|^2 + |q(\mathbf{y})|^2 + 2\operatorname{Re}(q(\mathbf{x}), q(\mathbf{y})) \end{aligned} \tag{13.13}$$

Now in any inner product space,

$$(x, iy) = \operatorname{Re}(x, iy) + i \operatorname{Im}(x, iy).$$

Also

$$(x, iy) = (-i)(x, y) = (-i)\operatorname{Re}(x, y) + \operatorname{Im}(x, y).$$

Therefore, equating the real parts, $\operatorname{Im}(x,y) = \operatorname{Re}(x, iy)$ and so

$$(x, y) = \operatorname{Re}(x, y) + i \operatorname{Re}(x, iy) \tag{13.14}$$

Now from 13.13, since q preserves distances, $.\operatorname{Re}(q(\mathbf{x}), q(\mathbf{y})) = \operatorname{Re}(\mathbf{x},\mathbf{y})$ which implies from 13.14 that

$$(\mathbf{x},\mathbf{y}) = (q(\mathbf{x}), q(\mathbf{y})). \tag{13.15}$$

Now consulting the diagram which gives the meaning for the matrix of a linear transformation, observe that $q \circ M(A) = A \circ q$ and $q \circ M(A^*) = A^* \circ q$. Therefore, from 13.15

$$(A(q(\mathbf{x})), q(\mathbf{y})) = (q(\mathbf{x}), A^* q(\mathbf{y})) = (q(\mathbf{x}), q(M(A^*)(\mathbf{y}))) = (\mathbf{x}, M(A^*)(\mathbf{y}))$$

but also

$$(A(q(\mathbf{x})), q(\mathbf{y})) = (q(M(A)(\mathbf{x})), q(\mathbf{y})) = (M(A)(\mathbf{x}), \mathbf{y}) = \left(\mathbf{x}, M(A)^*(\mathbf{y})\right).$$

Since $\mathbf{x}, \mathbf{y}$ are arbitrary, this shows that $M(A^*) = M(A)^*$ as claimed. Therefore, if A is self adjoint, $M(A) = M(A^*) = M(A)^*$ and so $M(A)$ is also self adjoint. If $M(A) = M(A)^*$ then $M(A) = M(A^*)$ and so $A = A^*$. This proves the lemma.

The following corollary is one of the items in the above proof.

Corollary 13.4.2 *Let X be a finite dimensional Hilbert space and let $\{v_1, \cdots, v_n\}$ be an orthonormal basis for X. Also, let q be the coordinate map associated with this basis satisfying $q(\mathbf{x}) \equiv \sum_i x_i v_i$. Then $(\mathbf{x}, \mathbf{y})_{\mathbb{F}^n} = (q(\mathbf{x}), q(\mathbf{y}))_X$. Also, if $A \in \mathcal{L}(X, X)$, and $M(A)$ is the matrix of A with respect to this basis,*

$$(Aq(\mathbf{x}), q(\mathbf{y}))_X = (M(A)\mathbf{x}, \mathbf{y})_{\mathbb{F}^n}.$$

Definition 13.4.3 *A self adjoint $A \in \mathcal{L}(X, X)$, is positive definite if whenever $\mathbf{x} \neq \mathbf{0}$, $(A\mathbf{x}, \mathbf{x}) > 0$ and A is negative definite if for all $\mathbf{x} \neq \mathbf{0}$, $(A\mathbf{x}, \mathbf{x}) < 0$. A is positive semidefinite or just nonnegative for short if for all $\mathbf{x}$, $(A\mathbf{x}, \mathbf{x}) \geq 0$. A is negative semidefinite or nonpositive for short if for all $\mathbf{x}$, $(A\mathbf{x}, \mathbf{x}) \leq 0$.*

The following lemma is of fundamental importance in determining which linear transformations are positive or negative definite.

Lemma 13.4.4 *Let X be a finite dimensional Hilbert space. A self adjoint $A \in \mathcal{L}(X, X)$ is positive definite if and only if all its eigenvalues are positive and negative definite if and only if all its eigenvalues are negative. It is positive semidefinite if all the eigenvalues are nonnegative and it is negative semidefinite if all the eigenvalues are nonpositive.*

Proof: Suppose first that A is positive definite and let λ be an eigenvalue. Then for $\mathbf{x}$ an eigenvector corresponding to λ, $\lambda(\mathbf{x}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, \mathbf{x}) > 0$. Therefore, $\lambda > 0$ as claimed.

Now suppose all the eigenvalues of A are positive. From Theorem 13.3.3 and Corollary 13.3.6, $A = \sum_{i=1}^n \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$ where the λ_i are the positive eigenvalues and $\{\mathbf{u}_i\}$ are an orthonormal set of eigenvectors. Therefore, letting $\mathbf{x} \neq \mathbf{0}$,

$$\begin{aligned}(A\mathbf{x}, \mathbf{x}) &= \left(\left(\sum_{i=1}^n \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i\right)\mathbf{x}, \mathbf{x}\right) = \left(\sum_{i=1}^n \lambda_i \mathbf{u}_i (\mathbf{x}, \mathbf{u}_i), \mathbf{x}\right) \\ &= \left(\sum_{i=1}^n \lambda_i (\mathbf{x}, \mathbf{u}_i)(\mathbf{u}_i, \mathbf{x})\right) = \sum_{i=1}^n \lambda_i |(\mathbf{u}_i, \mathbf{x})|^2 > 0\end{aligned}$$

because, since $\{\mathbf{u}_i\}$ is an orthonormal basis, $|\mathbf{x}|^2 = \sum_{i=1}^n |(\mathbf{u}_i, \mathbf{x})|^2$.

To establish the claim about negative definite, it suffices to note that A is negative definite if and only if $-A$ is positive definite and the eigenvalues of A are (-1) times the eigenvalues of $-A$. The claims about positive semidefinite and negative semidefinite are obtained similarly. This proves the lemma.

The next theorem is about a way to recognize whether a self adjoint $A \in \mathcal{L}(X, X)$ is positive or negative definite without having to find the eigenvalues. In order to state this theorem, here is some notation.

Definition 13.4.5 *Let A be an $n \times n$ matrix. Denote by A_k the $k \times k$ matrix obtained by deleting the $k+1, \cdots, n$ columns and the $k+1, \cdots, n$ rows from A. Thus $A_n = A$ and A_k is the $k \times k$ submatrix of A which occupies the upper left corner of A.*

The following theorem is proved in [7]

Theorem 13.4.6 *Let X be a finite dimensional Hilbert space and let $A \in \mathcal{L}(X, X)$ be self adjoint. Then A is positive definite if and only if $\det(M(A)_k) > 0$ for every $k = 1, \cdots, n$. Here $M(A)$ denotes the matrix of A with respect to some fixed orthonormal basis of X.*

Proof: This theorem is proved by induction on n. It is clearly true if $n = 1$. Suppose then that it is true for $n-1$ where $n \geq 2$. Since $\det(M(A)) > 0$, it follows that all the eigenvalues are nonzero. Are they all positive? Suppose not. Then there is some even number of them which are negative, even because the product of all the eigenvalues is known to be positive, equaling $\det(M(A))$. Pick two, λ_1 and λ_2 and let $M(A)\mathbf{u}_i = \lambda_i \mathbf{u}_i$ where $\mathbf{u}_i \neq \mathbf{0}$ for $i = 1, 2$ and $(\mathbf{u}_1, \mathbf{u}_2) = 0$. Now if $\mathbf{y} \equiv \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2$ is an element of $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)$, then since these are eigenvalues and $(\mathbf{u}_1, \mathbf{u}_2) = 0$, a short computation shows

$$(M(A)(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2), \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2)$$

$$= |\alpha_1|^2 \lambda_1 |\mathbf{u}_1|^2 + |\alpha_2|^2 \lambda_2 |\mathbf{u}_2|^2 < 0.$$

Now letting $\mathbf{x} \in \mathbb{C}^{n-1}$, the induction hypothesis implies

$$(\mathbf{x}^*, 0) M(A) \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = \mathbf{x}^* M(A)_{n-1} \mathbf{x} = (M(A)\mathbf{x}, \mathbf{x}) > 0.$$

Now the dimension of $\{\mathbf{z} \in \mathbb{C}^n : z_n = 0\}$ is $n-1$ and the dimension of $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2) = 2$ and so there must be some nonzero $\mathbf{x} \in \mathbb{C}^n$ which is in both of these subspaces of $\mathbb{C}^n$. However, the first computation would require that $(M(A)\mathbf{x}, \mathbf{x}) < 0$ while the second would require that $(M(A)\mathbf{x}, \mathbf{x}) > 0$. This contradiction shows that all the eigenvalues must be positive. This proves the if part of the theorem. The only if part is left to the reader.

Corollary 13.4.7 *Let X be a finite dimensional Hilbert space and let $A \in \mathcal{L}(X, X)$ be self adjoint. Then A is negative definite if and only if $\det(M(A)_k)(-1)^k > 0$ for every $k = 1, \cdots, n$. Here $M(A)$ denotes the matrix of A with respect to some fixed orthonormal basis of X.*

Proof: This is immediate from the above theorem by noting that, as in the proof of Lemma 13.4.4, A is negative definite if and only if $-A$ is positive definite. Therefore, if $\det(-M(A)_k) > 0$ for all $k = 1, \cdots, n$, it follows that A is negative definite. However, $\det(-M(A)_k) = (-1)^k \det(M(A)_k)$. This proves the corollary.

13.5 Fractional Powers

With the above theory, it is possible to take fractional powers of certain elements of $\mathcal{L}(X, X)$ where X is a finite dimensional Hilbert space. To begin with, consider the square root of a nonnegative self adjoint operator. This is easier than the general theory and it is the square root which is of most importance.

Theorem 13.5.1 *Let $A \in \mathcal{L}(X, X)$ be self adjoint and nonnegative. Then there exists a unique self adjoint nonnegative $B \in \mathcal{L}(X, X)$ such that $B^2 = A$ and B commutes with every element of $\mathcal{L}(X, X)$ which commutes with A.*

Proof: By Theorem 13.3.3, there exists an orthonormal basis of eigenvectors of A, say $\{v_i\}_{i=1}^n$ such that $Av_i = \lambda_i v_i$. Therefore, by Theorem 13.2.4, $A = \sum_i \lambda_i v_i \otimes v_i$ where each $\lambda_i \geq 0$.

Now by Lemma 13.4.4, each $\lambda_i \geq 0$. Therefore, it makes sense to define

$$B \equiv \sum_i \lambda_i^{1/2} v_i \otimes v_i.$$

It is easy to verify that

$$(v_i \otimes v_i)(v_j \otimes v_j) = \begin{cases} 0 \text{ if } i \neq j \\ v_i \otimes v_i \text{ if } i = j \end{cases}.$$

Therefore, a short computation verifies that $B^2 = \sum_i \lambda_i v_i \otimes v_i = A$. If C commutes with A, then for some c_{ij},

$$C = \sum_{ij} c_{ij} v_i \otimes v_j$$

and so since they commute,

$$\sum_{i,j,k} c_{ij} v_i \otimes v_j \lambda_k v_k \otimes v_k = \sum_{i,j,k} c_{ij} \lambda_k \delta_{jk} v_i \otimes v_k = \sum_{i,k} c_{ik} \lambda_k v_i \otimes v_k$$

$$\begin{aligned} &= \sum_{i,j,k} c_{ij} \lambda_k v_k \otimes v_k v_i \otimes v_j = \sum_{i,j,k} c_{ij} \lambda_k \delta_{ki} v_k \otimes v_j = \sum_{j,k} c_{kj} \lambda_k v_k \otimes v_j \\ &= \sum_{k,i} c_{ik} \lambda_i v_i \otimes v_k \end{aligned}$$

Then by independence,

$$c_{ik} \lambda_i = c_{ik} \lambda_k$$

Therefore, $c_{ik} \lambda_i^{1/2} = c_{ik} \lambda_k^{1/2}$ which ammounts to saying that B also commutes with C. It is clear that this operator is self adjoint. This proves existence.

Suppose B_1 is another square root which is self adjoint, nonnegative and commutes with every matrix which commutes with A. Since both B, B_1 are nonnegative,

$$(B(B - B_1)x, (B - B_1)x) \geq 0,$$

$$(B_1(B - B_1)x, (B - B_1)x) \geq 0 \tag{13.16}$$

Now, adding these together, and using the fact that the two commute,

$$\left(\left(B^2 - B_1^2\right)x, (B - B_1)x\right) = ((A - A)x, (B - B_1)x) = 0.$$

It follows that both inner products in 13.16 equal 0. Next use the existence part ot this to take the square root of B and B_1 which is denoted by $\sqrt{B}, \sqrt{B_1}$ respectively. Then

$$\begin{aligned} 0 &= \left(\sqrt{B}(B - B_1)x, \sqrt{B}(B - B_1)x\right) \\ 0 &= \left(\sqrt{B_1}(B - B_1)x, \sqrt{B_1}(B - B_1)x\right) \end{aligned}$$

which implies $\sqrt{B}(B - B_1)x = \sqrt{B_1}(B - B_1)x = 0$. Thus also,

$$B(B - B_1)x = B_1(B - B_1)x = 0$$

Hence

$$0 = (B(B - B_1)x - B_1(B - B_1)x, x) = ((B - B_1)x, (B - B_1)x)$$

and so, since x is arbitrary, $B_1 = B$. This proves the theorem.

The main result is the following theorem.

Theorem 13.5.2 *Let $A \in \mathcal{L}(X, X)$ be self adjoint and nonnegative and let k be a positive integer. Then there exists a unique self adjoint nonnegative $B \in \mathcal{L}(X, X)$ such that $B^k = A$.*

Proof: By Theorem 13.3.3, there exists an orthonormal basis of eigenvectors of A, say $\{v_i\}_{i=1}^n$ such that $Av_i = \lambda_i v_i$. Therefore, by Corollary 13.3.6 or Theorem 13.2.4, $A = \sum_i \lambda_i v_i \otimes v_i$ where each $\lambda_i \geq 0$.

Now by Lemma 13.4.4, each $\lambda_i \geq 0$. Therefore, it makes sense to define

$$B \equiv \sum_i \lambda_i^{1/k} v_i \otimes v_i.$$

It is easy to verify that

$$(v_i \otimes v_i)(v_j \otimes v_j) = \begin{cases} 0 \text{ if } i \neq j \\ v_i \otimes v_i \text{ if } i = j \end{cases}.$$

Therefore, a short computation verifies that $B^k = \sum_i \lambda_i v_i \otimes v_i = A$. This proves existence.

In order to prove uniqueness, let $p(t)$ be a polynomial which has the property that $p(\lambda_i) = \lambda_i^{1/k}$ for each i. In other words, goes through the ordered pairs $\left(\lambda_i, \lambda_i^{1/k}\right)$. Then a similar short computation shows

$$p(A) = \sum_i p(\lambda_i) v_i \otimes v_i = \sum_i \lambda_i^{1/k} v_i \otimes v_i = B.$$

Now suppose $C^k = A$ where $C \in \mathcal{L}(X, X)$ is self adjoint and nonnegative. Then

$$CB = Cp(A) = Cp\left(C^k\right) = p\left(C^k\right) C = p(A) C = BC.$$

Therefore, $\{B, C\}$ is a commuting family of linear transformations which are both self adjoint. Letting $M(B)$ and $M(C)$ denote matrices of these linear transformations taken with respect to some fixed orthonormal basis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, it follows that $M(B)$ and $M(C)$ commute and that both can be diagonalized (Lemma 13.4.1). See the diagram for a short verification of the claim the two matrices commute..

$$\begin{array}{ccccc} & B & & C & \\ X & \rightarrow & X & \rightarrow & X \\ q \uparrow & \circ & \uparrow q & \circ & \uparrow q \\ \mathbb{F}^n & \rightarrow & \mathbb{F}^n & \rightarrow & \mathbb{F}^n \\ & M(B) & & M(C) & \end{array}$$

Therefore, by Theorem 13.1.9, these two matrices can be simultaneously diagonalized. Thus

$$U^{-1} M(B) U = D_1,\ U^{-1} M(C) U = D_2 \tag{13.17}$$

where the D_i is a diagonal matrix consisting of the eigenvalues of B or C. Also it is clear that

$$M(C)^k = M(A)$$

because $M(C)^k$ is given by

$$\overbrace{q^{-1}Cqq^{-1}Cq\cdots q^{-1}Cq}^{k \text{ times}} = q^{-1}C^k q = q^{-1}Aq = M(A)$$

and similarly

$$M(B)^k = M(A).$$

Then raising these to powers,

$$U^{-1} M(A) U = U^{-1} M(B)^k U = D_1^k$$

and

$$U^{-1} M(A) U = U^{-1} M(C)^k U = D_2^k.$$

Therefore, $D_1^k = D_2^k$ and since the diagonal entries of D_i are nonnegative, this requires that $D_1 = D_2$. Therefore, from 13.17, $M(B) = M(C)$ and so $B = C$. This proves the theorem.

13.6 Polar Decompositions

An application of Theorem 13.3.3, is the following fundamental result, important in geometric measure theory and continuum mechanics. It is sometimes called the right polar decomposition. The notation used is that which is seen in continuum mechanics, see for example Gurtin [9]. Don't confuse the U in this theorem with a unitary transformation. It is not so. When the following theorem is applied in continuum mechanics, F is normally the deformation gradient, the derivative of a nonlinear map from some subset of three dimensional space to three dimensional space. In this context, U is called the right Cauchy Green strain tensor. It is a measure of how a body is stretched independent of rigid motions. First, here is a simple lemma.

Lemma 13.6.1 *Suppose $R \in \mathcal{L}(X,Y)$ where X, Y are Hilbert spaces and R preserves distances. Then $R^*R = I$.*

Proof: Since R preserves distances, $|R\mathbf{x}| = |\mathbf{x}|$ for every $\mathbf{x}$. Therefore from the axioms of the dot product,

$$\begin{aligned} & |\mathbf{x}|^2 + |\mathbf{y}|^2 + (\mathbf{x},\mathbf{y}) + (\mathbf{y},\mathbf{x}) \\ = \; & |\mathbf{x}+\mathbf{y}|^2 \\ = \; & (R(\mathbf{x}+\mathbf{y}), R(\mathbf{x}+\mathbf{y})) \\ = \; & (R\mathbf{x},R\mathbf{x}) + (R\mathbf{y},R\mathbf{y}) + (R\mathbf{x},R\mathbf{y}) + (R\mathbf{y},R\mathbf{x}) \\ = \; & |\mathbf{x}|^2 + |\mathbf{y}|^2 + (R^*R\mathbf{x},\mathbf{y}) + (\mathbf{y},R^*R\mathbf{x}) \end{aligned}$$

and so for all $\mathbf{x}, \mathbf{y}$,

$$(R^*R\mathbf{x}-\mathbf{x},\mathbf{y}) + (\mathbf{y},R^*R\mathbf{x}-\mathbf{x}) = 0$$

Hence for all $\mathbf{x}, \mathbf{y}$,

$$\mathrm{Re}\,(R^*R\mathbf{x}-\mathbf{x},\mathbf{y}) = 0$$

Now for $\mathbf{x}, \mathbf{y}$ given, choose $\alpha \in \mathbb{C}$ such that

$$\alpha\,(R^*R\mathbf{x}-\mathbf{x},\mathbf{y}) = |(R^*R\mathbf{x}-\mathbf{x},\mathbf{y})|$$

Then

$$\begin{aligned} 0 \; &= \; \mathrm{Re}\,(R^*R\mathbf{x}-\mathbf{x},\overline{\alpha}\mathbf{y}) = \mathrm{Re}\,\alpha\,(R^*R\mathbf{x}-\mathbf{x},\mathbf{y}) \\ &= \; |(R^*R\mathbf{x}-\mathbf{x},\mathbf{y})| \end{aligned}$$

Thus $|(R^*R\mathbf{x}-\mathbf{x},\mathbf{y})| = 0$ for all $\mathbf{x}, \mathbf{y}$ because the given $\mathbf{x}, \mathbf{y}$ were arbitrary. Let $\mathbf{y} = R^*R\mathbf{x}-\mathbf{x}$ to conclude that for all $\mathbf{x}$,

$$R^*R\mathbf{x}-\mathbf{x} = \mathbf{0}$$

which says $R^*R = I$ since $\mathbf{x}$ is arbitrary. This proves the lemma.

Theorem 13.6.2 *Let X be a Hilbert space of dimension n and let Y be a Hilbert space of dimension $m \geq n$ and let $F \in \mathcal{L}(X,Y)$. Then there exists $R \in \mathcal{L}(X,Y)$ and $U \in \mathcal{L}(X,X)$ such that*

$$F = RU,\ U = U^*, (U \text{ is Hermitian}),$$

all eigenvalues of U are non negative,

$$U^2 = F^*F, R^*R = I,$$

and $|R\mathbf{x}| = |\mathbf{x}|$.

Proof: $(F^*F)^* = F^*F$ and so by Theorem 13.3.3, there is an orthonormal basis of eigenvectors, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ such that

$$F^*F\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ F^*F = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i.$$

It is also clear that $\lambda_i \geq 0$ because

$$\lambda_i (\mathbf{v}_i, \mathbf{v}_i) = (F^*F\mathbf{v}_i, \mathbf{v}_i) = (F\mathbf{v}_i, F\mathbf{v}_i) \geq 0.$$

Let

$$U \equiv \sum_{i=1}^{n} \lambda_i^{1/2} \mathbf{v}_i \otimes \mathbf{v}_i.$$

Then $U^2 = F^*F$, $U = U^*$, and the eigenvalues of U, $\left\{\lambda_i^{1/2}\right\}_{i=1}^{n}$ are all non negative.

Let $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$ be an orthonormal basis for $U(X)$. By the Gram Schmidt procedure there exists an extension to an orthonormal basis for X,

$$\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}.$$

Next note that $\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r\}$ is also an orthonormal set of vectors in Y because

$$(F\mathbf{x}_k, F\mathbf{x}_j) = (F^*F\mathbf{x}_k, \mathbf{x}_j) = \left(U^2\mathbf{x}_k, \mathbf{x}_j\right) = (U\mathbf{x}_k, U\mathbf{x}_j) = \delta_{jk}.$$

By the Gram Schmidt procedure, there exists an extension of $\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r\}$ to an orthonormal basis for Y,

$$\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_m\}.$$

Since $m \geq n$, there are at least as many $\mathbf{z}_k$ as there are $\mathbf{y}_k$. Now for $\mathbf{x} \in X$, since

$$\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}$$

is an orthonormal basis for X, there exist unique scalars,

$$c_1, \cdots, c_r, d_{r+1}, \cdots, d_n$$

such that

$$\mathbf{x} = \sum_{k=1}^{r} c_k U\mathbf{x}_k + \sum_{k=r+1}^{n} d_k \mathbf{y}_k$$

Define

$$R\mathbf{x} \equiv \sum_{k=1}^{r} c_k F\mathbf{x}_k + \sum_{k=r+1}^{n} d_k \mathbf{z}_k \tag{13.18}$$

Thus

$$|R\mathbf{x}|^2 = \sum_{k=1}^{r} |c_k|^2 + \sum_{k=r+1}^{n} |d_k|^2 = |\mathbf{x}|^2.$$

Therefore, by Lemma 13.6.1 $R^*R = I$.

Then also there exist scalars b_k such that

$$U\mathbf{x} = \sum_{k=1}^{r} b_k U\mathbf{x}_k \tag{13.19}$$

and so from 13.18,

$$RU\mathbf{x} = \sum_{k=1}^{r} b_k F\mathbf{x}_k = F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right)$$

Is $F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) = F(\mathbf{x})$?

$$\left(F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) - F(\mathbf{x}), F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) - F(\mathbf{x})\right)$$

$$\begin{aligned}
&= \left((F^*F)\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(U^2\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(U\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right), U\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(\sum_{k=1}^{r} b_k U\mathbf{x}_k - U\mathbf{x}, \sum_{k=1}^{r} b_k U\mathbf{x}_k - U\mathbf{x}\right) = 0
\end{aligned}$$

Because from 13.19, $U\mathbf{x} = \sum_{k=1}^{r} b_k U\mathbf{x}_k$. Therefore, $RU\mathbf{x} = F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) = F(\mathbf{x})$ and this proves the theorem.

The following corollary follows as a simple consequence of this theorem. It is called the left polar decomposition.

Corollary 13.6.3 *Let $F \in \mathcal{L}(X,Y)$ and suppose $n \geq m$ where X is a Hilbert space of dimension n and Y is a Hilbert space of dimension m. Then there exists a Hermitian $U \in \mathcal{L}(X,X)$, and an element of $\mathcal{L}(X,Y)$, R, such that*

$$F = UR,\ RR^* = I.$$

Proof: Recall that $L^{**} = L$ and $(ML)^* = L^*M^*$. Now apply Theorem 13.6.2 to $F^* \in \mathcal{L}(Y,X)$. Thus,

$$F^* = R^*U$$

where R^* and U satisfy the conditions of that theorem. Then

$$F = UR$$

and $RR^* = R^{**}R^* = I$. This proves the corollary.

The following existence theorem for the polar decomposition of an element of $\mathcal{L}(X,X)$ is a corollary.

Corollary 13.6.4 *Let $F \in \mathcal{L}(X,X)$. Then there exists a Hermitian $W \in \mathcal{L}(X,X)$, and a unitary matrix, Q such that $F = WQ$, and there exists a Hermitian $U \in \mathcal{L}(X,X)$ and a unitary R, such that $F = RU$.*

This corollary has a fascinating relation to the question whether a given linear transformation is normal. Recall that an $n \times n$ matrix, A, is normal if $AA^* = A^*A$. Retain the same definition for an element of $\mathcal{L}(X,X)$.

Theorem 13.6.5 *Let $F \in \mathcal{L}(X,X)$. Then F is normal if and only if in Corollary 13.6.4 $RU = UR$ and $QW = WQ$.*

Proof: I will prove the statement about $RU = UR$ and leave the other part as an exercise. First suppose that $RU = UR$ and show F is normal. To begin with,

$$UR^* = (RU)^* = (UR)^* = R^*U.$$

Therefore,

$$\begin{aligned} F^*F &= UR^*RU = U^2 \\ FF^* &= RUUR^* = URR^*U = U^2 \end{aligned}$$

which shows F is normal.

Now suppose F is normal. Is $RU = UR$? Since F is normal,

$$FF^* = RUUR^* = RU^2R^*$$

and

$$F^*F = UR^*RU = U^2.$$

Therefore, $RU^2R^* = U^2$, and both are nonnegative and self adjoint. Therefore, the square roots of both sides must be equal by the uniqueness part of the theorem on fractional powers. It follows that the square root of the first, RUR^* must equal the square root of the second, U. Therefore, $RUR^* = U$ and so $RU = UR$. This proves the theorem in one case. The other case in which W and Q commute is left as an exercise.

13.7 An Application To Statistics

A random vector is a function $\mathbf{X} : \Omega \to \mathbb{R}^p$ where Ω is a probability space. This means that there exists a σ algebra of measurable sets $\mathcal{F}$ and a probability measure $P : \mathcal{F} \to [0,1]$. In practice, people often don't worry too much about the underlying probability space and instead pay more attention to the distribution measure of the random variable. For E a suitable subset of $\mathbb{R}^p$, this measure gives the probability that $\mathbf{X}$ has values in E. There are often excellent reasons for believing that a random vector is normally distributed. This means that the probability that $\mathbf{X}$ has values in a set E is given by

$$\int_E \frac{1}{(2\pi)^{p/2}\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})\right) d\mathbf{x}$$

The expression in the integral is called the normal probability density function. There are two parameters, $\mathbf{m}$ and Σ where $\mathbf{m}$ is called the mean and Σ is called the covariance matrix. It is a symmetric matrix which has all real eigenvalues which are all positive. While it may be reasonable to assume this is the distribution, in general, you won't know $\mathbf{m}$ and Σ and in order to use this formula to predict anything, you would need to know these quantities.

What people do to estimate these is to take n independent observations $\mathbf{x}_1, \cdots, \mathbf{x}_n$ and try to predict what $\mathbf{m}$ and Σ should be based on these observations. One criterion used for making this determination is the method of maximum likelyhood. In this method, you seek to choose the two parameters in such a way as to maximize the likelyhood which is given as

$$\prod_{i=1}^{n} \frac{1}{\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}_i-\mathbf{m})\right).$$

For convenience the term $(2\pi)^{p/2}$ was ignored. This leads to the estimate for $\mathbf{m}$ as

$$\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \equiv \overline{\mathbf{x}}.$$

This part follows fairly easily from taking the ln and then setting partial derivatives equal to 0. The estimation of Σ is harder. However, it is not too hard using the theorems presented above. I am following a nice discussion given in Wikipedia. It will make use of Theorem 7.5.3 on the trace as well as the theorem about the square root of a linear transformation given above. First note that by Theorem 7.5.3,

$$\begin{aligned}(\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1} (\mathbf{x}_i-\mathbf{m}) &= \operatorname{trace}\left((\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1} (\mathbf{x}_i-\mathbf{m})\right) \\ &= \operatorname{trace}\left((\mathbf{x}_i-\mathbf{m}) (\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1}\right)\end{aligned}$$

Therefore, the thing to maximize is

$$\prod_{i=1}^{n} \frac{1}{\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}\operatorname{trace}\left((\mathbf{x}_i-\mathbf{m}) (\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1}\right)\right)$$

$$\begin{aligned}&= \det\left(\Sigma^{-1}\right)^{n/2} \exp\left(-\frac{1}{2}\operatorname{trace}\sum_{i=1}^{n} (\mathbf{x}_i-\mathbf{m}) (\mathbf{x}_i-\mathbf{m})^* \Sigma^{-1}\right) \\ &= \det\left(\Sigma^{-1}\right)^{n/2} \exp\left(-\frac{1}{2}\operatorname{trace}\overbrace{\sum_{i=1}^{n} (\mathbf{x}_i-\mathbf{m}) (\mathbf{x}_i-\mathbf{m})^*}^{S}\Sigma^{-1}\right) \\ &\equiv \det\left(\Sigma^{-1}\right)^{n/2} \exp\left(-\frac{1}{2}\operatorname{trace}\left(S\Sigma^{-1}\right)\right)\end{aligned}$$

where S is the $p \times p$ matrix indicated above. Now S is symmetric and has eigenvalues which are all nonnegative because $(S\mathbf{y}, \mathbf{y}) \geq 0$. Therefore, S has a unique self adjoint square root. Using Theorem 7.5.3 again, the above equals

$$\det\left(\Sigma^{-1}\right)^{n/2} \exp\left(-\frac{1}{2}\operatorname{trace}\left(S^{1/2}\Sigma^{-1}S^{1/2}\right)\right)$$

Let $B = S^{1/2}\Sigma^{-1}S^{1/2}$ and assume $\det(S) \neq 0$. Then $\Sigma^{-1} = S^{-1/2}BS^{-1/2}$. The above equals

$$\det\left(S^{-1}\right) \det(B)^{n/2} \exp\left(-\frac{1}{2}\operatorname{trace}(B)\right)$$

Of course the thing to estimate is only found in B. Therefore, $\det\left(S^{-1}\right)$ can be discarded in trying to maximize things. Since B is symmetric, it is similar to a diagonal matrix D which has $\lambda_1, \cdots, \lambda_n$ down the diagonal. Thus it is desired to maximize

$$\left(\prod_{i=1}^{p} \lambda_i\right)^{n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{p} \lambda_i\right)$$

Taking ln it follows that it suffices to maximize

$$\frac{n}{2}\sum_{i=1}^{p} \ln \lambda_i - \frac{1}{2}\sum_{i=1}^{p} \lambda_i$$

Taking the derivative with respect to λ_i,

$$\frac{n}{2}\frac{1}{\lambda_i} - \frac{1}{2} = 0$$

and so $\lambda_i = n$. It follows from the above that

$$\Sigma = S^{1/2}B^{-1}S^{1/2}$$

where B^{-1} has only the eigenvalues $1/n$. It follows B^{-1} must equal the diagonal matrix which has $1/n$ down the diagonal. The reason for this is that B is similar to a diagonal matrix because it is symmetric. Thus $B = P^{-1}\frac{1}{n}IP = \frac{1}{n}I$ because the identity commutes with every matrix. But now it follows that

$$\Sigma = \frac{1}{n}S$$

Of course this is just an estimate and so we write $\hat{\Sigma}$ instead of Σ.

This has shown that the maximum likelyhood estimate for Σ is

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \mathbf{m})(\mathbf{x}_i - \mathbf{m})^*$$

13.8 The Singular Value Decomposition

In this section, A will be an $m \times n$ matrix. To begin with, here is a simple lemma.

Lemma 13.8.1 *Let A be an $m \times n$ matrix. Then A^*A is self adjoint and all its eigenvalues are nonnegative.*

Proof: It is obvious that A^*A is self adjoint. Suppose $A^*A\mathbf{x} = \lambda\mathbf{x}$. Then $\lambda|\mathbf{x}|^2 = (\lambda\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) \geq 0$.

Definition 13.8.2 *Let A be an $m \times n$ matrix. The singular values of A are the square roots of the positive eigenvalues of A^*A.*

With this definition and lemma here is the main theorem on the singular value decomposition. In all that follows, I will write the following partitioned matrix

$$\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

where σ denotes an $r \times r$ diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{pmatrix}$$

and the bottom row of zero matrices in the partitioned matrix, as well as the right columns of zero matrices are each of the right size so that the resulting matrix is $m \times n$. Either could vanish completely. However, I will write it in the above form. It is easy to make the necessary adjustments in the other two cases.

Theorem 13.8.3 *Let A be an $m \times n$ matrix. Then there exist unitary matrices, U and V of the appropriate size such that*

$$U^*AV = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

where σ is of the form

$$\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{pmatrix}$$

for the σ_i the singular values of A, arranged in order of decreasing size.

Proof: By the above lemma and Theorem 13.3.3 there exists an orthonormal basis, $\{\mathbf{v}_i\}_{i=1}^n$ such that $A^*A\mathbf{v}_i = \sigma_i^2\mathbf{v}_i$ where $\sigma_i^2 > 0$ for $i = 1, \cdots, k, (\sigma_i > 0)$, and equals zero if $i > k$. Thus for $i > k$, $A\mathbf{v}_i = \mathbf{0}$ because

$$(A\mathbf{v}_i, A\mathbf{v}_i) = (A^*A\mathbf{v}_i, \mathbf{v}_i) = (\mathbf{0}, \mathbf{v}_i) = 0.$$

For $i = 1, \cdots, k$, define $\mathbf{u}_i \in \mathbb{F}^m$ by

$$\mathbf{u}_i \equiv \sigma_i^{-1} A\mathbf{v}_i.$$

Thus $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$. Now

$$\begin{aligned} (\mathbf{u}_i, \mathbf{u}_j) &= \left(\sigma_i^{-1}A\mathbf{v}_i, \sigma_j^{-1}A\mathbf{v}_j\right) = \left(\sigma_i^{-1}\mathbf{v}_i, \sigma_j^{-1}A^*A\mathbf{v}_j\right) \\ &= \left(\sigma_i^{-1}\mathbf{v}_i, \sigma_j^{-1}\sigma_j^2\mathbf{v}_j\right) = \frac{\sigma_j}{\sigma_i}(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}. \end{aligned}$$

Thus $\{\mathbf{u}_i\}_{i=1}^k$ is an orthonormal set of vectors in $\mathbb{F}^m$. Also,

$$AA^*\mathbf{u}_i = AA^*\sigma_i^{-1}A\mathbf{v}_i = \sigma_i^{-1}AA^*A\mathbf{v}_i = \sigma_i^{-1}A\sigma_i^2\mathbf{v}_i = \sigma_i^2\mathbf{u}_i.$$

Now extend $\{\mathbf{u}_i\}_{i=1}^k$ to an orthonormal basis for all of $\mathbb{F}^m, \{\mathbf{u}_i\}_{i=1}^m$ and let

$$U \equiv \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{pmatrix}$$

while

$$V \equiv \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

Thus U is the matrix which has the $\mathbf{u}_i$ as columns and V is defined as the matrix which has the $\mathbf{v}_i$ as columns. Then

$$U^*AV = \begin{pmatrix} \mathbf{u}_1^* \\ \vdots \\ \mathbf{u}_k^* \\ \vdots \\ \mathbf{u}_m^* \end{pmatrix} A \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{u}_1^* \\ \vdots \\ \mathbf{u}_k^* \\ \vdots \\ \mathbf{u}_m^* \end{pmatrix} \begin{pmatrix} \sigma_1\mathbf{u}_1 & \cdots & \sigma_k\mathbf{u}_k & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

where σ is given in the statement of the theorem. This proves the theorem.

The singular value decomposition has as an immediate corollary the following interesting result.

Corollary 13.8.4 *Let A be an $m \times n$ matrix. Then the rank of A and A^* equals the number of singular values.*

Proof: Since V and U are unitary, they are each one to one and onto and so it follows that

$$\begin{aligned} \operatorname{rank}(A) &= \operatorname{rank}(U^*AV) \\ &= \operatorname{rank}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \\ &= \text{number of singular values.} \end{aligned}$$

Also since U, V are unitary,

$$\begin{aligned} \operatorname{rank}(A^*) &= \operatorname{rank}(V^*A^*U) \\ &= \operatorname{rank}\left((U^*AV)^*\right) \\ &= \operatorname{rank}\left(\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}^*\right) \\ &= \text{number of singular values.} \end{aligned}$$

This proves the corollary.

13.9 Approximation In The Frobenius Norm

The Frobenius norm is one of many norms for a matrix. It is arguably the most obvious of all norms. Here is its definition.

Definition 13.9.1 *Let A be a complex $m \times n$ matrix. Then*

$$||A||_F \equiv (\operatorname{trace}(AA^*))^{1/2}$$

Also this norm comes from the inner product

$$(A, B)_F \equiv \operatorname{trace}(AB^*)$$

Thus $||A||_F^2$ is easily seen to equal $\sum_{ij} |a_{ij}|^2$ so essentially, it treats the matrix as a vector in $\mathbb{F}^{m\times n}$.

Lemma 13.9.2 *Let A be an $m \times n$ complex matrix with singular matrix*

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

with σ as defined above. Then

$$||\Sigma||_F^2 = ||A||_F^2 \tag{13.20}$$

and the following hold for the Frobenius norm. If U, V are unitary and of the right size,

$$||UA||_F = ||A||_F, \ ||UAV||_F = ||A||_F. \tag{13.21}$$

Proof: From the definition and letting U, V be unitary and of the right size,

$$||UA||_F^2 \equiv \text{trace}\,(UAA^*U^*) = \text{trace}\,(AA^*) = ||A||_F^2$$

Also,

$$||AV||_F^2 \equiv \text{trace}\,(AVV^*A^*) = \text{trace}\,(AA^*) = ||A||_F^2\,.$$

It follows

$$||UAV||_F^2 = ||AV||_F^2 = ||A||_F^2\,.$$

Now consider 13.20. From what was just shown,

$$||A||_F^2 = ||U\Sigma V^*||_F^2 = ||\Sigma||_F^2\,.$$

This proves the lemma.

Of course, this shows that

$$||A||_F^2 = \sum_i \sigma_i^2,$$

the sum of the squares of the singular values of A.

Why is the singular value decomposition important? It implies

$$A = U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*$$

where σ is the diagonal matrix having the singular values down the diagonal. Now sometimes A is a huge matrix, 1000×2000 or something like that. This happens in applications to situations where the entries of A describe a picture. What also happens is that most of the singular values are very small. What if you deleted those which were very small, say for all $i \geq l$ and got a new matrix,

$$A' \equiv U \begin{pmatrix} \sigma' & 0 \\ 0 & 0 \end{pmatrix} V^*?$$

Then the entries of A' would end up being close to the entries of A but there is much less information to keep track of. This turns out to be very useful. More precisely, letting

$$\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix},\ U^*AV = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix},$$

$$||A - A'||_F^2 = \left|\left|U \begin{pmatrix} \sigma - \sigma' & 0 \\ 0 & 0 \end{pmatrix} V^*\right|\right|_F^2 = \sum_{k=l+1}^{r} \sigma_k^2$$

Thus A is approximated by A' where A' has rank $l < r$. In fact, it is also true that out of all matrices of rank l, this A' is the one which is closest to A in the Frobenius norm. Here is why.

Let B be a matrix which has rank l. Then from Lemma 13.9.2

$$\begin{aligned} ||A - B||_F^2 &= ||U^*(A - B)V||_F^2 = ||U^*AV - U^*BV||_F^2 \\ &= \left|\left|\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} - U^*BV\right|\right|_F^2 \end{aligned}$$

and since the singular values of A decrease from the upper left to the lower right, it follows that for B to be closest as possible to A in the Frobenius norm,

$$U^*BV = \begin{pmatrix} \sigma' & 0 \\ 0 & 0 \end{pmatrix}$$

which implies $B = A'$ above. This is really obvious if you look at a simple example. Say

$$\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for example. Then what rank 1 matrix would be closest to this one in the Frobenius norm? Obviously

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

13.10 Least Squares And Singular Value Decomposition

The singular value decomposition also has a very interesting connection to the problem of least squares solutions. Recall that it was desired to find $\mathbf{x}$ such that $|A\mathbf{x} - \mathbf{y}|$ is as small as possible. Lemma 12.5.1 shows that there is a solution to this problem which can be found by solving the system $A^*A\mathbf{x} = A^*\mathbf{y}$. Each $\mathbf{x}$ which solves this system solves the minimization problem as was shown in the lemma just mentioned. Now consider this equation for the solutions of the minimization problem in terms of the singular value decomposition.

$$\overbrace{V \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} U^* U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*}^{A^* \qquad A} \mathbf{x} = \overbrace{V \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} U^*}^{A^*} \mathbf{y}.$$

Therefore, this yields the following upon using block multiplication and multiplying on the left by V^*.

$$\begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} V^*\mathbf{x} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} U^*\mathbf{y}. \tag{13.22}$$

One solution to this equation which is very easy to spot is

$$\mathbf{x} = V \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*\mathbf{y}. \tag{13.23}$$

13.11 The Moore Penrose Inverse

The particular solution of the least squares problem given in 13.23 is important enough that it motivates the following definintion.

Definition 13.11.1 *Let A be an $m \times n$ matrix. Then the Moore Penrose inverse of A, denoted by A^+ is defined as*

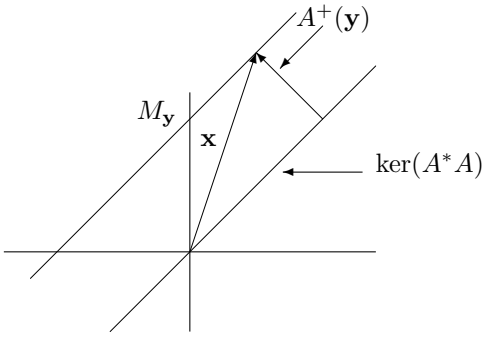
$$A^+ \equiv V \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

Here

$$U^*AV = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

as above.

Thus $A^+\mathbf{y}$ is a solution to the minimization problem to find $\mathbf{x}$ which minimizes $|A\mathbf{x} - \mathbf{y}|$. In fact, one can say more about this. In the following picture $M_{\mathbf{y}}$ denotes the set of least squares solutions $\mathbf{x}$ such that $A^*A\mathbf{x} = A^*\mathbf{y}$.



Then $A^+(\mathbf{y})$ is as given in the picture.

Proposition 13.11.2 *$A^+\mathbf{y}$ is the solution to the problem of minimizing $|A\mathbf{x}-\mathbf{y}|$ for all $\mathbf{x}$ which has smallest norm. Thus*

$$\left|AA^+\mathbf{y}-\mathbf{y}\right| \leq |A\mathbf{x}-\mathbf{y}| \text{ for all } \mathbf{x}$$

and if $\mathbf{x}_1$ satisfies $|A\mathbf{x}_1-\mathbf{y}| \leq |A\mathbf{x}-\mathbf{y}|$ for all $\mathbf{x}$, then $|A^+\mathbf{y}| \leq |\mathbf{x}_1|$.

Proof: Consider $\mathbf{x}$ satisfying 13.22, equivalently $A^*A\mathbf{x} = A^*\mathbf{y}$,

$$\begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} V^*\mathbf{x} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} U^*\mathbf{y}$$

which has smallest norm. This is equivalent to making $|V^*\mathbf{x}|$ as small as possible because V^* is unitary and so it preserves norms. For $\mathbf{z}$ a vector, denote by $(\mathbf{z})_k$ the vector in $\mathbb{F}^k$ which consists of the first k entries of $\mathbf{z}$. Then if $\mathbf{x}$ is a solution to 13.22

$$\begin{pmatrix} \sigma^2 (V^*\mathbf{x})_k \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \sigma (U^*\mathbf{y})_k \\ \mathbf{0} \end{pmatrix}$$

and so $(V^*\mathbf{x})_k = \sigma^{-1}(U^*\mathbf{y})_k$. Thus the first k entries of $V^*\mathbf{x}$ are determined. In order to make $|V^*\mathbf{x}|$ as small as possible, the remaining $n-k$ entries should equal zero. Therefore,

$$V^*\mathbf{x} = \begin{pmatrix} (V^*\mathbf{x})_k \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma^{-1}(U^*\mathbf{y})_k \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*\mathbf{y}$$

and so

$$\mathbf{x} = V \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*\mathbf{y} \equiv A^+\mathbf{y}$$

This proves the proposition.

Lemma 13.11.3 *The matrix, A^+ satisfies the following conditions.*

$$AA^+A = A,\ A^+AA^+ = A^+,\ A^+A \text{ and } AA^+ \text{ are Hermitian.} \tag{13.24}$$

Proof: This is routine. Recall

$$A = U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*$$

and

$$A^+ = V \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$$

so you just plug in and verify it works.

A much more interesting observation is that A^+ is characterized as being the unique matrix which satisfies 13.24. This is the content of the following Theorem.

Theorem 13.11.4 *Let A be an $m \times n$ matrix. Then a matrix, A_0, is the Moore Penrose inverse of A if and only if A_0 satisfies*

$$AA_0A = A,\ A_0AA_0 = A_0,\ A_0A \text{ and } AA_0 \text{ are Hermitian.} \tag{13.25}$$

Proof: From the above lemma, the Moore Penrose inverse satisfies 13.25. Suppose then that A_0 satisfies 13.25. It is necessary to verify that $A_0 = A^+$. Recall that from the singular value decomposition, there exist unitary matrices, U and V such that

$$U^*AV = \Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix},\ A = U\Sigma V^*.$$

Let

$$V^*A_0U = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{13.26}$$

where P is $k \times k$.

Next use the first equation of 13.25 to write

$$\overbrace{U\Sigma V^*}^{A} \overbrace{V \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*}^{A_0} \overbrace{U\Sigma V^*}^{A} = \overbrace{U\Sigma V^*}^{A}.$$

Then multiplying both sides on the left by U^* and on the right by V,

$$\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

Now this requires

$$\begin{pmatrix} \sigma P \sigma & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{13.27}$$

Therefore, $P = \sigma^{-1}$. From the requirement that AA_0 is Hermitian,

$$\overbrace{U\Sigma V^*}^{A} \overbrace{V \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*}^{A_0} = U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*$$

must be Hermitian. Therefore, it is necessary that

$$\begin{aligned} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} &= \begin{pmatrix} \sigma P & \sigma Q \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & \sigma Q \\ 0 & 0 \end{pmatrix} \end{aligned}$$

is Hermitian. Then

$$\begin{pmatrix} I & \sigma Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ Q^*\sigma & 0 \end{pmatrix}$$

Thus

$$Q^*\sigma = 0$$

and so multiplying both sides on the right by σ^{-1}, it follows $Q^* = 0$ and so $Q = 0$.

From the requirement that $A_0 A$ is Hermitian, it is necessary that

$$\begin{aligned} \overbrace{V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*}^{A_0}\overbrace{U\Sigma V^*}^{A} &= V\begin{pmatrix} P\sigma & 0 \\ R\sigma & 0 \end{pmatrix}V^* \\ &= V\begin{pmatrix} I & 0 \\ R\sigma & 0 \end{pmatrix}V^* \end{aligned}$$

is Hermitian. Therefore, also

$$\begin{pmatrix} I & 0 \\ R\sigma & 0 \end{pmatrix}$$

is Hermitian. Thus $R = 0$ because this equals

$$\begin{pmatrix} I & 0 \\ R\sigma & 0 \end{pmatrix}^* = \begin{pmatrix} I & \sigma^* R^* \\ 0 & 0 \end{pmatrix}$$

which requires $R\sigma = 0$. Now multiply on right by σ^{-1} to find that $R = 0$.

Use 13.26 and the second equation of 13.25 to write

$$\overbrace{V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*}^{A_0}\overbrace{U\Sigma V^*}^{A}\overbrace{V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*}^{A_0} = \overbrace{V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*}^{A_0}.$$

which implies

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

This yields from the above in which is was shown that R, Q are both 0

$$\begin{aligned} \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & S \end{pmatrix} &= \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} && (13.28) \\ &= \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & S \end{pmatrix}. && (13.29) \end{aligned}$$

Therefore, $S = 0$ also and so

$$V^* A_0 U \equiv \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

which says

$$A_0 = V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* \equiv A^+.$$

This proves the theorem.

The theorem is significant because there is no mention of eigenvalues or eigenvectors in the characterization of the Moore Penrose inverse given in 13.25. It also shows immediately that the Moore Penrose inverse is a generalization of the usual inverse. See Problem 3.

13.12 Exercises

1. ♠Show $(A^*)^* = A$ and $(AB)^* = B^*A^*$.

2. Prove Corollary 13.3.9.

3. ♠Show that if A is an $n \times n$ matrix which has an inverse then $A^+ = A^{-1}$.

4. ♠Using the singular value decomposition, show that for any square matrix, A, it follows that A^*A is unitarily similar to AA^*.

5. Let A, B be a $m \times n$ matrices. Define an inner product on the set of $m \times n$ matrices by
$$(A, B)_F \equiv \operatorname{trace}(AB^*).$$
Show this is an inner product satisfying all the inner product axioms. Recall for M an $n \times n$ matrix, $\operatorname{trace}(M) \equiv \sum_{i=1}^n M_{ii}$. The resulting norm, $||\cdot||_F$ is called the Frobenius norm and it can be used to measure the distance between two matrices.

6. Let A be an $m \times n$ matrix. Show
$$||A||_F^2 \equiv (A, A)_F = \sum_j \sigma_j^2$$
where the σ_j are the singular values of A.

7. ♠If A is a general $n \times n$ matrix having possibly repeated eigenvalues, show there is a sequence $\{A_k\}$ of $n \times n$ matrices having distinct eigenvalues which has the property that the ij^{th} entry of A_k converges to the ij^{th} entry of A for all ij. **Hint:** Use Schur's theorem.

8. ♠Prove the Cayley Hamilton theorem as follows. First suppose A has a basis of eigenvectors $\{\mathbf{v}_k\}_{k=1}^n, A\mathbf{v}_k = \lambda_k \mathbf{v}_k$. Let $p(\lambda)$ be the characteristic polynomial. Show $p(A)\mathbf{v}_k = p(\lambda_k)\mathbf{v}_k = \mathbf{0}$. Then since $\{\mathbf{v}_k\}$ is a basis, it follows $p(A)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ and so $p(A) = 0$. Next in the general case, use Problem 7 to obtain a sequence $\{A_k\}$ of matrices whose entries converge to the entries of A such that A_k has n distinct eigenvalues and therefore by Theorem 7.1.7 A_k has a basis of eigenvectors. Therefore, from the first part and for $p_k(\lambda)$ the characteristic polynomial for A_k, it follows $p_k(A_k) = 0$. Now explain why and the sense in which
$$\lim_{k\to\infty} p_k(A_k) = p(A).$$

9. Prove that Theorem 13.4.6 and Corollary 13.4.7 can be strengthened so that the condition on the A_k is necessary as well as sufficient. **Hint:** Consider vectors of the form $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$ where $\mathbf{x} \in \mathbb{F}^k$.

10. ♠ Show directly that if A is an $n \times n$ matrix and $A = A^*$ (A is Hermitian) then all the eigenvalues and eigenvectors are real and that eigenvectors associated with distinct eigenvalues are orthogonal, (their inner product is zero).

11. Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be an orthonormal basis for $\mathbb{F}^n$. Let Q be a matrix whose i^{th} column is $\mathbf{v}_i$. Show
$$Q^*Q = QQ^* = I.$$

12. ♠ Show that an $n \times n$ matrix, Q is unitary if and only if it preserves distances. This means $|Q\mathbf{v}| = |\mathbf{v}|$. This was done in the text but you should try to do it for yourself.

13. ♠ Suppose $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ are two orthonormal bases for $\mathbb{F}^n$ and suppose Q is an $n \times n$ matrix satisfying $Q\mathbf{v}_i = \mathbf{w}_i$. Then show Q is unitary. If $|\mathbf{v}| = 1$, show there is a unitary transformation which maps $\mathbf{v}$ to $\mathbf{e}_1$.

14. ♠Finish the proof of Theorem 13.6.5.

15. ♠ Let A be a Hermitian matrix so $A = A^*$ and suppose all eigenvalues of A are larger than δ^2. Show
$$(A\mathbf{v}, \mathbf{v}) \geq \delta^2 |\mathbf{v}|^2$$
Where here, the inner product is
$$(\mathbf{v}, \mathbf{u}) \equiv \sum_{j=1}^{n} v_j \overline{u_j}.$$

16. The discrete Fourier transform maps $\mathbb{C}^n \to \mathbb{C}^n$ as follows.
$$F(\mathbf{x}) = \mathbf{z} \text{ where } z_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{-i\frac{2\pi}{n}jk} x_j.$$
Show that F^{-1} exists and is given by the formula
$$F^{-1}(\mathbf{z}) = \mathbf{x} \text{ where } x_j = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{i\frac{2\pi}{n}jk} z_k$$
Here is one way to approach this problem. Note $\mathbf{z} = U\mathbf{x}$ where
$$U = \frac{1}{\sqrt{n}} \begin{pmatrix} e^{-i\frac{2\pi}{n}0\cdot 0} & e^{-i\frac{2\pi}{n}1\cdot 0} & e^{-i\frac{2\pi}{n}2\cdot 0} & \cdots & e^{-i\frac{2\pi}{n}(n-1)\cdot 0} \\ e^{-i\frac{2\pi}{n}0\cdot 1} & e^{-i\frac{2\pi}{n}1\cdot 1} & e^{-i\frac{2\pi}{n}2\cdot 1} & \cdots & e^{-i\frac{2\pi}{n}(n-1)\cdot 1} \\ e^{-i\frac{2\pi}{n}0\cdot 2} & e^{-i\frac{2\pi}{n}1\cdot 2} & e^{-i\frac{2\pi}{n}2\cdot 2} & \cdots & e^{-i\frac{2\pi}{n}(n-1)\cdot 2} \\ \vdots & \vdots & \vdots & & \vdots \\ e^{-i\frac{2\pi}{n}0\cdot (n-1)} & e^{-i\frac{2\pi}{n}1\cdot (n-1)} & e^{-i\frac{2\pi}{n}2\cdot (n-1)} & \cdots & e^{-i\frac{2\pi}{n}(n-1)\cdot (n-1)} \end{pmatrix}$$
Now argue U is unitary and use this to establish the result. To show this verify each row has length 1 and the dot product of two different rows gives 0. Now $U_{kj} = e^{-i\frac{2\pi}{n}jk}$ and so $(U^*)_{kj} = e^{i\frac{2\pi}{n}jk}$.

17. Let f be a periodic function having period 2π. The Fourier series of f is an expression of the form
$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \equiv \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}$$
and the idea is to find c_k such that the above sequence converges in some way to f. If
$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

and you formally multiply both sides by e^{-imx} and then integrate from 0 to 2π, interchanging the integral with the sum without any concern for whether this makes sense, show it is reasonable from this to expect

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx} dx.$$

Now suppose you only know $f(x)$ at equally spaced points $2\pi j/n$ for $j = 0, 1, \cdots, n$. Consider the Riemann sum for this integral obtained from using the left endpoint of the subintervals determined from the partition $\left\{\frac{2\pi}{n} j\right\}_{j=0}^{n}$. How does this compare with the discrete Fourier transform? What happens as $n \to \infty$ to this approximation?

18. ♠Suppose A is a real 3×3 orthogonal matrix (Recall this means $AA^T = A^T A = I$.) having determinant 1. Show it must have an eigenvalue equal to 1. Note this shows there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{x}$. **Hint:** Show first or recall that any orthogonal matrix must preserve lengths. That is, $|A\mathbf{x}| = |\mathbf{x}|$.

19. ♠Let A be a complex $m \times n$ matrix. Using the description of the Moore Penrose inverse in terms of the singular value decomposition, show that

$$\lim_{\delta \to 0+} \left(A^* A + \delta I\right)^{-1} A^* = A^+$$

where the convergence happens in the Frobenius norm. Also verify, using the singular value decomposition, that the inverse exists in the above formula.

Norms For Finite Dimensional Vector Spaces

In this chapter, X and Y are finite dimensional vector spaces which have a norm. The following is a definition.

Definition 14.0.1 *A linear space X is a normed linear space if there is a norm defined on X, $||\cdot||$ satisfying*

$$||\mathbf{x}|| \geq 0, \;\; ||\mathbf{x}|| = 0 \text{ if and only if } \mathbf{x} = 0,$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||,$$

$$||c\mathbf{x}|| = |c| \, ||\mathbf{x}||$$

whenever c is a scalar. A set, $U \subseteq X$, a normed linear space is open if for every $p \in U$, there exists $\delta > 0$ such that

$$B(p, \delta) \equiv \{x : ||x - p|| < \delta\} \subseteq U.$$

Thus, a set is open if every point of the set is an interior point.

To begin with recall the Cauchy Schwarz inequality which is stated here for convenience in terms of the inner product space, $\mathbb{C}^n$.

Theorem 14.0.2 *The following inequality holds for a_i and $b_i \in \mathbb{C}$.*

$$\left| \sum_{i=1}^{n} a_i \overline{b}_i \right| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{1/2}. \tag{14.1}$$

Definition 14.0.3 *Let $(X, ||\cdot||)$ be a normed linear space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors. Then this is called a Cauchy sequence if for all $\varepsilon > 0$ there exists N such that if $m, n \geq N$, then*

$$||x_n - x_m|| < \varepsilon.$$

This is written more briefly as

$$\lim_{m,n \to \infty} ||x_n - x_m|| = 0.$$

Definition 14.0.4 *A normed linear space, $(X, ||\cdot||)$ is called a Banach space if it is complete. This means that, whenever, $\{\mathbf{x}_n\}$ is a Cauchy sequence there exists a unique $\mathbf{x} \in X$ such that $\lim_{n \to \infty} ||\mathbf{x} - \mathbf{x}_n|| = 0$.*

Let X be a finite dimensional normed linear space with norm $||\cdot||$ where the field of scalars is denoted by $\mathbb{F}$ and is understood to be either $\mathbb{R}$ or $\mathbb{C}$. Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for X. If $\mathbf{x} \in X$, denote by x_i the i^{th} component of $\mathbf{x}$ with respect to this basis. Thus

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i.$$

Definition 14.0.5 *For $\mathbf{x} \in X$ and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ a basis, define a new norm by*

$$|\mathbf{x}| \equiv \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

where

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i.$$

Similarly, for $\mathbf{y} \in Y$ with basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$, and y_i its components with respect to this basis,

$$|\mathbf{y}| \equiv \left(\sum_{i=1}^{m} |y_i|^2 \right)^{1/2}$$

For $A \in \mathcal{L}(X, Y)$, the space of linear mappings from X to Y,

$$||A|| \equiv \sup\{|A\mathbf{x}| : |\mathbf{x}| \leq 1\}. \tag{14.2}$$

The first thing to show is that the two norms, $||\cdot||$ and $|\cdot|$, are equivalent. This means the conclusion of the following theorem holds.

Theorem 14.0.6 *Let $(X, ||\cdot||)$ be a finite dimensional normed linear space and let $|\cdot|$ be described above relative to a given basis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Then $|\cdot|$ is a norm and there exist constants $\delta, \Delta > 0$ independent of $\mathbf{x}$ such that*

$$\delta ||\mathbf{x}|| \leq |\mathbf{x}| \leq \Delta ||\mathbf{x}||. \tag{14.3}$$

Proof: All of the above properties of a norm are obvious except the second, the triangle inequality. To establish this inequality, use the Cauchy Schwartz inequality to write

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &\equiv \sum_{i=1}^{n} |x_i + y_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2 + 2 \operatorname{Re} \sum_{i=1}^{n} x_i \overline{y}_i \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2 \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2} \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2 |\mathbf{x}| |\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

and this proves the second property above.

It remains to show the equivalence of the two norms. By the Cauchy Schwartz inequality again,

$$\begin{aligned} ||\mathbf{x}|| &\equiv \left|\left| \sum_{i=1}^{n} x_i \mathbf{v}_i \right|\right| \leq \sum_{i=1}^{n} |x_i| \, ||\mathbf{v}_i|| \leq |\mathbf{x}| \left(\sum_{i=1}^{n} ||\mathbf{v}_i||^2 \right)^{1/2} \\ &\equiv \delta^{-1} |\mathbf{x}|. \end{aligned}$$

This proves the first half of the inequality.

Suppose the second half of the inequality is not valid. Then there exists a sequence $\mathbf{x}^k \in X$ such that

$$\left|\mathbf{x}^k\right| > k \left|\left|\mathbf{x}^k\right|\right|, \ k = 1, 2, \cdots .$$

Then define

$$\mathbf{y}^k \equiv \frac{\mathbf{x}^k}{\left|\mathbf{x}^k\right|}.$$

It follows

$$\left|\mathbf{y}^k\right| = 1, \quad \left|\mathbf{y}^k\right| > k \left|\left|\mathbf{y}^k\right|\right|. \tag{14.4}$$

Letting y_i^k be the components of $\mathbf{y}^k$ with respect to the given basis, it follows the vector

$$\left(y_1^k, \cdots, y_n^k\right)$$

is a unit vector in $\mathbb{F}^n$. By the Heine Borel theorem, there exists a subsequence, still denoted by k such that

$$\left(y_1^k, \cdots, y_n^k\right) \to \left(y_1, \cdots, y_n\right).$$

It follows from 14.4 and this that for

$$\mathbf{y} = \sum_{i=1}^n y_i \mathbf{v}_i,$$

$$0 = \lim_{k\to\infty} \left|\left|\mathbf{y}^k\right|\right| = \lim_{k\to\infty} \left|\left|\sum_{i=1}^n y_i^k \mathbf{v}_i\right|\right| = \left|\left|\sum_{i=1}^n y_i \mathbf{v}_i\right|\right|$$

but not all the y_i equal zero. This contradicts the assumption that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis and proves the second half of the inequality.

Corollary 14.0.7 *If $(X, ||\cdot||)$ is a finite dimensional normed linear space with the field of scalars $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, then X is complete.*

Proof: Let $\{\mathbf{x}^k\}$ be a Cauchy sequence. Then letting the components of $\mathbf{x}^k$ with respect to the given basis be

$$x_1^k, \cdots, x_n^k,$$

it follows from Theorem 14.0.6, that

$$\left(x_1^k, \cdots, x_n^k\right)$$

is a Cauchy sequence in $\mathbb{F}^n$ and so

$$\left(x_1^k, \cdots, x_n^k\right) \to \left(x_1, \cdots, x_n\right) \in \mathbb{F}^n.$$

Thus,

$$\mathbf{x}^k = \sum_{i=1}^n x_i^k \mathbf{v}_i \to \sum_{i=1}^n x_i \mathbf{v}_i \in X.$$

This proves the corollary.

Corollary 14.0.8 *Suppose X is a finite dimensional linear space with the field of scalars either $\mathbb{C}$ or $\mathbb{R}$ and $||\cdot||$ and $|||\cdot|||$ are two norms on X. Then there exist positive constants, δ and Δ, independent of $\mathbf{x} \in X$ such that*

$$\delta \left|||\mathbf{x}|||\right| \leq ||\mathbf{x}|| \leq \Delta \left|||\mathbf{x}|||\right|.$$

Thus any two norms are equivalent.

This is very important because it shows that all questions of convergence can be considered relative to any norm with the same outcome.

Proof: Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for X and let $|\cdot|$ be the norm taken with respect to this basis which was described earlier. Then by Theorem 14.0.6, there are positive constants $\delta_1, \Delta_1, \delta_2, \Delta_2$, all independent of $\mathbf{x} \in X$ such that

$$\delta_2 |||\mathbf{x}||| \leq |\mathbf{x}| \leq \Delta_2 |||\mathbf{x}|||,$$

$$\delta_1 ||\mathbf{x}|| \leq |\mathbf{x}| \leq \Delta_1 ||\mathbf{x}||.$$

Then

$$\delta_2 |||\mathbf{x}||| \leq |\mathbf{x}| \leq \Delta_1 ||\mathbf{x}|| \leq \frac{\Delta_1}{\delta_1} |\mathbf{x}| \leq \frac{\Delta_1 \Delta_2}{\delta_1} |||\mathbf{x}|||$$

and so

$$\frac{\delta_2}{\Delta_1} |||\mathbf{x}||| \leq ||\mathbf{x}|| \leq \frac{\Delta_2}{\delta_1} |||\mathbf{x}|||$$

which proves the corollary.

Definition 14.0.9 *Let X and Y be normed linear spaces with norms $||\cdot||_X$ and $||\cdot||_Y$ respectively. Then $\mathcal{L}(X,Y)$ denotes the space of linear transformations, called bounded linear transformations, mapping X to Y which have the property that*

$$||A|| \equiv \sup\{||Ax||_Y : ||x||_X \leq 1\} < \infty.$$

Then $||A||$ is referred to as the operator norm of the bounded linear transformation, A.

It is an easy exercise to verify that $||\cdot||$ is a norm on $\mathcal{L}(X,Y)$ and it is always the case that

$$||Ax||_Y \leq ||A|| \, ||x||_X.$$

Furthermore, you should verify that you can replace ≤ 1 with $= 1$ in the definition. Thus

$$||A|| \equiv \sup\{||Ax||_Y : ||x||_X = 1\}.$$

Theorem 14.0.10 *Let X and Y be finite dimensional normed linear spaces of dimension n and m respectively and denote by $||\cdot||$ the norm on either X or Y. Then if A is any linear function mapping X to Y, then $A \in \mathcal{L}(X,Y)$ and $(\mathcal{L}(X,Y), ||\cdot||)$ is a complete normed linear space of dimension nm with*

$$||A\mathbf{x}|| \leq ||A|| \, ||\mathbf{x}||.$$

Proof: It is necessary to show the norm defined on linear transformations really is a norm. Again the first and third properties listed above for norms are obvious. It remains to show the second and verify $||A|| < \infty$. Letting $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis and $|\cdot|$ defined with respect to this basis as above, there exist constants $\delta, \Delta > 0$ such that

$$\delta ||\mathbf{x}|| \leq |\mathbf{x}| \leq \Delta ||\mathbf{x}||.$$

Then,

$$\begin{aligned} ||A+B|| &\equiv \sup\{||(A+B)(\mathbf{x})|| : ||\mathbf{x}|| \leq 1\} \\ &\leq \sup\{||A\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} + \sup\{||B\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} \\ &\equiv ||A|| + ||B||. \end{aligned}$$

Next consider the claim that $||A|| < \infty$. This follows from

$$||A(\mathbf{x})|| = \left|\left| A\left(\sum_{i=1}^{n} x_i \mathbf{v}_i\right)\right|\right| \leq \sum_{i=1}^{n} |x_i| \, ||A(\mathbf{v}_i)||$$

$$\leq |\mathbf{x}| \left(\sum_{i=1}^{n} ||A(\mathbf{v}_i)||^2\right)^{1/2} \leq \Delta \, ||\mathbf{x}|| \left(\sum_{i=1}^{n} ||A(\mathbf{v}_i)||^2\right)^{1/2} < \infty.$$

Thus $||A|| \leq \Delta \left(\sum_{i=1}^{n} ||A(\mathbf{v}_i)||^2\right)^{1/2}$.

Next consider the assertion about the dimension of $\mathcal{L}(X,Y)$. Let the two sets of bases be

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \text{ and } \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$$

for X and Y respectively. Let $\mathbf{w}_i \otimes \mathbf{v}_k \in \mathcal{L}(X,Y)$ be defined by

$$\mathbf{w}_i \otimes \mathbf{v}_k \mathbf{v}_l \equiv \begin{cases} \mathbf{0} \text{ if } l \neq k \\ \mathbf{w}_i \text{ if } l = k \end{cases}$$

and let $L \in \mathcal{L}(X,Y)$. Then

$$L\mathbf{v}_r = \sum_{j=1}^{m} d_{jr} \mathbf{w}_j$$

for some d_{jk}. Also

$$\sum_{j=1}^{m} \sum_{k=1}^{n} d_{jk} \mathbf{w}_j \otimes \mathbf{v}_k (\mathbf{v}_r) = \sum_{j=1}^{m} d_{jr} \mathbf{w}_j.$$

It follows that

$$L = \sum_{j=1}^{m} \sum_{k=1}^{n} d_{jk} \mathbf{w}_j \otimes \mathbf{v}_k$$

because the two linear transformations agree on a basis. Since L is arbitrary this shows

$$\{\mathbf{w}_i \otimes \mathbf{v}_k : i = 1, \cdots, m, \ k = 1, \cdots, n\}$$

spans $\mathcal{L}(X,Y)$. If

$$\sum_{i,k} d_{ik} \mathbf{w}_i \otimes \mathbf{v}_k = \mathbf{0},$$

then

$$\mathbf{0} = \sum_{i,k} d_{ik} \mathbf{w}_i \otimes \mathbf{v}_k (\mathbf{v}_l) = \sum_{i=1}^{m} d_{il} \mathbf{w}_i$$

and so, since $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ is a basis, $d_{il} = 0$ for each $i = 1, \cdots, m$. Since l is arbitrary, this shows $d_{il} = 0$ for all i and l. Thus these linear transformations form a basis and this shows the dimension of $\mathcal{L}(X,Y)$ is mn as claimed. By Corollary 14.0.7 $(\mathcal{L}(X,Y), ||\cdot||)$ is complete. If $\mathbf{x} \neq \mathbf{0}$,

$$||A\mathbf{x}|| \frac{1}{||\mathbf{x}||} = \left|\left| A \frac{\mathbf{x}}{||\mathbf{x}||} \right|\right| \leq ||A||$$

This proves the theorem.

Note by Corollary 14.0.8 you can define a norm any way desired on any finite dimensional linear space which has the field of scalars $\mathbb{R}$ or $\mathbb{C}$ and any other way of defining a norm on this space yields an equivalent norm. Thus, it doesn't much matter as far as notions of

convergence are concerned which norm is used for a finite dimensional space. In particular in the space of $m \times n$ matrices, you can use the operator norm defined above, or some other way of giving this space a norm. A popular choice for a norm is the Frobenius norm discussed earlier but reviewed here.

Definition 14.0.11 *Make the space of $m \times n$ matrices into a Hilbert space by defining*

$$(A, B) \equiv tr\,(AB^*)\,.$$

Another way of describing a norm for an $n \times n$ matrix is as follows.

Definition 14.0.12 *Let A be an $m \times n$ matrix. Define the spectral norm of A, written as $||A||_2$ to be*

$$\max\left\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } A^*A\right\}.$$

That is, the largest singular value of A. (Note the eigenvalues of A^*A are all positive because if $A^*A\mathbf{x} = \lambda\mathbf{x}$, then

$$\lambda\,(\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) \geq 0.)$$

Actually, this is nothing new. It turns out that $||\cdot||_2$ is nothing more than the operator norm for A taken with respect to the usual Euclidean norm,

$$|\mathbf{x}| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}.$$

Proposition 14.0.13 *The following holds.*

$$||A||_2 = \sup\{|A\mathbf{x}| : |\mathbf{x}| = 1\} \equiv ||A||\,.$$

Proof: Note that A^*A is Hermitian and so by Corollary 13.3.5,

$$\begin{aligned} ||A||_2 &= \max\left\{(A^*A\mathbf{x}, \mathbf{x})^{1/2} : |\mathbf{x}| = 1\right\} \\ &= \max\left\{(A\mathbf{x}, A\mathbf{x})^{1/2} : |\mathbf{x}| = 1\right\} \\ &= \max\{|A\mathbf{x}| : |\mathbf{x}| = 1\} = ||A||\,. \end{aligned}$$

This proves the proposition.

Here is another proof of this proposition. Recall there are unitary matrices of the right size U, V such that $A = U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*$ where the matrix on the inside is as described in the section on the singular value decomposition. Then since unitary matrices preserve norms,

$$\begin{aligned} ||A|| &= \sup_{|\mathbf{x}|\leq 1} \left|U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*\mathbf{x}\right| = \sup_{|V^*\mathbf{x}|\leq 1} \left|U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*\mathbf{x}\right| \\ &= \sup_{|\mathbf{y}|\leq 1} \left|U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \mathbf{y}\right| = \sup_{|\mathbf{y}|\leq 1} \left|\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \mathbf{y}\right| = \sigma_1 \equiv ||A||_2 \end{aligned}$$

This completes the alternate proof.

From now on, $||A||_2$ will mean either the operator norm of A taken with respect to the usual Euclidean norm or the largest singular value of A, whichever is most convenient.

An interesting application of the notion of equivalent norms on $\mathbb{R}^n$ is the process of giving a norm on a finite Cartesian product of normed linear spaces.

Definition 14.0.14 *Let X_i, $i = 1, \cdots, n$ be normed linear spaces with norms, $||\cdot||_i$. For*

$$\mathbf{x} \equiv (x_1, \cdots, x_n) \in \prod_{i=1}^{n} X_i$$

define $\theta : \prod_{i=1}^{n} X_i \to \mathbb{R}^n$ by

$$\theta(\mathbf{x}) \equiv (||x_1||_1, \cdots, ||x_n||_n)$$

Then if $||\cdot||$ is any norm on $\mathbb{R}^n$, define a norm on $\prod_{i=1}^{n} X_i$, also denoted by $||\cdot||$ by

$$||\mathbf{x}|| \equiv ||\theta\mathbf{x}||.$$

The following theorem follows immediately from Corollary 14.0.8.

Theorem 14.0.15 *Let X_i and $||\cdot||_i$ be given in the above definition and consider the norms on $\prod_{i=1}^{n} X_i$ described there in terms of norms on $\mathbb{R}^n$. Then any two of these norms on $\prod_{i=1}^{n} X_i$ obtained in this way are equivalent.*

For example, define

$$||\mathbf{x}||_1 \equiv \sum_{i=1}^{n} |x_i|,$$

$$||\mathbf{x}||_\infty \equiv \max\{|x_i|, i = 1, \cdots, n\},$$

or

$$||\mathbf{x}||_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

and all three are equivalent norms on $\prod_{i=1}^{n} X_i$.

14.1 The p Norms

In addition to $||\cdot||_1$ and $||\cdot||_\infty$ mentioned above, it is common to consider the so called p norms for $\mathbf{x} \in \mathbb{C}^n$.

Definition 14.1.1 *Let $\mathbf{x} \in \mathbb{C}^n$. Then define for $p \geq 1$,*

$$||\mathbf{x}||_p \equiv \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

The following inequality is called Holder's inequality.

Proposition 14.1.2 *For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,*

$$\sum_{i=1}^{n} |x_i|\,|y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^{p'}\right)^{1/p'}$$

The proof will depend on the following lemma.

Lemma 14.1.3 *If $a, b \geq 0$ and p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Proof of the Proposition: If $\mathbf{x}$ or $\mathbf{y}$ equals the zero vector there is nothing to prove. Therefore, assume they are both nonzero. Let $A = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ and $B = \left(\sum_{i=1}^n |y_i|^{p'}\right)^{1/p'}$. Then using Lemma 14.1.3,

$$\begin{aligned}\sum_{i=1}^n \frac{|x_i|}{A}\frac{|y_i|}{B} &\leq \sum_{i=1}^n \left[\frac{1}{p}\left(\frac{|x_i|}{A}\right)^p + \frac{1}{p'}\left(\frac{|y_i|}{B}\right)^{p'}\right] \\ &= \frac{1}{p}\frac{1}{A^p}\sum_{i=1}^n |x_i|^p + \frac{1}{p'}\frac{1}{B^p}\sum_{i=1}^n |y_i|^{p'} \\ &= \frac{1}{p} + \frac{1}{p'} = 1\end{aligned}$$

and so

$$\sum_{i=1}^n |x_i|\,|y_i| \leq AB = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\left(\sum_{i=1}^n |y_i|^{p'}\right)^{1/p'}.$$

This proves the proposition.

Theorem 14.1.4 *The p norms do indeed satisfy the axioms of a norm.*

Proof: It is obvious that $||\cdot||_p$ does indeed satisfy most of the norm axioms. The only one that is not clear is the triangle inequality. To save notation write $||\cdot||$ in place of $||\cdot||_p$ in what follows. Note also that $\frac{p}{p'} = p - 1$. Then using the Holder inequality,

$$\begin{aligned}||\mathbf{x}+\mathbf{y}||^p &= \sum_{i=1}^n |x_i + y_i|^p \\ &\leq \sum_{i=1}^n |x_i + y_i|^{p-1}|x_i| + \sum_{i=1}^n |x_i + y_i|^{p-1}|y_i| \\ &= \sum_{i=1}^n |x_i + y_i|^{\frac{p}{p'}}|x_i| + \sum_{i=1}^n |x_i + y_i|^{\frac{p}{p'}}|y_i| \\ &\leq \left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p'}\left[\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}\right] \\ &= ||\mathbf{x}+\mathbf{y}||^{p/p'}\left(||\mathbf{x}||_p + ||\mathbf{y}||_p\right)\end{aligned}$$

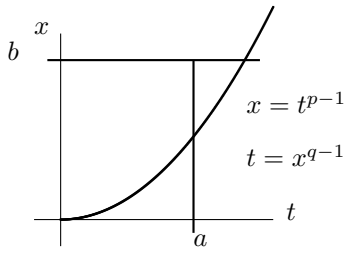
so dividing by $||\mathbf{x}+\mathbf{y}||^{p/p'}$, it follows

$$||\mathbf{x}+\mathbf{y}||^p\,||\mathbf{x}+\mathbf{y}||^{-p/p'} = ||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}||_p + ||\mathbf{y}||_p$$

$\left(p - \frac{p}{p'} = p\left(1 - \frac{1}{p'}\right) = p\frac{1}{p} = 1.\right)$. This proves the theorem.

It only remains to prove Lemma 14.1.3.

Proof of the lemma: Let $p' = q$ to save on notation and consider the following picture:



$$ab \leq \int_0^a t^{p-1}dt + \int_0^b x^{q-1}dx = \frac{a^p}{p} + \frac{b^q}{q}.$$

Note equality occurs when $a^p = b^q$.

Alternate proof of the lemma: Let

$$f(t) \equiv \frac{1}{p}(at)^p + \frac{1}{q}\left(\frac{b}{t}\right)^q,\ t > 0$$

You see right away it is decreasing for a while, having an assymptote at $t = 0$ and then reaches a minimum and increases from then on. Take its derivative.

$$f'(t) = (at)^{p-1}a + \left(\frac{b}{t}\right)^{q-1}\left(\frac{-b}{t^2}\right)$$

Set it equal to 0. This happens when

$$t^{p+q} = \frac{b^q}{a^p}. \tag{14.5}$$

Thus

$$t = \frac{b^{q/(p+q)}}{a^{p/(p+q)}}$$

and so at this value of t,

$$at = (ab)^{q/(p+q)},\ \left(\frac{b}{t}\right) = (ab)^{p/(p+q)}.$$

Thus the minimum of f is

$$\frac{1}{p}\left((ab)^{q/(p+q)}\right)^p + \frac{1}{q}\left((ab)^{p/(p+q)}\right)^q = (ab)^{pq/(p+q)}$$

but recall $1/p + 1/q = 1$ and so $pq/(p+q) = 1$. Thus the minimum value of f is ab. Letting $t = 1$, this shows

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Note that equality occurs when the minimum value happens for $t = 1$ and this indicates from 14.5 that $a^p = b^q$. This proves the lemma.

Now $||A||_p$ may be considered as the operator norm of A taken with respect to $||\cdot||_p$. In the case when $p = 2$, this is just the spectral norm. There is an easy estimate for $||A||_p$ in terms of the entries of A.

Theorem 14.1.5 *The following holds.*

$$||A||_p \leq \left(\sum_k \left(\sum_j |A_{jk}|^p \right)^{q/p} \right)^{1/q}$$

Proof: Let $||\mathbf{x}||_p \leq 1$ and let $A = (\mathbf{a}_1, \cdots, \mathbf{a}_n)$ where the $\mathbf{a}_k$ are the columns of A. Then

$$A\mathbf{x} = \left(\sum_k x_k \mathbf{a}_k \right)$$

and so by Holder's inequality,

$$\begin{aligned} ||A\mathbf{x}||_p &\equiv \left|\left| \sum_k x_k \mathbf{a}_k \right|\right|_p \leq \sum_k |x_k| \, ||\mathbf{a}_k||_p \\ &\leq \left(\sum_k |x_k|^p \right)^{1/p} \left(\sum_k ||\mathbf{a}_k||_p^q \right)^{1/q} \\ &\leq \left(\sum_k \left(\sum_j |A_{jk}|^p \right)^{q/p} \right)^{1/q} \end{aligned}$$

and this shows $||A||_p \leq \left(\sum_k \left(\sum_j |A_{jk}|^p \right)^{q/p} \right)^{1/q}$ and proves the theorem.

14.2 The Condition Number

Let $A \in \mathcal{L}(X, X)$ be a linear transformation where X is a finite dimensional vector space and consider the problem $Ax = b$ where it is assumed there is a unique solution to this problem. How does the solution change if A is changed a little bit and if b is changed a little bit? This is clearly an interesting question because you often do not know A and b exactly. If a small change in these quantities results in a large change in the solution, x, then it seems clear this would be undesirable. In what follows $||\cdot||$ when applied to a linear transformation will always refer to the operator norm.

Lemma 14.2.1 *Let $A, B \in \mathcal{L}(X, X)$ where X is a normed vector space as above. Then for $||\cdot||$ denoting the operator norm,*

$$||AB|| \leq ||A|| \, ||B|| \, .$$

Proof: This follows from the definition. Letting $||x|| \leq 1$, it follows from Theorem 14.0.10

$$||ABx|| \leq ||A|| \, ||Bx|| \leq ||A|| \, ||B|| \, ||x|| \leq ||A|| \, ||B||$$

and so

$$||AB|| \equiv \sup_{||x|| \leq 1} ||ABx|| \leq ||A|| \, ||B|| \, .$$

This proves the lemma.

Lemma 14.2.2 *Let $A, B \in \mathcal{L}(X,X), A^{-1} \in \mathcal{L}(X,X)$, and suppose $||B|| < 1/\left|\left|A^{-1}\right|\right|$. Then $(A+B)^{-1}$ exists and*

$$\left|\left|(A+B)^{-1}\right|\right| \leq \left|\left|A^{-1}\right|\right| \left|\frac{1}{1-||A^{-1}B||}\right|.$$

The above formula makes sense because $\left|\left|A^{-1}B\right|\right| < 1$.

Proof: By Lemma 14.2.1,

$$\left|\left|A^{-1}B\right|\right| \leq \left|\left|A^{-1}\right|\right| ||B|| < \left|\left|A^{-1}\right|\right| \frac{1}{||A^{-1}||} = 1$$

Suppose $(A+B)x = 0$. Then $0 = A\left(I + A^{-1}B\right)x$ and so since A is one to one, $\left(I + A^{-1}B\right)x = 0$. Therefore,

$$\begin{aligned} 0 &= \left|\left|\left(I + A^{-1}B\right)x\right|\right| \geq ||x|| - \left|\left|A^{-1}Bx\right|\right| \\ &\geq ||x|| - \left|\left|A^{-1}B\right|\right| ||x|| = \left(1 - \left|\left|A^{-1}B\right|\right|\right) ||x|| > 0 \end{aligned}$$

a contradiction. This also shows $\left(I + A^{-1}B\right)$ is one to one. Therefore, both $(A+B)^{-1}$ and $\left(I + A^{-1}B\right)^{-1}$ are in $\mathcal{L}(X,X)$. Hence

$$(A+B)^{-1} = \left(A\left(I + A^{-1}B\right)\right)^{-1} = \left(I + A^{-1}B\right)^{-1} A^{-1}$$

Now if

$$x = \left(I + A^{-1}B\right)^{-1} y$$

for $||y|| \leq 1$, then

$$\left(I + A^{-1}B\right)x = y$$

and so

$$||x|| \left(1 - \left|\left|A^{-1}B\right|\right|\right) \leq \left|\left|x + A^{-1}Bx\right|\right| \leq ||y|| = 1$$

and so

$$||x|| = \left|\left|\left(I + A^{-1}B\right)^{-1} y\right|\right| \leq \frac{1}{1 - ||A^{-1}B||}$$

Since $||y|| \leq 1$ is arbitrary, this shows

$$\left|\left|\left(I + A^{-1}B\right)^{-1}\right|\right| \leq \frac{1}{1 - ||A^{-1}B||}$$

Therefore,

$$\begin{aligned} \left|\left|(A+B)^{-1}\right|\right| &= \left|\left|\left(I + A^{-1}B\right)^{-1} A^{-1}\right|\right| \\ &\leq \left|\left|A^{-1}\right|\right| \left|\left|\left(I + A^{-1}B\right)^{-1}\right|\right| \leq \left|\left|A^{-1}\right|\right| \frac{1}{1 - ||A^{-1}B||} \end{aligned}$$

This proves the lemma.

Proposition 14.2.3 *Suppose A is invertible, $b \neq 0$, $Ax = b$, and $A_1 x_1 = b_1$ where $||A - A_1|| < 1/\left|\left|A^{-1}\right|\right|$. Then*

$$\frac{||x_1 - x||}{||x||} \leq \frac{1}{\left(1 - ||A^{-1}(A_1 - A)||\right)} ||A|| \left|\left|A^{-1}\right|\right| \left(\frac{||A_1 - A||}{||A||} + \frac{||b - b_1||}{||b||}\right). \tag{14.6}$$

Proof: It follows from the assumptions that

$$Ax - A_1x + A_1x - A_1x_1 = b - b_1.$$

Hence

$$A_1(x - x_1) = (A_1 - A)x + b - b_1.$$

Now $A_1 = (A + (A_1 - A))$ and so by the above lemma, A_1^{-1} exists and so

$$(x - x_1) = A_1^{-1}(A_1 - A)x + A_1^{-1}(b - b_1)$$

$$= (A + (A_1 - A))^{-1}(A_1 - A)x + (A + (A_1 - A))^{-1}(b - b_1).$$

By the estimate in Lemma 14.2.2,

$$||x - x_1|| \leq \frac{\left|\left|A^{-1}\right|\right|}{1 - ||A^{-1}(A_1 - A)||}\left(||A_1 - A||\,||x|| + ||b - b_1||\right).$$

Dividing by $||x||$,

$$\frac{||x - x_1||}{||x||} \leq \frac{\left|\left|A^{-1}\right|\right|}{1 - ||A^{-1}(A_1 - A)||}\left(||A_1 - A|| + \frac{||b - b_1||}{||x||}\right) \tag{14.7}$$

Now $b = Ax = A\left(A^{-1}b\right)$ and so $||b|| \leq ||A||\left|\left|A^{-1}b\right|\right|$ and so

$$||x|| = \left|\left|A^{-1}b\right|\right| \geq ||b|| / ||A||.$$

Therefore, from 14.7,

$$\begin{aligned}\frac{||x - x_1||}{||x||} &\leq \frac{\left|\left|A^{-1}\right|\right|}{1 - ||A^{-1}(A_1 - A)||}\left(\frac{||A||\,||A_1 - A||}{||A||} + \frac{||A||\,||b - b_1||}{||b||}\right)\\ &\leq \frac{\left|\left|A^{-1}\right|\right|\,||A||}{1 - ||A^{-1}(A_1 - A)||}\left(\frac{||A_1 - A||}{||A||} + \frac{||b - b_1||}{||b||}\right)\end{aligned}$$

which proves the proposition.

This shows that the number, $\left|\left|A^{-1}\right|\right|\,||A||$, controls how sensitive the relative change in the solution of $Ax = b$ is to small changes in A and b. This number is called the condition number. It is bad when it is large because a small relative change in b, for example could yield a large relative change in x.

Recall that for A an $n \times n$ matrix, $||A||_2 = \sigma_1$ where σ_1 is the largest singular value. The largest singular value of A^{-1} is therefore, $1/\sigma_n$ where σ_n is the smallest singular value of A. Therefore, the condition number reduces to σ_1/σ_n, the ratio of the largest to the smallest singular value of A.

14.3 The Spectral Radius

Even though it is in general impractical to compute the Jordan form, its existence is all that is needed in order to prove an important theorem about something which is relatively easy to compute. This is the spectral radius of a matrix.

Definition 14.3.1 *Define $\sigma(A)$ to be the eigenvalues of A. Also,*

$$\rho(A) \equiv \max(|\lambda| : \lambda \in \sigma(A))$$

The number, $\rho(A)$ is known as the spectral radius of A.

Recall the following symbols and their meaning.

$$\limsup_{n\to\infty} a_n,\ \liminf_{n\to\infty} a_n$$

They are respectively the largest and smallest limit points of the sequence $\{a_n\}$ where $\pm\infty$ is allowed in the case where the sequence is unbounded. They are also defined as

$$\begin{aligned}\limsup_{n\to\infty} a_n &\equiv \lim_{n\to\infty} \left(\sup\left\{a_k : k \geq n\right\}\right),\\ \liminf_{n\to\infty} a_n &\equiv \lim_{n\to\infty} \left(\inf\left\{a_k : k \geq n\right\}\right).\end{aligned}$$

Thus, the limit of the sequence exists if and only if these are both equal to the same real number.

Lemma 14.3.2 *Let J be a $p \times p$ Jordan matrix*

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{pmatrix}$$

where each J_k is of the form

$$J_k = \lambda_k I + N_k$$

in which N_k is a nilpotent matrix having zeros down the main diagonal and ones down the super diagonal. Then

$$\lim_{n\to\infty} ||J^n||^{1/n} = \rho$$

where $\rho = \max\left\{|\lambda_k|, k = 1, \ldots, n\right\}$. Here the norm is defined to equal

$$||B|| = \max\left\{|B_{ij}|, i, j\right\}.$$

Proof: Suppose first that $\rho \neq 0$. First note that for this norm, if B, B are $p \times p$ matrices,

$$||BC|| \leq p\,||B||\,||C||$$

which follows from a simple computation. Now

$$\begin{aligned}||J^n||^{1/n} &= \left|\left|\begin{pmatrix} (\lambda_1 I + N_1)^n & & \\ & \ddots & \\ & & (\lambda_s I + N_s)^n \end{pmatrix}\right|\right|^{1/n}\\ &= \rho\left|\left|\begin{pmatrix} \left(\frac{\lambda_1}{\rho} I + \frac{1}{\rho} N_1\right)^n & & \\ & \ddots & \\ & & \left(\frac{\lambda_2}{\rho} I + \frac{1}{\rho} N_2\right)^n \end{pmatrix}\right|\right|^{1/n} \qquad (14.8)\end{aligned}$$

From the definition of ρ, at least one of the λ_k/ρ has absolute value equal to 1. Therefore,

$$\left|\left|\begin{pmatrix} \left(\frac{\lambda_1}{\rho} I + \frac{1}{\rho} N_1\right)^n & & \\ & \ddots & \\ & & \left(\frac{\lambda_2}{\rho} I + \frac{1}{\rho} N_2\right)^n \end{pmatrix}\right|\right|^{1/n} - 1 \equiv e_n \geq 0$$

because each N_k has only zero terms on the main diagonal. Therefore, some term in the matrix has absolute value at least as large as 1. Now also, since $N_k^p = 0$, the norm of the matrix in the above is dominated by an expression of the form Cn^p where C is some constant which does not depend on n. This is because a typical block in the above matrix is of the form

$$\sum_{i=1}^{p} \binom{n}{i} \left(\frac{\lambda_k}{\rho}\right)^{n-i} N_k^i$$

and each $|\lambda_k| \leq \rho$. It follows that for $n > p+1$,

$$Cn^p \geq (1+e_n)^n \geq \binom{n}{p+1} e_n^{p+1}$$

and so

$$\left(\frac{Cn^p}{\binom{n}{p+1}}\right)^{1/(p+1)} \geq e_n \geq 0$$

Therefore, $\lim_{n\to\infty} e_n = 0$. It follows from 14.8 that the expression in the norms in this equation converges to 1 and so

$$\lim_{n\to\infty} ||J^n||^{1/n} = \rho.$$

In case $\rho = 0$ so that all the eigenvalues equal zero, it follows that $J^n = 0$ for all $n > p$. Therefore, the limit still exists and equals ρ. This proves the theorem.

The following theorem is due to Gelfand around 1941.

Theorem 14.3.3 *(Gelfand) Let A be a complex $p \times p$ matrix. Then if ρ is the absolute value of its largest eigenvalue,*

$$\lim_{n\to\infty} ||A^n||^{1/n} = \rho.$$

Here $||\cdot||$ is any norm on $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$.

Proof: First assume $||\cdot||$ is the special norm of the above lemma. Then letting J denote the Jordan form of A, $S^{-1}AS = J$, it follows from Lemma 14.3.2

$$\begin{aligned}
\limsup_{n\to\infty} ||A^n||^{1/n} &= \limsup_{n\to\infty} \left|\left|SJ^nS^{-1}\right|\right|^{1/n} \\
&\leq \limsup_{n\to\infty} \left(\left(p^2\right) ||S|| \left|\left|S^{-1}\right|\right|\right)^{1/n} ||J^n||^{1/n} = \rho
\end{aligned}$$

$$\begin{aligned}
&= \liminf_{n\to\infty} ||J^n||^{1/n} = \liminf_{n\to\infty} \left|\left|S^{-1}A^nS\right|\right|^n \\
&= \liminf_{n\to\infty} \left(\left(p^2\right) ||S|| \left|\left|S^{-1}\right|\right|\right)^{1/n} ||A^n||^{1/n} = \liminf_{n\to\infty} ||A^n||^{1/n}
\end{aligned}$$

If follows that $\liminf_{n\to\infty} ||A^n||^{1/n} = \limsup_{n\to\infty} ||A^n||^{1/n} = \lim_{n\to\infty} ||A^n||^{1/n} = \rho$.

Now by equivalence of norms, if $|||\cdot|||$ is any other norm for the set of complex $p \times p$ matrices, there exist constants δ, Δ such that

$$\delta ||A^n|| \leq |||A^n||| \leq \Delta ||A^n||$$

Then raising to the $1/n$ power and taking a limit,

$$\rho \leq \liminf_{n\to\infty} |||A^n|||^{1/n} \leq \limsup_{n\to\infty} |||A^n|||^{1/n} \leq \rho$$

which proves the theorem.

Example 14.3.4 *Consider* $\begin{pmatrix} 9 & -1 & 2 \\ -2 & 8 & 4 \\ 1 & 1 & 8 \end{pmatrix}$. *Estimate the absolute value of the largest eigenvalue.*

A laborious computation reveals the eigenvalues are 5, and 10. Therefore, the right answer in this case is 10. Consider $\left|\left|A^7\right|\right|^{1/7}$ where the norm is obtained by taking the maximum of all the absolute values of the entries. Thus

$$\begin{pmatrix} 9 & -1 & 2 \\ -2 & 8 & 4 \\ 1 & 1 & 8 \end{pmatrix}^7 = \begin{pmatrix} 8015\,625 & -1984\,375 & 3968\,750 \\ -3968\,750 & 6031\,250 & 7937\,500 \\ 1984\,375 & 1984\,375 & 6031\,250 \end{pmatrix}$$

and taking the seventh root of the largest entry gives

$$\rho(A) \approx 8015\,625^{1/7} = 9.\,688\,951\,236\,71.$$

Of course the interest lies primarily in matrices for which the exact roots to the characteristic equation are not known and in the theoretical significance.

14.4 Series And Sequences Of Linear Operators

Before beginning this discussion, it is necessary to define what is meant by convergence in $\mathcal{L}(X,Y)$.

Definition 14.4.1 *Let* $\{A_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{L}(X,Y)$ *where* X, Y *are finite dimensional normed linear spaces. Then* $\lim_{n\to\infty} A_k = A$ *if for every* $\varepsilon > 0$ *there exists* N *such that if* $n > N$, *then*

$$||A - A_n|| < \varepsilon.$$

Here the norm refers to any of the norms defined on $\mathcal{L}(X,Y)$. *By Corollary 14.0.8 and Theorem 9.2.3 it doesn't matter which one is used. Define the symbol for an infinite sum in the usual way. Thus*

$$\sum_{k=1}^{\infty} A_k \equiv \lim_{n\to\infty} \sum_{k=1}^{n} A_k$$

Lemma 14.4.2 *Suppose* $\{A_k\}_{k=1}^{\infty}$ *is a sequence in* $\mathcal{L}(X,Y)$ *where* X, Y *are finite dimensional normed linear spaces. Then if*

$$\sum_{k=1}^{\infty} ||A_k|| < \infty,$$

It follows that

$$\sum_{k=1}^{\infty} A_k \tag{14.9}$$

exists. In words, absolute convergence implies convergence.

Proof: For $p \le m \le n$,

$$\left|\left|\sum_{k=1}^{n} A_k - \sum_{k=1}^{m} A_k\right|\right| \le \sum_{k=p}^{\infty} ||A_k||$$

and so for p large enough, this term on the right in the above inequality is less than ε. Since ε is arbitrary, this shows the partial sums of 14.9 are a Cauchy sequence. Therefore by Corollary 14.0.7 it follows that these partial sums converge.

As a special case, suppose $\lambda \in \mathbb{C}$ and consider

$$\sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!}$$

where $t \in \mathbb{R}$. In this case, $A_k = \frac{t^k \lambda^k}{k!}$ and you can think of it as being in $\mathcal{L}(\mathbb{C}, \mathbb{C})$. Then the following corollary is of great interest.

Corollary 14.4.3 *Let*

$$f(t) \equiv \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} \equiv 1 + \sum_{k=1}^{\infty} \frac{t^k \lambda^k}{k!}$$

Then this function is a well defined complex valued function and furthermore, it satisfies the initial value problem,

$$y' = \lambda y,\ y(0) = 1$$

Furthermore, if $\lambda = a + ib$,

$$|f|(t) = e^{at}.$$

Proof: That $f(t)$ makes sense follows right away from Lemma 14.4.2.

$$\sum_{k=0}^{\infty} \left| \frac{t^k \lambda^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{|t|^k |\lambda|^k}{k!} = e^{|t||\lambda|}$$

It only remains to verify f satisfies the differential equation because it is obvious from the series that $f(0) = 1$.

$$\frac{f(t+h) - f(t)}{h} = \frac{1}{h} \sum_{k=1}^{\infty} \frac{\left((t+h)^k - t^k \right) \lambda^k}{k!}$$

and by the mean value theorem this equals an expression of the following form where θ_k is a number between 0 and 1.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k (t + \theta_k h)^{k-1} \lambda^k}{k!} &= \sum_{k=1}^{\infty} \frac{(t + \theta_k h)^{k-1} \lambda^k}{(k-1)!} \\ &= \lambda \sum_{k=0}^{\infty} \frac{(t + \theta_k h)^k \lambda^k}{k!} \end{aligned}$$

It only remains to verify this converges to

$$\lambda \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} = \lambda f(t)$$

as $h \to 0$.

$$\left| \sum_{k=0}^{\infty} \frac{(t + \theta_k h)^k \lambda^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} \right| = \left| \sum_{k=0}^{\infty} \frac{\left((t + \theta_k h)^k - t^k \right) \lambda^k}{k!} \right|$$

and by the mean value theorem again and the triangle inequality

$$\leq \left| \sum_{k=0}^{\infty} \frac{k\left|(t+\eta_k)\right|^{k-1}|h|\,|\lambda|^k}{k!} \right| \leq |h| \sum_{k=0}^{\infty} \frac{k\left|(t+\eta_k)\right|^{k-1}|\lambda|^k}{k!}$$

where η_k is between 0 and 1. Thus

$$\leq |h| \sum_{k=0}^{\infty} \frac{k\left(|t|+1\right)^{k-1}|\lambda|^k}{k!} = |h|\, C(t)$$

It follows $f'(t) = \lambda f(t)$. This proves the first part.

Next note that for $f(t) = u(t) + iv(t)$, both u, v are differentiable. This is because

$$u = \frac{f+\overline{f}}{2},\ v = \frac{f-\overline{f}}{2i}.$$

Then from the differential equation,

$$(a+ib)(u+iv) = u' + iv'$$

and equating real and imaginary parts,

$$u' = au - bv,\ v' = av + bu.$$

Then a short computation shows

$$\left(u^2+v^2\right)' = 2a\left(u^2+v^2\right),\ \left(u^2+v^2\right)(0) = 1.$$

Now in general, if

$$y' = cy,\ y(0) = 1,$$

with c real it follows $y(t) = e^{ct}$. To see this,

$$y' - cy = 0$$

and so, multiplying both sides by e^{-ct} you get

$$\frac{d}{dt}\left(ye^{-ct}\right) = 0$$

and so ye^{-ct} equals a constant which must be 1 because of the initial condition $y(0) = 1$. Thus

$$\left(u^2+v^2\right)(t) = e^{2at}$$

and taking square roots yields the desired conclusion. This proves the corollary.

Definition 14.4.4 *The function in Corollary 14.4.3 given by that power series is denoted as*

$$\exp(\lambda t) \ \text{ or } e^{\lambda t}.$$

The next lemma is normally discussed in advanced calculus courses but is proved here for the convenience of the reader. It is known as the root test.

Definition 14.4.5 *For $\{a_n\}$ any sequence of real numbers*

$$\limsup_{n\to\infty} a_n \equiv \lim_{n\to\infty} \left(\sup\{a_k : k \geq n\}\right)$$

Similarly

$$\liminf_{n\to\infty} a_n \equiv \lim_{n\to\infty} \left(\inf\{a_k : k \geq n\}\right)$$

In case A_n is an increasing (decreasing) sequence which is unbounded above (below) then it is understood that $\lim_{n\to\infty} A_n = \infty(-\infty)$ respectively. Thus either of $\limsup$ or $\liminf$ can equal $+\infty$ or $-\infty$. However, the important thing about these is that unlike the limit, these always exist.

It is convenient to think of these as the largest point which is the limit of some subsequence of $\{a_n\}$ and the smallest point which is the limit of some subsequence of $\{a_n\}$ respectively. Thus $\lim_{n\to\infty} a_n$ exists and equals some point of $[-\infty,\infty]$ if and only if the two are equal.

Lemma 14.4.6 *Let $\{a_p\}$ be a sequence of nonnegative terms and let*

$$r = \limsup_{p\to\infty} a_p^{1/p}.$$

Then if $r < 1$, it follows the series, $\sum_{k=1}^{\infty} a_k$ converges and if $r > 1$, then a_p fails to converge to 0 so the series diverges. If A is an $n \times n$ matrix and

$$1 < \limsup_{p\to\infty} ||A^p||^{1/p}, \tag{14.10}$$

then $\sum_{k=0}^{\infty} A^k$ fails to converge.

Proof: Suppose $r < 1$. Then there exists N such that if $p > N$,

$$a_p^{1/p} < R$$

where $r < R < 1$. Therefore, for all such p, $a_p < R^p$ and so by comparison with the geometric series, $\sum R^p$, it follows $\sum_{p=1}^{\infty} a_p$ converges.

Next suppose $r > 1$. Then letting $1 < R < r$, it follows there are infinitely many values of p at which

$$R < a_p^{1/p}$$

which implies $R^p < a_p$, showing that a_p cannot converge to 0 and so the series cannot converge either.

To see the last claim, if 14.10 holds, then from the first part of this lemma, $||A^p||$ fails to converge to 0 and so $\left\{\sum_{k=0}^{m} A^k\right\}_{m=0}^{\infty}$ is not a Cauchy sequence. Hence $\sum_{k=0}^{\infty} A^k \equiv \lim_{m\to\infty} \sum_{k=0}^{m} A^k$ cannot exist.

Now denote by $\sigma(A)^p$ the collection of all numbers of the form λ^p where $\lambda \in \sigma(A)$.

Lemma 14.4.7 $\sigma(A^p) = \sigma(A)^p$

Proof: In dealing with $\sigma(A^p)$, is suffices to deal with $\sigma(J^p)$ where J is the Jordan form of A because J^p and A^p are similar. Thus if $\lambda \in \sigma(A^p)$, then $\lambda \in \sigma(J^p)$ and so $\lambda = \alpha$ where α is one of the entries on the main diagonal of J^p. These entries are of the form λ^p where $\lambda \in \sigma(A)$. Thus $\lambda \in \sigma(A)^p$ and this shows $\sigma(A^p) \subseteq \sigma(A)^p$.

Now take $\alpha \in \sigma(A)$ and consider α^p.

$$\alpha^p I - A^p = \left(\alpha^{p-1} I + \cdots + \alpha A^{p-2} + A^{p-1}\right)(\alpha I - A)$$

and so $\alpha^p I - A^p$ fails to be one to one which shows that $\alpha^p \in \sigma(A^p)$ which shows that $\sigma(A)^p \subseteq \sigma(A^p)$. This proves the lemma.

14.5 Iterative Methods For Linear Systems

Consider the problem of solving the equation

$$A\mathbf{x} = \mathbf{b} \tag{14.11}$$

where A is an $n \times n$ matrix. In many applications, the matrix A is huge and composed mainly of zeros. For such matrices, the method of Gauss elimination (row operations) is not a good way to solve the system because the row operations can destroy the zeros and storing all those zeros takes a lot of room in a computer. These systems are called sparse. To solve them, it is common to use an iterative technique. I am following the treatment given to this subject by Nobel and Daniel [14].

Definition 14.5.1 *The Jacobi iterative technique, also called the method of simultaneous corrections is defined as follows. Let* $\mathbf{x}^1$ *be an initial vector, say the zero vector or some other vector. The method generates a succession of vectors,* $\mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4, \cdots$ *and hopefully this sequence of vectors will converge to the solution to 14.11. The vectors in this list are called iterates and they are obtained according to the following procedure. Letting* $A = (a_{ij})$,

$$a_{ii}x_i^{r+1} = -\sum_{j \neq i} a_{ij}x_j^r + b_i. \tag{14.12}$$

In terms of matrices, letting

$$A = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}$$

The iterates are defined as

$$\begin{aligned} &\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ \vdots \\ x_n^{r+1} \end{pmatrix} \\ = \; &-\begin{pmatrix} 0 & * & \cdots & * \\ * & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ \vdots \\ x_n^r \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \end{aligned} \tag{14.13}$$

The matrix on the left in 14.13 is obtained by retaining the main diagonal of A and setting every other entry equal to zero. The matrix on the right in 14.13 is obtained from A by setting every diagonal entry equal to zero and retaining all the other entries unchanged.

Example 14.5.2 *Use the Jacobi method to solve the system*

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Of course this is solved most easily using row reductions. The Jacobi method is useful when the matrix is 1000×1000 or larger. This example is just to illustrate how the method works. First lets solve it using row operations. The augmented matrix is

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & 2 \\ 0 & 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 & 4 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{6}{29} \\ 0 & 1 & 0 & 0 & \frac{11}{29} \\ 0 & 0 & 1 & 0 & \frac{8}{29} \\ 0 & 0 & 0 & 1 & \frac{25}{29} \end{pmatrix}$$

which in terms of decimals is approximately equal to

$$\begin{pmatrix} 1.0 & 0 & 0 & 0 & .206 \\ 0 & 1.0 & 0 & 0 & .379 \\ 0 & 0 & 1.0 & 0 & .275 \\ 0 & 0 & 0 & 1.0 & .862 \end{pmatrix}.$$

In terms of the matrices, the Jacobi iteration is of the form

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ x_3^{r+1} \\ x_4^{r+1} \end{pmatrix} = - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ x_3^r \\ x_4^r \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Multiplying by the invese of the matrix on the left, [1]this iteration reduces to

$$\begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ x_3^{r+1} \\ x_4^{r+1} \end{pmatrix} = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ x_3^r \\ x_4^r \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix}. \tag{14.14}$$

Now iterate this starting with

$$\mathbf{x}^1 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{x}^2 = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix}$$

Then

$$\mathbf{x}^3 = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \overbrace{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix}}^{\mathbf{x}_2} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} .166 \\ .26 \\ .2 \\ .7 \end{pmatrix}$$

[1]You certainly would not compute the invese in solving a large system. This is just to show you how the method works for this simple example. You would use the first description in terms of indices.

$$\mathbf{x}^4 = -\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \overbrace{\begin{pmatrix} .166 \\ .26 \\ .2 \\ .7 \end{pmatrix}}^{\mathbf{x}_3} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} .24 \\ .408\,5 \\ .356 \\ .9 \end{pmatrix}$$

$$\mathbf{x}^5 = -\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \overbrace{\begin{pmatrix} .24 \\ .408\,5 \\ .356 \\ .9 \end{pmatrix}}^{\mathbf{x}_4} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} .197 \\ .351 \\ .256\,6 \\ .822 \end{pmatrix}$$

$$\mathbf{x}^6 = -\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \overbrace{\begin{pmatrix} .197 \\ .351 \\ .256\,6 \\ .822 \end{pmatrix}}^{\mathbf{x}^5} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} .216 \\ .386 \\ .295 \\ .871 \end{pmatrix}.$$

You can keep going like this. Recall the solution is approximately equal to

$$\begin{pmatrix} .206 \\ .379 \\ .275 \\ .862 \end{pmatrix}$$

so you see that with no care at all and only 6 iterations, an approximate solution has been obtained which is not too far off from the actual solution.

It is important to realize that a computer would use 14.12 directly. Indeed, writing the problem in terms of matrices as I have done above destroys every benefit of the method. However, it makes it a little easier to see what is happening and so this is why I have presented it in this way.

Definition 14.5.3 *The Gauss Seidel method, also called the method of successive corrections is given as follows. For $A = (a_{ij})$, the iterates for the problem $A\mathbf{x} = \mathbf{b}$ are obtained according to the formula*

$$\sum_{j=1}^{i} a_{ij} x_j^{r+1} = -\sum_{j=i+1}^{n} a_{ij} x_j^r + b_i. \tag{14.15}$$

In terms of matrices, letting

$$A = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}$$

The iterates are defined as

$$\begin{aligned} &\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & * \end{pmatrix} \begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ \vdots \\ x_n^{r+1} \end{pmatrix} \\ = \; &-\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ \vdots \\ x_n^r \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \end{aligned} \tag{14.16}$$

In words, you set every entry in the original matrix which is strictly above the main diagonal equal to zero to obtain the matrix on the left. To get the matrix on the right, you set every entry of A which is on or below the main diagonal equal to zero. Using the iteration procedure of 14.15 directly, the Gauss Seidel method makes use of the very latest information which is available at that stage of the computation.

The following example is the same as the example used to illustrate the Jacobi method.

Example 14.5.4 *Use the Gauss Seidel method to solve the system*

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

In terms of matrices, this procedure is

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ x_3^{r+1} \\ x_4^{r+1} \end{pmatrix} = - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ x_3^r \\ x_4^r \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Multiplying by the inverse of the matrix on the left[2] this yields

$$\begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ x_3^{r+1} \\ x_4^{r+1} \end{pmatrix} = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{30} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{60} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ x_3^r \\ x_4^r \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix}$$

As before, I will be totally unoriginal in the choice of $\mathbf{x}^1$. Let it equal the zero vector. Therefore,

$$\mathbf{x}^2 = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix}.$$

Now

$$\mathbf{x}^3 = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{30} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{60} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \overbrace{\begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix}}^{\mathbf{x}^2} + \begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix} = \begin{pmatrix} .194 \\ .343 \\ .306 \\ .846 \end{pmatrix}.$$

It follows

$$\mathbf{x}^4 = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{30} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{60} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} .194 \\ .343 \\ .306 \\ .846 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix} = \begin{pmatrix} .219 \\ .368\,75 \\ .283\,3 \\ .858\,35 \end{pmatrix}$$

and so

$$\mathbf{x}^5 = - \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{30} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{60} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} .219 \\ .368\,75 \\ .283\,3 \\ .858\,35 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{5}{12} \\ \frac{13}{30} \\ \frac{47}{60} \end{pmatrix} = \begin{pmatrix} .210\,42 \\ .376\,57 \\ .277\,7 \\ .861\,15 \end{pmatrix}.$$

[2] As in the case of the Jacobi iteration, the computer would not do this. It would use the iteration procedure in terms of the entries of the matrix directly. Otherwise all benefit to using this method is lost.

Recall the answer is

$$\begin{pmatrix} .206 \\ .379 \\ .275 \\ .862 \end{pmatrix}$$

so the iterates are already pretty close to the answer. You could continue doing these iterates and it appears they converge to the solution. Now consider the following example.

Example 14.5.5 *Use the Gauss Seidel method to solve the system*

$$\begin{pmatrix} 1 & 4 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

The exact solution is given by doing row operations on the augmented matrix. When this is done the row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

and so the solution is approximately

$$\begin{pmatrix} 6 \\ -\frac{5}{4} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 6.0 \\ -1.25 \\ 1.0 \\ .5 \end{pmatrix}$$

The Gauss Seidel iterations are of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1^{r+1} \\ x_2^{r+1} \\ x_3^{r+1} \\ x_4^{r+1} \end{pmatrix} = - \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^r \\ x_2^r \\ x_3^r \\ x_4^r \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

and so, multiplying by the inverse of the matrix on the left, the iteration reduces to the following in terms of matrix multiplication.

$$\mathbf{x}^{r+1} = - \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \mathbf{x}^r + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}.$$

This time, I will pick an initial vector close to the answer. Let

$$\mathbf{x}^1 = \begin{pmatrix} 6 \\ -1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

This is very close to the answer. Now lets see what the Gauss Seidel iteration does to it.

$$\mathbf{x}^2 = - \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 5.0 \\ -1.0 \\ .9 \\ .55 \end{pmatrix}$$

You can't expect to be real close after only one iteration. Lets do another.

$$\mathbf{x}^3 = -\begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} 5.0 \\ -1.0 \\ .9 \\ .55 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 5.0 \\ -.975 \\ .88 \\ .56 \end{pmatrix}$$

$$\mathbf{x}^4 = -\begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{10} & \frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} 5.0 \\ -.975 \\ .88 \\ .56 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 4.9 \\ -.945 \\ .866 \\ .567 \end{pmatrix}$$

The iterates seem to be getting farther from the actual solution. Why is the process which worked so well in the other examples not working here? A better question might be: Why does either process ever work at all?

Both iterative procedures for solving

$$A\mathbf{x} = \mathbf{b} \tag{14.17}$$

are of the form

$$B\mathbf{x}^{r+1} = -C\mathbf{x}^r + \mathbf{b}$$

where $A = B + C$. In the Jacobi procedure, the matrix C was obtained by setting the diagonal of A equal to zero and leaving all other entries the same while the matrix, B was obtained by making every entry of A equal to zero other than the diagonal entries which are left unchanged. In the Gauss Seidel procedure, the matrix B was obtained from A by making every entry strictly above the main diagonal equal to zero and leaving the others unchanged and C was obtained from A by making every entry on or below the main diagonal equal to zero and leaving the others unchanged. Thus in the Jacobi procedure, B is a diagonal matrix while in the Gauss Seidel procedure, B is lower triangular. Using matrices to explicitly solve for the iterates, yields

$$\mathbf{x}^{r+1} = -B^{-1}C\mathbf{x}^r + B^{-1}\mathbf{b}. \tag{14.18}$$

This is what you would never have the computer do but this is what will allow the statement of a theorem which gives the condition for convergence of these and all other similar methods. Recall the definition of the spectral radius of $M, \rho(M)$, in Definition 14.3.1 on Page 358.

Theorem 14.5.6 *Suppose $\rho\left(B^{-1}C\right) < 1$. Then the iterates in 14.18 converge to the unique solution of 14.17.*

I will prove this theorem in the next section. The proof depends on analysis which should not be surprising because it involves a statement about convergence of sequences.

What is an easy to verify sufficient condition which will imply the above holds? It is easy to give one in the case of the Jacobi method. Suppose the matrix A is diagonally dominent. That is $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$. Then B would be the diagonal matrix consisting of the entries $|a_{ii}|$. You can see then that every entry of $B^{-1}C$ has absolute value less than 1. Thus if you let the norm $\left|\left|B^{-1}C\right|\right|_\infty$ be given by the maximum of the absolute values of the entries of the matrix, then $\left|\left|B^{-1}C\right|\right|_\infty = r < 1$. Also, by equivalence of norms it follows there exist positive constants δ, Δ such that

$$\delta \left|\left|\cdot\right|\right| \leq \left|\left|\cdot\right|\right|_\infty \leq \Delta \left|\left|\cdot\right|\right|$$

where here $||\cdot||$ is an operator norm. It follows that if $|\lambda| \geq 1$, then $\left(\lambda I - B^{-1}C\right)^{-1}$ exists. In fact it equals

$$\sum_{k=0}^{\infty} \lambda^{-1} \left(\frac{B^{-1}C}{\lambda}\right)^k,$$

the series converging because

$$\left|\left|\sum_{k=m}^{n} \left(\frac{B^{-1}C}{\lambda}\right)^k\right|\right|_\infty \leq \sum_{k=m}^{\infty} \left|\left|\left(\frac{B^{-1}C}{\lambda}\right)^k\right|\right|_\infty$$

$$\leq \sum_{k=m}^{\infty} \Delta \left|\left|\left(\frac{B^{-1}C}{\lambda}\right)^k\right|\right|_\infty \leq \sum_{k=m}^{\infty} \Delta \left|\left|\left(\frac{B^{-1}C}{\lambda}\right)\right|\right|^k$$

$$\leq \sum_{k=m}^{\infty} \frac{\Delta}{\delta} \left|\left|\left(\frac{B^{-1}C}{\lambda}\right)\right|\right|^k_\infty \leq \frac{\Delta}{\delta} \sum_{k=m}^{\infty} r^k \leq \frac{\Delta}{\delta} \left(\frac{r^m}{1-r}\right)$$

which shows the partial sums form a Cauchy sequence. Therefore, $\rho\left(B^{-1}C\right) < 1$ in this case.

You might try a similar argument in the case of the Gauss Seidel method.

14.6 Theory Of Convergence

Definition 14.6.1 *A normed vector space, E with norm $||\cdot||$ is called a Banach space if it is also complete. This means that every Cauchy sequence converges. Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that whenever $m, n > N$,*

$$||x_n - x_m|| < \varepsilon.$$

Thus whenever $\{x_n\}$ is a Cauchy sequence, there exists x such that

$$\lim_{n\to\infty} ||x - x_n|| = 0.$$

Example 14.6.2 *Let Ω be a nonempty subset of a normed linear space, F. Denote by $BC(\Omega; E)$ the set of bounded continuous functions having values in E where E is a Banach space. Then define the norm on $BC(\Omega; E)$ by*

$$||f|| \equiv \sup\left\{||f(x)||_E : x \in \Omega\right\}.$$

Lemma 14.6.3 *The space $BC(\Omega; E)$ with the given norm is a Banach space.*

Proof: It is obvious $||\cdot||$ is a norm. It only remains to verify $BC(\Omega; E)$ is complete. Let $\{f_n\}$ be a Cauchy sequence. Then pick $x \in \Omega$.

$$||f_n(x) - f_m(x)||_E \leq ||f_n - f_m|| < \varepsilon$$

whenever m, n are large enough. Thus, for each $x, \{f_n(x)\}$ is a Cauchy sequence in E. Since E is complete, it follows there exists a function, f defined on Ω such that $f(x) = \lim_{n\to\infty} f_n(x)$.

It remains to verify that $f \in BC(\Omega; E)$ and that $||f - f_n|| \to 0$. I will first show that

$$\lim_{n\to\infty} \left(\sup_{x\in\Omega} \left\{||f(x) - f_n(x)||_E\right\}\right) = 0. \tag{14.19}$$

From this it will follow that f is bounded. Then I will show that f is continuous and $||f - f_n|| \to 0$. Let $\varepsilon > 0$ be given and let N be such that for $m, n > N$

$$||f_n - f_m|| < \varepsilon/3.$$

Then it follows that for all x,

$$||f(x) - f_m(x)||_E = \lim_{n\to\infty} ||f_n(x) - f_m(x)||_E \leq \varepsilon/3$$

Therefore, for $m > N$,

$$\sup_{x\in\Omega} \{||f(x) - f_m(x)||_E\} \leq \frac{\varepsilon}{3} < \varepsilon.$$

This proves 14.19. Then by the triangle inequality and letting N be as just described, pick $m > N$. Then for any $x \in \Omega$

$$||f(x)||_E \leq ||f_m(x)||_E + \varepsilon \leq ||f_m|| + \varepsilon.$$

Hence f is bounded. Now pick $x \in \Omega$ and let $\varepsilon > 0$ be given and N be as above. Then

$$\begin{aligned} ||f(x) - f(y)||_E &\leq ||f(x) - f_m(x)||_E + ||f_m(x) - f_m(y)||_E + ||f_m(y) - f(y)||_E \\ &\leq \frac{\varepsilon}{3} + ||f_m(x) - f_m(y)||_E + \frac{\varepsilon}{3}. \end{aligned}$$

Now by continuity of f_m, the middle term is less than $\varepsilon/3$ whenever $||x - y||$ is sufficiently small. Therefore, f is also continuous. Finally, from the above,

$$||f - f_n|| \leq \frac{\varepsilon}{3}$$

whenever $n > N$ and so $\lim_{n\to\infty} ||f - f_n|| = 0$ as claimed. This proves the lemma.

The most familiar example of a Banach space is $\mathbb{F}^n$. The following lemma is of great importance so it is stated in general.

Lemma 14.6.4 *Suppose $T : E \to E$ where E is a Banach space with norm $|\cdot|$. Also suppose*

$$|T\mathbf{x} - T\mathbf{y}| \leq r|\mathbf{x} - \mathbf{y}| \tag{14.20}$$

for some $r \in (0, 1)$. Then there exists a unique fixed point, $\mathbf{x} \in E$ such that

$$T\mathbf{x} = \mathbf{x}. \tag{14.21}$$

Letting $\mathbf{x}^1 \in E$, this fixed point, $\mathbf{x}$, is the limit of the sequence of iterates,

$$\mathbf{x}^1, T\mathbf{x}^1, T^2\mathbf{x}^1, \cdots. \tag{14.22}$$

In addition to this, there is a nice estimate which tells how close $\mathbf{x}^1$ is to $\mathbf{x}$ in terms of things which can be computed.

$$\left|\mathbf{x}^1 - \mathbf{x}\right| \leq \frac{1}{1-r}\left|\mathbf{x}^1 - T\mathbf{x}^1\right|. \tag{14.23}$$

Proof: This follows easily when it is shown that the above sequence, $\left\{T^k\mathbf{x}^1\right\}_{k=1}^{\infty}$ is a Cauchy sequence. Note that

$$\left|T^2\mathbf{x}^1 - T\mathbf{x}^1\right| \leq r\left|T\mathbf{x}^1 - \mathbf{x}^1\right|.$$

Suppose

$$\left|T^k\mathbf{x}^1 - T^{k-1}\mathbf{x}^1\right| \leq r^{k-1}\left|T\mathbf{x}^1 - \mathbf{x}^1\right|. \tag{14.24}$$

Then

$$\begin{aligned}\left|T^{k+1}\mathbf{x}^1 - T^k\mathbf{x}^1\right| &\leq r\left|T^k\mathbf{x}^1 - T^{k-1}\mathbf{x}^1\right| \\ &\leq rr^{k-1}\left|T\mathbf{x}^1 - \mathbf{x}^1\right| = r^k\left|T\mathbf{x}^1 - \mathbf{x}^1\right|.\end{aligned}$$

By induction, this shows that for all $k \geq 2$, 14.24 is valid. Now let $k > l \geq N$.

$$\begin{aligned}\left|T^k\mathbf{x}^1 - T^l\mathbf{x}^1\right| &= \left|\sum_{j=l}^{k-1}\left(T^{j+1}\mathbf{x}^1 - T^j\mathbf{x}^1\right)\right| \\ &\leq \sum_{j=l}^{k-1}\left|T^{j+1}\mathbf{x}^1 - T^j\mathbf{x}^1\right| \\ &\leq \sum_{j=N}^{k-1} r^j\left|T\mathbf{x}^1 - \mathbf{x}^1\right| \leq \left|T\mathbf{x}^1 - \mathbf{x}^1\right|\frac{r^N}{1-r}\end{aligned}$$

which converges to 0 as $N \to \infty$. Therefore, this is a Cauchy sequence so it must converge to $\mathbf{x} \in E$. Then

$$\mathbf{x} = \lim_{k\to\infty} T^k\mathbf{x}^1 = \lim_{k\to\infty} T^{k+1}\mathbf{x}^1 = T\lim_{k\to\infty} T^k\mathbf{x}^1 = T\mathbf{x}.$$

This shows the existence of the fixed point. To show it is unique, suppose there were another one, $\mathbf{y}$. Then

$$|\mathbf{x} - \mathbf{y}| = |T\mathbf{x} - T\mathbf{y}| \leq r|\mathbf{x} - \mathbf{y}|$$

and so $\mathbf{x} = \mathbf{y}$.

It remains to verify the estimate.

$$\begin{aligned}\left|\mathbf{x}^1 - \mathbf{x}\right| &\leq \left|\mathbf{x}^1 - T\mathbf{x}^1\right| + \left|T\mathbf{x}^1 - \mathbf{x}\right| \\ &= \left|\mathbf{x}^1 - T\mathbf{x}^1\right| + \left|T\mathbf{x}^1 - T\mathbf{x}\right| \\ &\leq \left|\mathbf{x}^1 - T\mathbf{x}^1\right| + r\left|\mathbf{x}^1 - \mathbf{x}\right|\end{aligned}$$

and solving the inequality for $\left|\mathbf{x}^1 - \mathbf{x}\right|$ gives the estimate desired. This proves the lemma.

The following corollary is what will be used to prove the convergence condition for the various iterative procedures.

Corollary 14.6.5 *Suppose $T : E \to E$, for some constant C*

$$|T\mathbf{x} - T\mathbf{y}| \leq C|\mathbf{x} - \mathbf{y}|,$$

for all $\mathbf{x}, \mathbf{y} \in E$, and for some $N \in \mathbb{N}$,

$$\left|T^N\mathbf{x} - T^N\mathbf{y}\right| \leq r|\mathbf{x} - \mathbf{y}|,$$

for all $\mathbf{x}, \mathbf{y} \in E$ where $r \in (0, 1)$. Then there exists a unique fixed point for T and it is still the limit of the sequence, $\left\{T^k\mathbf{x}^1\right\}$ for any choice of $\mathbf{x}^1$.

Proof: From Lemma 14.6.4 there exists a unique fixed point for T^N denoted here as $\mathbf{x}$. Therefore, $T^N\mathbf{x} = \mathbf{x}$. Now doing T to both sides,

$$T^NT\mathbf{x} = T\mathbf{x}.$$

By uniqueness, $T\mathbf{x} = \mathbf{x}$ because the above equation shows $T\mathbf{x}$ is a fixed point of T^N and there is only one fixed point of T^N. In fact, there is only one fixed point of T because a fixed point of T is automatically a fixed point of T^N.

It remains to show $T^k\mathbf{x}^1 \to \mathbf{x}$, the unique fixed point of T^N. If this does not happen, there exists $\varepsilon > 0$ and a subsequence, still denoted by T^k such that

$$\left|T^k\mathbf{x}^1 - \mathbf{x}\right| \geq \varepsilon$$

Now $k = j_k N + r_k$ where $r_k \in \{0, \cdots, N-1\}$ and j_k is a positive integer such that $\lim_{k\to\infty} j_k = \infty$. Then there exists a single $r \in \{0, \cdots, N-1\}$ such that for infinitely many $k, r_k = r$. Taking a further subsequence, still denoted by T^k it follows

$$\left|T^{j_k N + r}\mathbf{x}^1 - \mathbf{x}\right| \geq \varepsilon \tag{14.25}$$

However,

$$T^{j_k N + r}\mathbf{x}^1 = T^r T^{j_k N}\mathbf{x}^1 \to T^r\mathbf{x} = \mathbf{x}$$

and this contradicts 14.25. This proves the corollary.

Theorem 14.6.6 *Suppose $\rho\left(B^{-1}C\right) < 1$. Then the iterates in 14.18 converge to the unique solution of 14.17.*

Proof: Consider the iterates in 14.18. Let $T\mathbf{x} = B^{-1}C\mathbf{x} + \mathbf{b}$. Then

$$\begin{aligned}\left|T^k\mathbf{x} - T^k\mathbf{y}\right| &= \left|\left(B^{-1}C\right)^k\mathbf{x} - \left(B^{-1}C\right)^k\mathbf{y}\right| \\ &\leq \left|\left|\left(B^{-1}C\right)^k\right|\right| |\mathbf{x} - \mathbf{y}|.\end{aligned}$$

Here $||\cdot||$ refers to any of the operator norms. It doesn't matter which one you pick because they are all equivalent. I am writing the proof to indicate the operator norm taken with respect to the usual norm on E. Since $\rho\left(B^{-1}C\right) < 1$, it follows from Gelfand's theorem, Theorem 14.3.3 on Page 360, there exists N such that if $k \geq N$, then for some $r^{1/k} < 1$,

$$\left|\left|\left(B^{-1}C\right)^k\right|\right|^{1/k} < r^{1/k} < 1.$$

Consequently,

$$\left|T^N\mathbf{x} - T^N\mathbf{y}\right| \leq r\,|\mathbf{x} - \mathbf{y}|.$$

Also $|T\mathbf{x} - T\mathbf{y}| \leq \left|\left|B^{-1}C\right|\right| |\mathbf{x} - \mathbf{y}|$ and so Corollary 14.6.5 applies and gives the conclusion of this theorem.

14.7 Exercises

1. ♠Solve the system

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

2. Solve the system
$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

3. Solve the system
$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

4. If you are considering a system of the form $A\mathbf{x} = \mathbf{b}$ and A^{-1} does not exist, will either the Gauss Seidel or Jacobi methods work? Explain. What does this indicate about finding eigenvectors for a given eigenvalue?

5. ♠For $||\mathbf{x}||_\infty \equiv \max\{|x_j| : j = 1, 2, \cdots, n\}$, the parallelogram identity does not hold. Explain.

6. ♠A norm $||\cdot||$ is said to be strictly convex if whenever $||x|| = ||y||, x \neq y$, it follows
$$\left|\left|\frac{x+y}{2}\right|\right| < ||x|| = ||y||.$$
Show the norm $|\cdot|$ which comes from an inner product is strictly convex.

7. ♠A norm $||\cdot||$ is said to be uniformly convex if whenever $||x_n||, ||y_n||$ are equal to 1 for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} ||x_n + y_n|| = 2$, it follows $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Show the norm $|\cdot|$ coming from an inner product is always uniformly convex. Also show that uniform convexity implies strict convexity which is defined in Problem 6.

8. ♠Suppose $A : \mathbb{C}^n \to \mathbb{C}^n$ is a one to one and onto matrix. Define
$$||\mathbf{x}|| \equiv |A\mathbf{x}|.$$
Show this is a norm.

9. ♠If X is a finite dimensional normed vector space and $A, B \in \mathcal{L}(X, X)$ such that $||B|| < ||A||$, can it be concluded that $\left|\left|A^{-1}B\right|\right| < 1$?

10. ♠Let X be a vector space with a norm $||\cdot||$ and let $V = \text{span}(v_1, \cdots, v_m)$ be a finite dimensional subspace of X such that $\{v_1, \cdots, v_m\}$ is a basis for V. Show V is a closed subspace of X. This means that if $w_n \to w$ and each $w_n \in V$, then so is w. Next show that if $w \notin V$,
$$\text{dist}(w, V) \equiv \inf\{||w - v|| : v \in V\} > 0.$$
Next show that if $w \notin V$ and $||w|| = 1$, there exists z such that $||z|| = 1$ and $\text{dist}(z, V) > 1/2$. For those who know some advanced calculus, show that if X is an infinite dimensional vector space having norm $||\cdot||$, then the closed unit ball in X cannot be compact. Thus closed and bounded is never compact in an infinite dimensional normed vector space.

11. ♠Suppose $\rho(A) < 1$ for $A \in \mathcal{L}(V,V)$ where V is a p dimensional vector space having a norm $||\cdot||$. You can use $\mathbb{R}^p$ or $\mathbb{C}^p$ if you like. Show there exists a new norm $|||\cdot|||$ such that with respect to this new norm, $|||A||| < 1$ where $|||A|||$ denotes the operator norm of A taken with respect to this new norm on V,

$$|||A||| \equiv \sup\{|||A\mathbf{x}||| : |||\mathbf{x}||| \leq 1\}$$

Hint: You know from Gelfand's theorem that

$$||A^n||^{1/n} < r < 1$$

provided n is large enough, this operator norm taken with respect to $||\cdot||$. Show there exists $0 < \lambda < 1$ such that

$$\rho\left(\frac{A}{\lambda}\right) < 1.$$

You can do this by arguing the eigenvalues of A/λ are the scalars μ/λ where $\mu \in \sigma(A)$. Now let $\mathbb{Z}_+$ denote the nonnegative integers.

$$|||\mathbf{x}||| \equiv \sup_{n\in\mathbb{Z}_+} \left|\left|\frac{A^n}{\lambda^n}\mathbf{x}\right|\right|$$

First show this is actually a norm. Next explain why

$$|||A\mathbf{x}||| \equiv \lambda \sup_{n\in\mathbb{Z}_+} \left|\left|\frac{A^{n+1}}{\lambda^{n+1}}\mathbf{x}\right|\right| \leq \lambda\,|||\mathbf{x}|||.$$

12. Establish a similar result to Problem 11 without using Gelfand's theorem. Use an argument which depends directly on the Jordan form or a modification of it.

13. ♠Using Problem 11 give an easier proof of Theorem 14.6.6 without having to use Corollary 14.6.5. It would suffice to use a different norm of this problem and the contraction mapping principle of Lemma 14.6.4.

14. ♠A matrix A is diagonally dominant if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$. Show that the Gauss Seidel method converges if A is diagonally dominant.

15. ♠Suppose $f(\lambda) = \sum_{k=0}^{\infty} a_n\lambda^n$ converges if $|\lambda| < R$. Show that if $\rho(A) < R$ where A is an $n \times n$ matrix, then

$$f(A) \equiv \sum_{k-0}^{\infty} a_n A^n$$

converges in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$. **Hint:** Use Gelfand's theorem and the root test.

16. Referring to Corollary 14.4.3, for $\lambda = a + ib$ show

$$\exp(\lambda t) = e^{at}(\cos(bt) + i\sin(bt)).$$

Hint: Let $y(t) = \exp(\lambda t)$ and let $z(t) = e^{-at}y(t)$. Show

$$z'' + b^2 z = 0,\ z(0) = 1, z'(0) = ib.$$

Now letting $z = u + iv$ where u, v are real valued, show

$$\begin{aligned} u'' + b^2 u &= 0,\ u(0) = 1, u'(0) = 0 \\ v'' + b^2 v &= 0,\ v(0) = 0, v'(0) = b. \end{aligned}$$

Next show $u(t) = \cos(bt)$ and $v(t) = \sin(bt)$ work in the above and that there is at most one solution to

$$w'' + b^2 w = 0 \; w(0) = \alpha, w'(0) = \beta.$$

Thus $z(t) = \cos(bt) + i\sin(bt)$ and so $y(t) = e^{at}(\cos(bt) + i\sin(bt))$. To show there is at most one solution to the above problem, suppose you have two, w_1, w_2. Subtract them. Let $f = w_1 - w_2$. Thus

$$f'' + b^2 f = 0$$

and f is real valued. Multiply both sides by f' and conclude

$$\frac{d}{dt}\left(\frac{(f')^2}{2} + b^2\frac{f^2}{2}\right) = 0$$

Thus the expression in parenthesis is constant. Explain why this constant must equal 0.

17. ♠Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Show the following power series converges in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

$$\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

You might want to use Lemma 14.4.2. This is how you can define $\exp(tA)$. Next show using arguments like those of Corollary 14.4.3

$$\frac{d}{dt}\exp(tA) = A\exp(tA)$$

so that this is a matrix valued solution to the differential equation and initial condition

$$\Psi'(t) = A\Psi(t), \; \Psi(0) = I.$$

This $\Psi(t)$ is called a fundamental matrix for the differential equation $\mathbf{y}' = A\mathbf{y}$. Show $t \to \Psi(t)\mathbf{y}_0$ gives a solution to the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \; \mathbf{y}(0) = \mathbf{y}_0.$$

18. In Problem 17 $\Psi(t)$ is defined by the given series. Denote by $\exp(t\sigma(A))$ the numbers $\exp(t\lambda)$ where $\lambda \in \sigma(A)$. Show $\exp(t\sigma(A)) = \sigma(\Psi(t))$. This is like Lemma 14.4.7. Letting J be the Jordan canonical form for A, explain why

$$\Psi(t) \equiv \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = S\sum_{k=0}^{\infty} \frac{t^k J^k}{k!} S^{-1}$$

and you note that in J^k, the diagonal entries are of the form λ^k for λ an eigenvalue of A. Also $J = D + N$ where N is nilpotent and commutes with D. Argue then that

$$\sum_{k=0}^{\infty} \frac{t^k J^k}{k!}$$

is an upper triangular matrix which has on the diagonal the expressions $e^{\lambda t}$ where $\lambda \in \sigma(A)$. Thus conclude

$$\sigma(\Psi(t)) \subseteq \exp(t\sigma(A))$$

Next take $e^{t\lambda} \in \exp(t\sigma(A))$ and argue it must be in $\sigma(\Psi(t))$. You can do this as follows:

$$\begin{aligned}\Psi(t) - e^{t\lambda}I &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} I \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(A^k - \lambda^k I\right) \\ &= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=1}^{k-1} A^{k-j}\lambda^j\right)(A - \lambda I) \qquad (14.26)\end{aligned}$$

Now you need to argue

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=1}^{k-1} A^{k-j}\lambda^j$$

converges to something in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. To do this, use the ratio test and Lemma 14.4.2 after first using the triangle inequality. Since $\lambda \in \sigma(A)$, $\Psi(t) - e^{t\lambda}I$ is not one to one and so this establishes the other inclusion. You fill in the details. This theorem is a special case of theorems which go by the name "spectral mapping theorem".

19. Suppose $\Psi(t) \in \mathcal{L}(V, W)$ where V, W are finite dimensional inner product spaces and $t \to \Psi(t)$ is continuous for $t \in [a, b]$: For every $\varepsilon > 0$ there there exists $\delta > 0$ such that if $|s - t| < \delta$ then $||\Psi(t) - \Psi(s)|| < \varepsilon$. Show $t \to (\Psi(t)v, w)$ is continuous. Here it is the inner product in W. Also define what it means for $t \to \Psi(t)v$ to be continuous and show this is continuous. Do it all for differentiable in place of continuous. Next show $t \to ||\Psi(t)||$ is continuous.

20. ♠If $\mathbf{z}(t) \in W$, an inner product space, what does it mean for $t \to \mathbf{z}(t)$ to be continuous or differentiable? If $\mathbf{z}$ is continuous, define

$$\int_a^b \mathbf{z}(t)\,dt \in W$$

as follows.

$$\left(\int_a^b \mathbf{z}(t)\,dt, w\right) \equiv \int_a^b (\mathbf{z}(t), w)\,dt.$$

Show that this definition is well defined and furthermore the triangle inequality,

$$\left|\int_a^b \mathbf{z}(t)\,dt\right| \leq \int_a^b |\mathbf{z}(t)|\,dt,$$

and fundamental theorem of calculus,

$$\frac{d}{dt}\left(\int_a^t \mathbf{z}(s)\,ds\right) = \mathbf{z}(t)$$

hold along with any other interesting properties of integrals which are true.

21. ♠In the situation of Problem 19 define

$$\int_a^b \Psi(t)\,dt \in \mathcal{L}(V, W)$$

as follows.

$$\left(\int_a^b \Psi(t)\,dt\,(v),w\right) \equiv \int_a^b (\Psi(t)\,v,w)\,dt.$$

Show this is well defined and does indeed give $\int_a^b \Psi(t)\,dt \in \mathcal{L}(V,W)$. Also show the triangle inequality

$$\left|\left|\int_a^b \Psi(t)\,dt\right|\right| \leq \int_a^b ||\Psi(t)||\,dt$$

where $||\cdot||$ is the operator norm and verify the fundamental theorem of calculus holds.

$$\left(\int_a^t \Psi(s)\,ds\right)' = \Psi(t).$$

Also verify the usual properties of integrals continue to hold such as the fact the integral is linear and

$$\int_a^b \Psi(t)\,dt + \int_b^c \Psi(t)\,dt = \int_a^c \Psi(t)\,dt$$

and similar things. **Hint:** On showing the triangle inequality, it will help if you use the fact that

$$|w|_W = \sup_{|v|\leq 1} |(w,v)|.$$

You should show this also.

22. ♠Prove Gronwall's inequality. Suppose $u(t) \geq 0$ and for all $t \in [0,T]$,

$$u(t) \leq u_0 + \int_0^t Ku(s)\,ds.$$

where K is some constant. Then

$$u(t) \leq u_0 e^{Kt}.$$

Hint: $w(t) = \int_0^t u(s)\,ds$. Then using the fundamental theorem of calculus, $w(t)$ satisfies the following.

$$u(t) - Kw(t) = w'(t) - Kw(t) \leq u_0,\ w(0) = 0.$$

Now use the usual techniques you saw in an introductory differential equations class. Multipy both sides of the above inequality by e^{-Kt} and note the resulting left side is now a total derivative. Integrate both sides from 0 to t and see what you have got. If you have problems, look ahead in the book. This inequality is proved later in Theorem B.4.3.

23. ♠With Gronwall's inequality and the integral defined in Problem 21 with its properties listed there, prove there is at most one solution to the initial value problem

$$\mathbf{y}' = A\mathbf{y},\ \mathbf{y}(0) = \mathbf{y}_0.$$

Hint: If there are two solutions, subtract them and call the result $\mathbf{z}$. Then

$$\mathbf{z}' = A\mathbf{z},\ \mathbf{z}(0) = \mathbf{0}.$$

It follows

$$\mathbf{z}(t) = \mathbf{0} + \int_0^t \mathbf{z}(s)\, ds$$

and so

$$||\mathbf{z}(t)|| \leq \int_0^t ||\mathbf{z}(s)||\, ds$$

Now consider Gronwall's inequality of Problem 22. Use Problem 20 as needed.

24. ♠Suppose A is a matrix which has the property that whenever $\mu \in \sigma(A)$, $\operatorname{Re}\mu < 0$. Consider the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \mathbf{y}(0) = \mathbf{y}_0.$$

The existence and uniqueness of a solution to this equation has been established above in preceeding problems, Problem 17 to 23. Show that in this case where the real parts of the eigenvalues are all negative, the solution to the initial value problem satisfies

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}.$$

Hint: A nice way to approach this problem is to show you can reduce it to the consideration of the initial value problem

$$\mathbf{z}' = J_\varepsilon \mathbf{z},\ \mathbf{z}(0) = \mathbf{z}_0$$

where J_ε is the modified Jordan canonical form where instead of ones down the main diagonal, there are ϵ down the main diagonal (Problem 16). Then

$$\mathbf{z}' = D\mathbf{z} + N_\varepsilon \mathbf{z}$$

where D is the diagonal matrix obtained from the eigenvalues of A and N_ε is a nilpotent matrix commuting with D which is very small provided ε is chosen very small. Now let $\Psi(t)$ be the solution of

$$\Psi' = -D\Psi,\ \Psi(0) = I$$

described earlier as

$$\sum_{k=0}^{\infty} \frac{(-1)^k t^k D^k}{k!}.$$

Thus $\Psi(t)$ commutes with D and N_ε. Tell why. Next argue

$$(\Psi(t)\mathbf{z})' = \Psi(t) N_\varepsilon \mathbf{z}(t)$$

and integrate from 0 to t. Then

$$\Psi(t)\mathbf{z}(t) - \mathbf{z}_0 = \int_0^t \Psi(s) N_\varepsilon \mathbf{z}(s)\, ds.$$

It follows

$$||\Psi(t)\mathbf{z}(t)|| \leq ||z_0|| + \int_0^t ||N_\varepsilon||\, ||\Psi(s)\mathbf{z}(s)||\, ds.$$

It follows from Gronwall's inequality

$$||\Psi(t)\mathbf{z}(t)|| \leq ||z_0||\, e^{||N_\varepsilon|| t}$$

Now look closely at the form of $\Psi(t)$ to get an estimate which is interesting. Explain why

$$\Psi(t) = \begin{pmatrix} e^{\mu_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n t} \end{pmatrix}$$

and now observe that if ε is chosen small enough, $||N_\varepsilon||$ is so small that each component of $\mathbf{z}(t)$ converges to 0.

25. Using Problem 24 show that if A is a matrix having the real parts of all eigenvalues less than 0 then if

$$\Psi'(t) = A\Psi(t),\ \Psi(0) = I$$

it follows

$$\lim_{t\to\infty} \Psi(t) = 0.$$

Hint: Consider the columns of $\Psi(t)$?

26. Let $\Psi(t)$ be a fundamental matrix satisfying

$$\Psi'(t) = A\Psi(t),\ \Psi(0) = I.$$

Show $\Psi(t)^n = \Psi(nt)$. **Hint:** Subtract and show the difference satisfies $\Phi' = A\Phi,\ \Phi(0) = 0$. Use uniqueness.

27. If the real parts of the eigenvalues of A are all negative, show that for every positive t,

$$\lim_{n\to\infty} \Psi(nt) = 0.$$

Hint: Pick $\mathrm{Re}(\sigma(A)) < -\lambda < 0$ and use Problem 18 about the spectrum of $\Psi(t)$ and Gelfand's theorem for the spectral radius along with Problem 26 to argue that $\left|\left|\Psi(nt)/e^{-\lambda nt}\right|\right| < 1$ for all n large enough.

Numerical Methods For Finding Eigenvalues

15.1 The Power Method For Eigenvalues

Let A be a complex $p \times p$ matrix and suppose that it has distinct eigenvaues

$$\{\lambda_1, \cdots, \lambda_m\}$$

and that $|\lambda_1| > |\lambda_k|$ for all k. Also let the Jordan form of A be

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}$$

with

$$J_k = \lambda_k I_k + N_k$$

where $N_k^{r_k} \neq 0$ but $N_k^{r_k+1} = 0$. Also let

$$P^{-1}AP = J,\ A = PJP^{-1}.$$

Now fix $\mathbf{x} \in \mathbb{F}^p$. Take $A\mathbf{x}$ and let s_1 be the entry of the vector $A\mathbf{x}$ which has largest absolute value. Thus $A\mathbf{x}/s_1$ is a vector $\mathbf{y}_1$ which has a component of 1 and every other entry of this vector has magnitude no larger than 1. If the scalars $\{s_1, \cdots, s_{n-1}\}$ and vectors $\{\mathbf{y}_1, \cdots, \mathbf{y}_{n-1}\}$ have been obtained, let

$$\mathbf{y}_n \equiv \frac{A\mathbf{y}_{n-1}}{s_n}$$

where s_n is the top entry of $A\mathbf{y}_{n-1}$ which has largest absolute value. Thus

$$\mathbf{y}_n = \frac{AA\mathbf{y}_{n-2}}{s_n s_{n-1}} \cdots = \frac{A^n \mathbf{x}}{s_n s_{n-1} \cdots s_1} \tag{15.1}$$

Consider one of the blocks in the Jordan form.

$$J_k^n = \lambda_1^n \sum_{i=0}^{r_k} \binom{n}{i} \frac{\lambda_k^{n-i}}{\lambda_1^n} N_k^i \equiv \lambda_1^n K(k,n)$$

Then from the above,

$$\frac{A^n}{s_n s_{n-1}\cdots s_1} = P\frac{\lambda_1^n}{s_n s_{n-1}\cdots s_1}\begin{pmatrix} K(1,n) & & \\ & \ddots & \\ & & K(m,n)\end{pmatrix}P^{-1}$$

Consider one of the terms in the sum for $K(k,n)$ for $k>1$. Letting the norm of a matrix be the maximum of the absolute values of its entries,

$$\left|\left|\binom{n}{i}\frac{\lambda_k^{n-i}}{\lambda_1^n}N_k^i\right|\right| \le n^{r_k}\left|\frac{\lambda_k}{\lambda_1}\right|^n p^{r_k}C$$

where C depends on the eigenvalues but is independent of n. Then this converges to 0 because the infinite sum of these converges due to the root test. Thus each of the matrices $K(k,n)$ converges to 0 for each $k>1$ as $n\to\infty$.

Now what about $K(1,n)$? It equals

$$\binom{n}{r_1}\sum_{i=0}^{r_1}\left(\binom{n}{i}/\binom{n}{r_1}\right)\lambda_1^{-i}N_1^i$$

$$=\binom{n}{r_1}\left(\lambda_1^{-r_1}N_1^{r_1}+m(n)\right)$$

where $\lim_{n\to\infty} m(n)=0$. This follows from

$$\lim_{n\to\infty}\left(\binom{n}{i}/\binom{n}{r_1}\right)=0,\ i<r_1$$

It follows that 15.1 is of the form

$$\begin{aligned}\mathbf{y}_n &= \frac{\lambda_1^n}{s_n s_{n-1}\cdots s_1}\binom{n}{r_1}P\begin{pmatrix}\left(\lambda_1^{-r_1}N_1^{r_1}+m(n)\right) & 0\\ 0 & E_n\end{pmatrix}P^{-1}\mathbf{x}\\ &= \frac{A\mathbf{y}_{n-1}}{s_n}\end{aligned}$$

where the entries of E_n converge to 0 as $n\to\infty$. Now denote by $\left(P^{-1}\mathbf{x}\right)_{m_1}$ the first m_1 entries of $P^{-1}\mathbf{x}$ where it is assumed that λ_1 has multiplicity m_1. Assume that

$$\left(P^{-1}\mathbf{x}\right)_{m_1}\notin \ker N_1^{r_1}$$

This will be the case unless you have made an extremely unfortunate choice of $\mathbf{x}$. Then $\mathbf{y}_n$ is of the form

$$\mathbf{y}_n = \frac{\lambda_1^n}{s_n s_{n-1}\cdots s_1}\binom{n}{r_1}P\begin{pmatrix}\left(\lambda_1^{-r_1}N_1^{r_1}+m(n)\right)\left(P^{-1}\mathbf{x}\right)_{m_1}\\ \mathbf{z}_n\end{pmatrix} \tag{15.2}$$

where $\binom{n}{r_1}\mathbf{z}_n\to\mathbf{0}$. Also, from the construction, there is a single entry of $\mathbf{y}_n$ equal to 1 and all other entries of the above vector have absolute value less than 1. It follows that

$$\frac{\lambda_1^n}{s_n s_{n-1}\cdots s_1}\binom{n}{r_1}$$

must be bounded independent of n.

Then it follows from this observation, that for large n, the above vector $\mathbf{y}_n$ is approximately equal to

$$\frac{\lambda_1^n}{s_n s_{n-1}\cdots s_1}\binom{n}{r_1} P\begin{pmatrix} \lambda_1^{-r_1} N_1^{r_1}\left(P^{-1}\mathbf{x}\right)_{m_1} \\ \mathbf{0} \end{pmatrix}$$

$$= \frac{1}{s_n s_{n-1}\cdots s_1} P\begin{pmatrix} \lambda_1^{n-r_1}\binom{n}{r_1} N_1^{r_1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}\mathbf{x} \tag{15.3}$$

If $\left(P^{-1}\mathbf{x}\right)_{m_1} \notin \ker\left(N_1^{r_1}\right)$, Then the above vector is also not equal to $\mathbf{0}$. What happens when it is multiplied on the left by $A - \lambda_1 I = P\left(J - \lambda_1 I\right) P^{-1}$? This results in

$$\frac{1}{s_n s_{n-1}\cdots s_1} P\begin{pmatrix} \lambda_1^{n-r_1} N_1 \binom{n}{r_1} N_1^{r_1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}\mathbf{x} = \mathbf{0}$$

because $N_1^{r_1+1} = 0$. Therefore, the vector in 15.3 is an eigenvector.

With this preparation, here is a theorem.

Theorem 15.1.1 *Let A be a complex $p \times p$ matrix such that the eigenvalues are*

$$\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$$

with $|\lambda_1| > |\lambda_j|$ for all $j \neq 1$. Then for $\mathbf{x}$ a given vector, let

$$\mathbf{y}_1 = \frac{A\mathbf{x}}{s_1}$$

where s_1 is an entry of $A\mathbf{x}$ which has the largest absolute value. If the scalars $\{s_1, \cdots, s_{n-1}\}$ and vectors $\{\mathbf{y}_1, \cdots, \mathbf{y}_{n-1}\}$ have been obtained, let

$$\mathbf{y}_n \equiv \frac{A\mathbf{y}_{n-1}}{s_n}$$

where s_n is the entry of $A\mathbf{y}_{n-1}$ which has largest absolute value. Then it is probably the case that $\{s_n\}$ will converge to λ_1 and $\{\mathbf{y}_n\}$ will converge to an eigenvector associated with λ_1.

Proof: Consider the claim about s_{n+1}. It was shown above that

$$\mathbf{z} \equiv P\begin{pmatrix} \lambda_1^{-r_1} N_1^{r_1}\left(P^{-1}\mathbf{x}\right)_{m_1} \\ \mathbf{0} \end{pmatrix}$$

is an eigenvector for λ_1. Let z_l be the top entry of $\mathbf{z}$ which has largest absolute value. Then for large n, it will probably be the case that the top entry of $\mathbf{y}_n$ which has largest absolute value will also be in the l^{th} slot. This follows from 15.2 because for large $n, \mathbf{z}_n$ will be very small, smaller than the largest entry of the top part of the vector in that expression. Then, since $m(n)$ is very small, the result follows if $\mathbf{z}$ has a well defined entry which has largest absolute value. Now from the above construction,

$$s_{n+1}\mathbf{y}_{n+1} \equiv A\mathbf{y}_n \approx \frac{\lambda_1^{n+1}}{s_n \cdots s_1}\binom{n}{r_1}\mathbf{z}$$

Applying a similar formula to s_n and the above observation, about the largest entry, it follows that for large n

$$s_{n+1} \approx \frac{\lambda_1^{n+1}}{s_n \cdots s_1}\binom{n}{r_1} z_l, \; s_n \approx \frac{\lambda_1^n}{s_{n-1} \cdots s_1}\binom{n-1}{r_1} z_l$$

Therefore, for large n,

$$\frac{s_{n+1}}{s_n} \approx \frac{\lambda_1}{s_n} \frac{n \cdots (n - r_1 + 1)}{(n-1) \cdots (n - r_1)} \approx \frac{\lambda_1}{s_n}$$

which shows that $s_{n+1} \approx \lambda_1$.

Now from the construction and formula in 15.2, then for large n

$$\begin{aligned}
\mathbf{y}_{n+1} &= \frac{\lambda_1^{n+1}}{s_{n+1} s_{n-1} \cdots s_1} \binom{n+1}{r_1} P \begin{pmatrix} \left(\lambda_1^{-r_1} N_1^{r_1} + m(n)\right) \left(P^{-1}\mathbf{x}\right)_{m_1} \\ \mathbf{z}_n \end{pmatrix} \\
&= \frac{\lambda_1}{s_{n+1}} \frac{\lambda_1^n}{s_n s_{n-1} \cdots s_1} \binom{n+1}{r_1} P \begin{pmatrix} \left(\lambda_1^{-r_1} N_1^{r_1} + m(n)\right) \left(P^{-1}\mathbf{x}\right)_{m_1} \\ \mathbf{z}_n \end{pmatrix} \\
&\approx \frac{\binom{n+1}{r_1}}{\binom{n}{r_1}} \frac{\lambda_1^n}{s_n s_{n-1} \cdots s_1} \binom{n}{r_1} P \begin{pmatrix} \left(\lambda_1^{-r_1} N_1^{r_1} + m(n)\right) \left(P^{-1}\mathbf{x}\right)_{m_1} \\ \mathbf{z}_n \end{pmatrix} \\
&= \frac{\binom{n+1}{r_1}}{\binom{n}{r_1}} \mathbf{y}_n \approx \mathbf{y}_n
\end{aligned}$$

Thus $\{\mathbf{y}_n\}$ is a Cauchy sequence and must converge to a vector $\mathbf{v}$. Now from the construction,

$$\lambda_1 \mathbf{v} = \lim_{n\to\infty} s_{n+1}\mathbf{y}_{n+1} = \lim_{n\to\infty} A\mathbf{y}_n = A\mathbf{v}.$$

This proves the theorem.

In summary, here is the procedure.

Finding the largest eigenvalue with its eigenvector.

1. Start with a vector, $\mathbf{u}_1$ which you hope is not unlucky.
2. If $\mathbf{u}_k$ is known,
$$\mathbf{u}_{k+1} = \frac{A\mathbf{u}_k}{s_{k+1}}$$
where s_{k+1} is the entry of $A\mathbf{u}_k$ which has largest absolute value.
3. When the scaling factors, s_k are not changing much, s_{k+1} will be close to the eigenvalue and $\mathbf{u}_{k+1}$ will be close to an eigenvector.
4. Check your answer to see if it worked well.

Example 15.1.2 *Find the largest eigenvalue of* $A = \begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix}$.

The power method will now be applied to find the largest eigenvalue for the above matrix. Letting $\mathbf{u}_1 = (1, \cdots, 1)^T$, we will consider $A\mathbf{u}_1$ and scale it.

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}.$$

Scaling this vector by dividing by the largest entry gives

$$\frac{1}{6} \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} = \mathbf{u}_2$$

Now lets do it again.

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ -8 \\ -6 \end{pmatrix}$$

Then

$$\mathbf{u}_3 = \frac{1}{22} \begin{pmatrix} 22 \\ -8 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{4}{11} \\ -\frac{3}{11} \end{pmatrix} = \begin{pmatrix} 1.0 \\ -.363\,636\,36 \\ -.272\,727\,27 \end{pmatrix}.$$

Continue doing this

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.363\,636\,36 \\ -.272\,727\,27 \end{pmatrix} = \begin{pmatrix} 7.090\,909\,1 \\ -4.363\,636\,4 \\ 1.636\,363\,7 \end{pmatrix}$$

Then

$$\mathbf{u}_4 = \begin{pmatrix} 1.0 \\ -.615\,38 \\ .230\,77 \end{pmatrix}$$

So far the scaling factors are changing fairly noticeably so continue.

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.615\,38 \\ .230\,77 \end{pmatrix} = \begin{pmatrix} 16.154 \\ -7.384\,6 \\ -1.384\,6 \end{pmatrix}$$

$$\mathbf{u}_5 = \begin{pmatrix} 1.0 \\ -.457\,14 \\ -8.571\,3 \times 10^{-2} \end{pmatrix}$$

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.457\,14 \\ -8.571\,3 \times 10^{-2} \end{pmatrix} = \begin{pmatrix} 10.457 \\ -5.485\,7 \\ .514\,3 \end{pmatrix}$$

Finally, after doing this iteration 11 times, you get

$$\mathbf{u}_{11} = \begin{pmatrix} 1.0 \\ -.499\,26 \\ -1.466\,2 \times 10^{-3} \end{pmatrix}$$

At this point, you could stop because the scaling factors are not changing by much. They went from 11.895 to 12.053. It looks like the eigenvalue is something like 12 which is in fact the case. The eigenvector is approximately $\mathbf{u}_{11}$. The true eigenvector for $\lambda = 12$ is

$$\begin{pmatrix} 1 \\ -.5 \\ 0 \end{pmatrix}$$

and so you see this is pretty close. If you didn't know this, observe

$$\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.499\,26 \\ -1.466\,2 \times 10^{-3} \end{pmatrix} = \begin{pmatrix} 11.974 \\ -5.991\,2 \\ 8.838\,6 \times 10^{-3} \end{pmatrix}$$

and

$$12.053 \begin{pmatrix} 1.0 \\ -.499\,26 \\ -1.466\,2 \times 10^{-3} \end{pmatrix} = \begin{pmatrix} 12.053 \\ -6.017\,6 \\ -1.767\,2 \times 10^{-2} \end{pmatrix}.$$

15.1.1 The Shifted Inverse Power Method

This method can find various eigenvalues and eigenvectors. It is a significant generalization of the above simple procedure and yields very good results. One can find complex eigenvalues using this method. The situation is this: You have a number, α which is close to λ, some eigenvalue of an $n \times n$ matrix, A. You don't know λ but you know that α is closer to λ than to any other eigenvalue. Your problem is to find both λ and an eigenvector which goes with λ. Another way to look at this is to start with α and seek the eigenvalue, λ, which is closest to α along with an eigenvector associated with λ. If α is an eigenvalue of A, then you have what you want. Therefore, I will always assume α is not an eigenvalue of A and so $(A-\alpha I)^{-1}$ exists. The method is based on the following lemma.

Lemma 15.1.3 *Let $\{\lambda_k\}_{k=1}^n$ be the eigenvalues of A. If $\mathbf{x}_k$ is an eigenvector of A for the eigenvalue λ_k, then $\mathbf{x}_k$ is an eigenvector for $(A-\alpha I)^{-1}$ corresponding to the eigenvalue $\frac{1}{\lambda_k-\alpha}$. Conversely, if*

$$(A-\alpha I)^{-1}\mathbf{y} = \frac{1}{\lambda-\alpha}\mathbf{y} \tag{15.4}$$

and $\mathbf{y} \neq \mathbf{0}$, then $A\mathbf{y} = \lambda\mathbf{y}$.

Proof: Let λ_k and $\mathbf{x}_k$ be as described in the statement of the lemma. Then

$$(A-\alpha I)\,\mathbf{x}_k = (\lambda_k-\alpha)\,\mathbf{x}_k$$

and so

$$\frac{1}{\lambda_k-\alpha}\mathbf{x}_k = (A-\alpha I)^{-1}\,\mathbf{x}_k.$$

Suppose 15.4. Then $\mathbf{y} = \frac{1}{\lambda-\alpha}\left[A\mathbf{y}-\alpha\mathbf{y}\right]$. Solving for $A\mathbf{y}$ leads to $A\mathbf{y} = \lambda\mathbf{y}$. This proves the lemma.

Now assume α is closer to λ than to any other eigenvalue. Then the magnitude of $\frac{1}{\lambda-\alpha}$ is greater than the magnitude of all the other eigenvalues of $(A-\alpha I)^{-1}$. Therefore, the power method applied to $(A-\alpha I)^{-1}$ will yield $\frac{1}{\lambda-\alpha}$. You end up with $s_{n+1} \approx \frac{1}{\lambda-\alpha}$ and solve for λ.

15.1.2 The Explicit Description Of The Method

Here is how you use this method to find the eigenvalue and eigenvector closest to α.

1. Find $(A-\alpha I)^{-1}$.
2. Pick $\mathbf{u}_1$. If you are not phenomenally unlucky, the iterations will converge.
3. If $\mathbf{u}_k$ has been obtained,
$$\mathbf{u}_{k+1} = \frac{(A-\alpha I)^{-1}\,\mathbf{u}_k}{s_{k+1}}$$
where s_{k+1} is the entry of $(A-\alpha I)^{-1}\,\mathbf{u}_k$ which has largest absolute value.
4. When the scaling factors, s_k are not changing much and the $\mathbf{u}_k$ are not changing much, find the approximation to the eigenvalue by solving
$$s_{k+1} = \frac{1}{\lambda-\alpha}$$
for λ. The eigenvector is approximated by $\mathbf{u}_{k+1}$.

5. Check your work by multiplying by the original matrix to see how well what you have found works.

Example 15.1.4 *Find the eigenvalue of* $A = \begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix}$ *which is closest to* -7. *Also find an eigenvector which goes with this eigenvalue.*

In this case the eigenvalues are $-6, 0$, and 12 so the correct answer is -6 for the eigenvalue. Then from the above procedure, I will start with an initial vector,

$$\mathbf{u}_1 \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then I must solve the following equation.

$$\left(\left(\begin{pmatrix} 5 & -14 & 11 \\ -4 & 4 & -4 \\ 3 & 6 & -3 \end{pmatrix} + 7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right.$$

Simplifying the matrix on the left, I must solve

$$\begin{pmatrix} 12 & -14 & 11 \\ -4 & 11 & -4 \\ 3 & 6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and then divide by the entry which has largest absolute value to obtain

$$\mathbf{u}_2 = \begin{pmatrix} 1.0 \\ .184 \\ -.76 \end{pmatrix}$$

Now solve

$$\begin{pmatrix} 12 & -14 & 11 \\ -4 & 11 & -4 \\ 3 & 6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.0 \\ .184 \\ -.76 \end{pmatrix}$$

and divide by the largest entry, 1.0515 to get

$$\mathbf{u}_3 = \begin{pmatrix} 1.0 \\ .0266 \\ -.97061 \end{pmatrix}$$

Solve

$$\begin{pmatrix} 12 & -14 & 11 \\ -4 & 11 & -4 \\ 3 & 6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.0 \\ .0266 \\ -.97061 \end{pmatrix}$$

and divide by the largest entry, 1.01 to get

$$\mathbf{u}_4 = \begin{pmatrix} 1.0 \\ 3.8454 \times 10^{-3} \\ -.99604 \end{pmatrix}.$$

These scaling factors are pretty close after these few iterations. Therefore, the predicted eigenvalue is obtained by solving the following for λ.

$$\frac{1}{\lambda + 7} = 1.01$$

which gives $\lambda = -6.01$. You see this is pretty close. In this case the eigenvalue closest to -7 was -6.

How would you know what to start with for an initial guess? You might apply Gerschgorin's theorem.

Example 15.1.5 *Consider the symmetric matrix,* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix}$. *Find the middle eigenvalue and an eigenvector which goes with it.*

Since A is symmetric, it follows it has three real eigenvalues which are solutions to

$$\begin{aligned} p(\lambda) &= \det\left(\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix}\right) \\ &= \lambda^3 - 4\lambda^2 - 24\lambda - 17 = 0 \end{aligned}$$

If you use your graphing calculator to graph this polynomial, you find there is an eigenvalue somewhere between $-.9$ and $-.8$ and that this is the middle eigenvalue. Of course you could zoom in and find it very accurately without much trouble but what about the eigenvector which goes with it? If you try to solve

$$\left((-.8)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

there will be only the zero solution because the matrix on the left will be invertible and the same will be true if you replace $-.8$ with a better approximation like $-.86$ or $-.855$. This is because all these are only approximations to the eigenvalue and so the matrix in the above is nonsingular for all of these. Therefore, you will only get the zero solution and

Eigenvectors are never equal to zero!

However, there exists such an eigenvector and you can find it using the shifted inverse power method. Pick $\alpha = -.855$. Then you solve

$$\left(\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix} + .855\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

or in other words,

$$\begin{pmatrix} 1.855 & 2.0 & 3.0 \\ 2.0 & 1.855 & 4.0 \\ 3.0 & 4.0 & 2.855 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and divide by the largest entry, -67.944, to obtain

$$\mathbf{u}_2 = \begin{pmatrix} 1.0 \\ -.589\,21 \\ -.230\,44 \end{pmatrix}$$

Now solve

$$\begin{pmatrix} 1.855 & 2.0 & 3.0 \\ 2.0 & 1.855 & 4.0 \\ 3.0 & 4.0 & 2.855 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.0 \\ -.589\,21 \\ -.230\,44 \end{pmatrix}$$

, Solution is : $\begin{pmatrix} -514.01 \\ 302.12 \\ 116.75 \end{pmatrix}$ and divide by the largest entry, -514.01, to obtain

$$\mathbf{u}_3 = \begin{pmatrix} 1.0 \\ -.587\,77 \\ -.227\,14 \end{pmatrix} \tag{15.5}$$

Then doing it again, the scaling factor is -513.42 and the next iterate is

$$\mathbf{u}_4 = \begin{pmatrix} 1.0 \\ -.587\,78 \\ -.227\,14 \end{pmatrix}$$

Clearly the $\mathbf{u}_k$ are not changing much. This suggests an approximate eigenvector for this eigenvalue which is close to $-.855$ is the above $\mathbf{u}_3$ and an eigenvalue is obtained by solving

$$\frac{1}{\lambda + .855} = -514.01,$$

which yields $\lambda = -.856\,9$ Lets check this.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.587\,77 \\ -.227\,14 \end{pmatrix} = \begin{pmatrix} -.856\,96 \\ .503\,67 \\ .194\,64 \end{pmatrix}.$$

$$-.856\,9 \begin{pmatrix} 1.0 \\ -.587\,77 \\ -.227\,14 \end{pmatrix} = \begin{pmatrix} -.856\,9 \\ .503\,7 \\ .194\,6 \end{pmatrix}$$

Thus the vector of 15.5 is very close to the desired eigenvector, just as $-.856\,9$ is very close to the desired eigenvalue. For practical purposes, I have found both the eigenvector and the eigenvalue.

Example 15.1.6 *Find the eigenvalues and eigenvectors of the matrix,* $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

This is only a 3×3 matrix and so it is not hard to estimate the eigenvalues. Just get the characteristic equation, graph it using a calculator and zoom in to find the eigenvalues. If you do this, you find there is an eigenvalue near -1.2, one near $-.4$, and one near 5.5. (The characteristic equation is $2 + 8\lambda + 4\lambda^2 - \lambda^3 = 0$.) Of course I have no idea what the eigenvectors are.

Lets first try to find the eigenvector and a better approximation for the eigenvalue near -1.2. In this case, let $\alpha = -1.2$. Then

$$(A - \alpha I)^{-1} = \begin{pmatrix} -25.357\,143 & -33.928\,571 & 50.0 \\ 12.5 & 17.5 & -25.0 \\ 23.214\,286 & 30.357\,143 & -45.0 \end{pmatrix}.$$

Then for the first iteration, letting $\mathbf{u}_1 = (1,1,1)^T$,

$$\begin{pmatrix} -25.357\,143 & -33.928\,571 & 50.0 \\ 12.5 & 17.5 & -25.0 \\ 23.214\,286 & 30.357\,143 & -45.0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -9.285\,714 \\ 5.0 \\ 8.571\,429 \end{pmatrix}$$

To get $\mathbf{u}_2$, I must divide by $-9.285\,714$. Thus

$$\mathbf{u}_2 = \begin{pmatrix} 1.0 \\ -.538\,461\,56 \\ -.923\,077 \end{pmatrix}.$$

Do another iteration.

$$\begin{pmatrix} -25.357\,143 & -33.928\,571 & 50.0 \\ 12.5 & 17.5 & -25.0 \\ 23.214\,286 & 30.357\,143 & -45.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.538\,461\,56 \\ -.923\,077 \end{pmatrix}$$

$$= \begin{pmatrix} -53.241\,762 \\ 26.153\,848 \\ 48.406\,596 \end{pmatrix}$$

Then to get $\mathbf{u}_3$ you divide by $-53.241\,762$. Thus

$$\mathbf{u}_3 = \begin{pmatrix} 1.0 \\ -.491\,228\,07 \\ -.909\,184\,71 \end{pmatrix}.$$

Now iterate again because the scaling factors are still changing quite a bit.

$$\begin{pmatrix} -25.357\,143 & -33.928\,571 & 50.0 \\ 12.5 & 17.5 & -25.0 \\ 23.214\,286 & 30.357\,143 & -45.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.491\,228\,07 \\ -.909\,184\,71 \end{pmatrix}$$

$$= \begin{pmatrix} -54.149\,712 \\ 26.633\,127 \\ 49.215\,317 \end{pmatrix}.$$

This time the scaling factor didn't change too much. It is $-54.149\,712$. Thus

$$\mathbf{u}_4 = \begin{pmatrix} 1.0 \\ -.491\,842\,45 \\ -.908\,874\,95 \end{pmatrix}.$$

Lets do one more iteration.

$$\begin{pmatrix} -25.357\,143 & -33.928\,571 & 50.0 \\ 12.5 & 17.5 & -25.0 \\ 23.214\,286 & 30.357\,143 & -45.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ -.491\,842\,45 \\ -.908\,874\,95 \end{pmatrix}$$

$$= \begin{pmatrix} -54.113\,379 \\ 26.614\,631 \\ 49.182\,727 \end{pmatrix}.$$

You see at this point the scaling factors have definitely settled down and so it seems our eigenvalue would be obtained by solving

$$\frac{1}{\lambda - (-1.2)} = -54.113\,379$$

and this yields $\lambda = -1.218\,479\,7$ as an approximation to the eigenvalue and the eigenvector would be obtained by dividing by $-54.113\,379$ which gives

$$\mathbf{u}_5 = \begin{pmatrix} 1.000\,000\,2 \\ -.491\,830\,97 \\ -.908\,883\,09 \end{pmatrix}.$$

How well does it work?

$$\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1.000\,000\,2 \\ -.491\,830\,97 \\ -.908\,883\,09 \end{pmatrix} = \begin{pmatrix} -1.218\,479\,8 \\ .599\,286\,34 \\ 1.107\,455\,6 \end{pmatrix}$$

while

$$-1.218\,479\,7 \begin{pmatrix} 1.000\,000\,2 \\ -.491\,830\,97 \\ -.908\,883\,09 \end{pmatrix} = \begin{pmatrix} -1.218\,479\,9 \\ .599\,286\,05 \\ 1.107\,455\,6 \end{pmatrix}.$$

For practical purposes, this has found the eigenvalue near -1.2 as well as an eigenvector associated with it.

Next I shall find the eigenvector and a more precise value for the eigenvalue near $-.4$. In this case,

$$(A-\alpha I)^{-1} = \begin{pmatrix} 8.064\,516\,1 \times 10^{-2} & -9.274\,193\,5 & 6.451\,612\,9 \\ -.403\,225\,81 & 11.370\,968 & -7.258\,064\,5 \\ .403\,225\,81 & 3.629\,032\,3 & -2.741\,935\,5 \end{pmatrix}.$$

As before, I have no idea what the eigenvector is so I will again try $(1,1,1)^T$. Then to find $\mathbf{u}_2$,

$$\begin{pmatrix} 8.064\,516\,1 \times 10^{-2} & -9.274\,193\,5 & 6.451\,612\,9 \\ -.403\,225\,81 & 11.370\,968 & -7.258\,064\,5 \\ .403\,225\,81 & 3.629\,032\,3 & -2.741\,935\,5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2.741\,935\,4 \\ 3.709\,677\,7 \\ 1.290\,322\,6 \end{pmatrix}$$

The scaling factor is $3.709\,677\,7$. Thus

$$\mathbf{u}_2 = \begin{pmatrix} -.739\,130\,36 \\ 1.0 \\ .347\,826\,07 \end{pmatrix}.$$

Now lets do another iteration.

$$\begin{pmatrix} 8.064\,516\,1 \times 10^{-2} & -9.274\,193\,5 & 6.451\,612\,9 \\ -.403\,225\,81 & 11.370\,968 & -7.258\,064\,5 \\ .403\,225\,81 & 3.629\,032\,3 & -2.741\,935\,5 \end{pmatrix} \begin{pmatrix} -.739\,130\,36 \\ 1.0 \\ .347\,826\,07 \end{pmatrix}$$

$$= \begin{pmatrix} -7.089\,761\,6 \\ 9.144\,460\,4 \\ 2.377\,279\,2 \end{pmatrix}.$$

The scaling factor is $9.144\,460\,4$. Thus

$$\mathbf{u}_3 = \begin{pmatrix} -.775\,306\,72 \\ 1.0 \\ .259\,969\,33 \end{pmatrix}.$$

Lets do another iteration. The scaling factors are still changing quite a bit.

$$\begin{pmatrix} 8.064\,516\,1 \times 10^{-2} & -9.274\,193\,5 & 6.451\,612\,9 \\ -.403\,225\,81 & 11.370\,968 & -7.258\,064\,5 \\ .403\,225\,81 & 3.629\,032\,3 & -2.741\,935\,5 \end{pmatrix} \begin{pmatrix} -.775\,306\,72 \\ 1.0 \\ .259\,969\,33 \end{pmatrix}$$

$$= \begin{pmatrix} -7.659\,496\,8 \\ 9.796\,717\,5 \\ 2.603\,589\,5 \end{pmatrix}.$$

The scaling factor is now $9.796\,717\,5$. Therefore,

$$\mathbf{u}_4 = \begin{pmatrix} -.781\,843\,18 \\ 1.0 \\ .265\,761\,41 \end{pmatrix}.$$

Lets do another iteration.

$$\begin{pmatrix} 8.064\,516\,1 \times 10^{-2} & -9.274\,193\,5 & 6.451\,612\,9 \\ -.403\,225\,81 & 11.370\,968 & -7.258\,064\,5 \\ .403\,225\,81 & 3.629\,032\,3 & -2.741\,935\,5 \end{pmatrix} \begin{pmatrix} -.781\,843\,18 \\ 1.0 \\ .265\,761\,41 \end{pmatrix}$$

$$= \begin{pmatrix} -7.622\,655\,6 \\ 9.757\,313\,9 \\ 2.585\,072\,3 \end{pmatrix}.$$

Now the scaling factor is $9.757\,313\,9$ and so

$$\mathbf{u}_5 = \begin{pmatrix} -.781\,224\,8 \\ 1.0 \\ .264\,936\,88 \end{pmatrix}.$$

I notice the scaling factors are not changing by much so the approximate eigenvalue is

$$\frac{1}{\lambda + .4} = 9.757\,313\,9$$

which shows $\lambda = -.297\,512\,78$ is an approximation to the eigenvalue near .4. How well does it work?

$$\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} -.781\,224\,8 \\ 1.0 \\ .264\,936\,88 \end{pmatrix} = \begin{pmatrix} .232\,361\,04 \\ -.297\,512\,72 \\ -.0\,787\,375\,2 \end{pmatrix}.$$

$$-.297\,512\,78 \begin{pmatrix} -.781\,224\,8 \\ 1.0 \\ .264\,936\,88 \end{pmatrix} = \begin{pmatrix} .232\,424\,36 \\ -.297\,512\,78 \\ -7.882\,210\,8 \times 10^{-2} \end{pmatrix}.$$

It works pretty well. For practical purposes, the eigenvalue and eigenvector have now been found. If you want better accuracy, you could just continue iterating.

Next I will find the eigenvalue and eigenvector for the eigenvalue near 5.5. In this case,

$$(A - \alpha I)^{-1} = \begin{pmatrix} 29.2 & 16.8 & 23.2 \\ 19.2 & 10.8 & 15.2 \\ 28.0 & 16.0 & 22.0 \end{pmatrix}.$$

As before, I have no idea what the eigenvector is but I am tired of always using $(1,1,1)^T$ and I don't want to give the impression that you always need to start with this vector. Therefore, I shall let $\mathbf{u}_1 = (1,2,3)^T$. What follows is the iteration without all the comments between steps.

$$\begin{pmatrix} 29.2 & 16.8 & 23.2 \\ 19.2 & 10.8 & 15.2 \\ 28.0 & 16.0 & 22.0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1.324 \times 10^2 \\ 86.4 \\ 1.26 \times 10^2 \end{pmatrix}.$$

$s_2 = 86.4$.

$$\mathbf{u}_2 = \begin{pmatrix} 1.532\,407\,4 \\ 1.0 \\ 1.458\,333\,3 \end{pmatrix}.$$

$$\begin{pmatrix} 29.2 & 16.8 & 23.2 \\ 19.2 & 10.8 & 15.2 \\ 28.0 & 16.0 & 22.0 \end{pmatrix} \begin{pmatrix} 1.532\,407\,4 \\ 1.0 \\ 1.458\,333\,3 \end{pmatrix} = \begin{pmatrix} 95.379\,629 \\ 62.388\,888 \\ 90.990\,74 \end{pmatrix}$$

$s_3 = 95.379\,629$.

$$\mathbf{u}_3 = \begin{pmatrix} 1.0 \\ .654\,111\,25 \\ .953\,985\,05 \end{pmatrix}$$

$$\begin{pmatrix} 29.2 & 16.8 & 23.2 \\ 19.2 & 10.8 & 15.2 \\ 28.0 & 16.0 & 22.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ .654\,111\,25 \\ .953\,985\,05 \end{pmatrix} = \begin{pmatrix} 62.321\,522 \\ 40.764\,974 \\ 59.453\,451 \end{pmatrix}$$

$s_4 = 62.321\,522$.

$$\mathbf{u}_4 = \begin{pmatrix} 1.0 \\ .654\,107\,48 \\ .953\,979\,45 \end{pmatrix}$$

$$\begin{pmatrix} 29.2 & 16.8 & 23.2 \\ 19.2 & 10.8 & 15.2 \\ 28.0 & 16.0 & 22.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ .654\,107\,48 \\ .953\,979\,45 \end{pmatrix} = \begin{pmatrix} 62.321\,329 \\ 40.764\,848 \\ 59.453\,268 \end{pmatrix}$$

$s_5 = 62.321\,329$. Looks like it is time to stop because this scaling factor is not changing much from s_3.

$$\mathbf{u}_5 = \begin{pmatrix} 1.0 \\ .654\,107\,49 \\ .953\,979\,46 \end{pmatrix}.$$

Then the approximation of the eigenvalue is gotten by solving

$$62.321\,329 = \frac{1}{\lambda - 5.5}$$

which gives $\lambda = 5.516\,045\,9$. Lets see how well it works.

$$\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1.0 \\ .654\,107\,49 \\ .953\,979\,46 \end{pmatrix} = \begin{pmatrix} 5.516\,045\,9 \\ 3.608\,087 \\ 5.262\,194\,4 \end{pmatrix}$$

$$5.516\,045\,9 \begin{pmatrix} 1.0 \\ .654\,107\,49 \\ .953\,979\,46 \end{pmatrix} = \begin{pmatrix} 5.516\,045\,9 \\ 3.608\,086\,9 \\ 5.262\,194\,5 \end{pmatrix}.$$

15.1.3 Complex Eigenvalues

What about complex eigenvalues? If your matrix is real, you won't see these by graphing the characteristic equation on your calculator. Will the shifted inverse power method find these eigenvalues and their associated eigenvectors? The answer is yes. However, for a real matrix, you must pick α to be complex. This is because the eigenvalues occur in conjugate pairs so if you don't pick it complex, it will be the same distance between any conjugate pair of complex numbers and so nothing in the above argument for convergence implies you will get convergence to a complex number. Also, the process of iteration will yield only real vectors and scalars.

Example 15.1.7 *Find the complex eigenvalues and corresponding eigenvectors for the matrix,*

$$\begin{pmatrix} 5 & -8 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here the characteristic equation is $\lambda^3 - 5\lambda^2 + 8\lambda - 6 = 0$. One solution is $\lambda = 3$. The other two are $1+i$ and $1-i$. I will apply the process to $\alpha = i$ to find the eigenvalue closest to i.

$$(A-\alpha I)^{-1} = \begin{pmatrix} -.02 - .14i & 1.24 + .68i & -.84 + .12i \\ -.14 + .02i & .68 - .24i & .12 + .84i \\ .02 + .14i & -.24 - .68i & .84 + .88i \end{pmatrix}$$

Then let $\mathbf{u}_1 = (1,1,1)^T$ for lack of any insight into anything better.

$$\begin{pmatrix} -.02 - .14i & 1.24 + .68i & -.84 + .12i \\ -.14 + .02i & .68 - .24i & .12 + .84i \\ .02 + .14i & -.24 - .68i & .84 + .88i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} .38 + .66i \\ .66 + .62i \\ .62 + .34i \end{pmatrix}$$

$s_2 = .66 + .62i$.

$$\mathbf{u}_2 = \begin{pmatrix} .804\,878\,05 + .243\,902\,44i \\ 1.0 \\ .756\,097\,56 - .195\,121\,95i \end{pmatrix}$$

$$\begin{pmatrix} -.02 - .14i & 1.24 + .68i & -.84 + .12i \\ -.14 + .02i & .68 - .24i & .12 + .84i \\ .02 + .14i & -.24 - .68i & .84 + .88i \end{pmatrix} \cdot \begin{pmatrix} .804\,878\,05 + .243\,902\,44i \\ 1.0 \\ .756\,097\,56 - .195\,121\,95i \end{pmatrix}$$

$$= \begin{pmatrix} .646\,341\,46 + .817\,073\,17i \\ .817\,073\,17 + .353\,658\,54i \\ .548\,780\,49 - 6.097\,560\,9 \times 10^{-2} i \end{pmatrix}$$

$s_3 = .646\,341\,46 + .817\,073\,17i$. After more iterations, of this sort, you find $s_9 = 1.002\,748\,5 + 2.137\,621\,7 \times 10^{-4} i$ and

$$\mathbf{u}_9 = \begin{pmatrix} 1.0 \\ .501\,514\,17 - .499\,807\,33i \\ 1.562\,088\,1 \times 10^{-3} - .499\,778\,55i \end{pmatrix}.$$

Then

$$\begin{pmatrix} -.02 - .14i & 1.24 + .68i & -.84 + .12i \\ -.14 + .02i & .68 - .24i & .12 + .84i \\ .02 + .14i & -.24 - .68i & .84 + .88i \end{pmatrix} \cdot \begin{pmatrix} 1.0 \\ .501\,514\,17 - .499\,807\,33i \\ 1.562\,088\,1 \times 10^{-3} - .499\,778\,55i \end{pmatrix}$$

$$= \begin{pmatrix} 1.000\,407\,8 + 1.269\,979 \times 10^{-3} i \\ .501\,077\,31 - .498\,893\,66i \\ 8.848\,928 \times 10^{-4} - .499\,515\,22i \end{pmatrix}$$

$s_{10} = 1.000\,407\,8 + 1.269\,979 \times 10^{-3} i$.

$$\mathbf{u}_{10} = \begin{pmatrix} 1.0 \\ .500\,239\,18 - .499\,325\,33i \\ 2.506\,749\,2 \times 10^{-4} - .499\,311\,92i \end{pmatrix}$$

The scaling factors are not changing much at this point

$$1.000\,407\,8 + 1.269\,979 \times 10^{-3} i = \frac{1}{\lambda - i}$$

The approximate eigenvalue is then $\lambda = .999\,590\,76 + .998\,731\,06i$. This is pretty close to $1 + i$. How well does the eigenvector work?

$$\begin{pmatrix} 5 & -8 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1.0 \\ .500\,239\,18 - .499\,325\,33i \\ 2.506\,749\,2 \times 10^{-4} - .499\,311\,92i \end{pmatrix}$$
$$= \begin{pmatrix} .999\,590\,61 + .998\,731\,12i \\ 1.0 \\ .500\,239\,18 - .499\,325\,33i \end{pmatrix}$$

$$(.999\,590\,76 + .998\,731\,06i) \begin{pmatrix} 1.0 \\ .500\,239\,18 - .499\,325\,33i \\ 2.506\,749\,2 \times 10^{-4} - .499\,311\,92i \end{pmatrix}$$
$$= \begin{pmatrix} .999\,590\,76 + .998\,731\,06i \\ .998\,726\,18 + 4.834\,203\,9 \times 10^{-4} i \\ .498\,928\,9 - .498\,857\,22i \end{pmatrix}$$

It took more iterations than before because α was not very close to $1 + i$.

This illustrates an interesting topic which leads to many related topics. If you have a polynomial, $x^4 + ax^3 + bx^2 + cx + d$, you can consider it as the characteristic polynomial of a certain matrix, called a **companion matrix**. In this case,

$$\begin{pmatrix} -a & -b & -c & -d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The above example was just a companion matrix for $\lambda^3 - 5\lambda^2 + 8\lambda - 6$. You can see the pattern which will enable you to obtain a companion matrix for any polynomial of the form $\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$. This illustrates that one way to find the complex zeros of a polynomial is to use the shifted inverse power method on a companion matrix for the polynomial. Doubtless there are better ways but this does illustrate how impressive this procedure is. Do you have a better way?

Note that the shifted inverse power method is a way you can begin with something close but not equal to an eigenvalue and end up with something close to an eigenvector.

15.1.4 Rayleigh Quotients And Estimates for Eigenvalues

There are many specialized results concerning the eigenvalues and eigenvectors for Hermitian matrices. Recall a matrix, A is Hermitian if $A = A^*$ where A^* means to take the transpose

of the conjugate of A. In the case of a real matrix, Hermitian reduces to symmetric. Recall also that for $\mathbf{x} \in \mathbb{F}^n$,

$$|\mathbf{x}|^2 = \mathbf{x}^*\mathbf{x} = \sum_{j=1}^{n} |x_j|^2 .$$

Recall the following corollary found on Page 183 which is stated here for convenience.

Corollary 15.1.8 *If A is Hermitian, then all the eigenvalues of A are real and there exists an orthonormal basis of eigenvectors.*

Thus for $\{\mathbf{x}_k\}_{k=1}^n$ this orthonormal basis,

$$\mathbf{x}_i^*\mathbf{x}_j = \delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

For $\mathbf{x} \in \mathbb{F}^n$, $\mathbf{x} \neq \mathbf{0}$, the Rayleigh quotient is defined by

$$\frac{\mathbf{x}^* A \mathbf{x}}{|\mathbf{x}|^2}.$$

Now let the eigenvalues of A be $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $A\mathbf{x}_k = \lambda_k \mathbf{x}_k$ where $\{\mathbf{x}_k\}_{k=1}^n$ is the above orthonormal basis of eigenvectors mentioned in the corollary. Then if $\mathbf{x}$ is an arbitrary vector, there exist constants, a_i such that

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{x}_i.$$

Also,

$$\begin{aligned} |\mathbf{x}|^2 &= \sum_{i=1}^{n} \overline{a}_i \mathbf{x}_i^* \sum_{j=1}^{n} a_j \mathbf{x}_j \\ &= \sum_{ij} \overline{a}_i a_j \mathbf{x}_i^* \mathbf{x}_j = \sum_{ij} \overline{a}_i a_j \delta_{ij} = \sum_{i=1}^{n} |a_i|^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\mathbf{x}^* A \mathbf{x}}{|\mathbf{x}|^2} &= \frac{\left(\sum_{i=1}^n \overline{a}_i \mathbf{x}_i^*\right)\left(\sum_{j=1}^n a_j \lambda_j \mathbf{x}_j\right)}{\sum_{i=1}^n |a_i|^2} \\ &= \frac{\sum_{ij} \overline{a}_i a_j \lambda_j \mathbf{x}_i^* \mathbf{x}_j}{\sum_{i=1}^n |a_i|^2} = \frac{\sum_{ij} \overline{a}_i a_j \lambda_j \delta_{ij}}{\sum_{i=1}^n |a_i|^2} \\ &= \frac{\sum_{i=1}^n |a_i|^2 \lambda_i}{\sum_{i=1}^n |a_i|^2} \in [\lambda_1, \lambda_n] . \end{aligned}$$

In other words, the Rayleigh quotient is always between the largest and the smallest eigenvalues of A. When $\mathbf{x} = \mathbf{x}_n$, the Rayleigh quotient equals the largest eigenvalue and when $\mathbf{x} = \mathbf{x}_1$ the Rayleigh quotient equals the smallest eigenvalue. Suppose you calculate a Rayleigh quotient. How close is it to some eigenvalue?

Theorem 15.1.9 *Let $\mathbf{x} \neq \mathbf{0}$ and form the Rayleigh quotient,*

$$\frac{\mathbf{x}^* A \mathbf{x}}{|\mathbf{x}|^2} \equiv q.$$

Then there exists an eigenvalue of A, denoted here by λ_q such that

$$|\lambda_q - q| \leq \frac{|A\mathbf{x} - q\mathbf{x}|}{|\mathbf{x}|}. \tag{15.6}$$

Proof: Let $\mathbf{x} = \sum_{k=1}^{n} a_k \mathbf{x}_k$ where $\{\mathbf{x}_k\}_{k=1}^{n}$ is the orthonormal basis of eigenvectors.

$$\begin{aligned}
|A\mathbf{x} - q\mathbf{x}|^2 &= (A\mathbf{x} - q\mathbf{x})^* (A\mathbf{x} - q\mathbf{x}) \\
&= \left(\sum_{k=1}^{n} a_k \lambda_k \mathbf{x}_k - q a_k \mathbf{x}_k\right)^* \left(\sum_{k=1}^{n} a_k \lambda_k \mathbf{x}_k - q a_k \mathbf{x}_k\right) \\
&= \left(\sum_{j=1}^{n} (\lambda_j - q)\, \overline{a}_j \mathbf{x}_j^*\right) \left(\sum_{k=1}^{n} (\lambda_k - q)\, a_k \mathbf{x}_k\right) \\
&= \sum_{j,k} (\lambda_j - q)\, \overline{a}_j\, (\lambda_k - q)\, a_k \mathbf{x}_j^* \mathbf{x}_k \\
&= \sum_{k=1}^{n} |a_k|^2 (\lambda_k - q)^2
\end{aligned}$$

Now pick the eigenvalue, λ_q which is closest to q. Then

$$|A\mathbf{x} - q\mathbf{x}|^2 = \sum_{k=1}^{n} |a_k|^2 (\lambda_k - q)^2 \geq (\lambda_q - q)^2 \sum_{k=1}^{n} |a_k|^2 = (\lambda_q - q)^2 |\mathbf{x}|^2$$

which implies 15.6.

Example 15.1.10 *Consider the symmetric matrix, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$. Let $\mathbf{x} = (1,1,1)^T$. How close is the Rayleigh quotient to some eigenvalue of A? Find the eigenvector and eigenvalue to several decimal places.*

Everything is real and so there is no need to worry about taking conjugates. Therefore, the Rayleigh quotient is

$$\frac{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{3} = \frac{19}{3}$$

According to the above theorem, there is some eigenvalue of this matrix, λ_q such that

$$\begin{aligned}
\left|\lambda_q - \frac{19}{3}\right| &\leq \frac{\left|\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{19}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|}{\sqrt{3}} \\
&= \frac{1}{\sqrt{3}} \begin{pmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ \frac{5}{3} \end{pmatrix} \\
&= \frac{\sqrt{\frac{1}{9} + \left(\frac{4}{3}\right)^2 + \left(\frac{5}{3}\right)^2}}{\sqrt{3}} = 1.247\,2
\end{aligned}$$

Could you find this eigenvalue and associated eigenvector? Of course you could. This is what the shifted inverse power method is all about.

Solve

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} - \frac{19}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

In other words solve

$$\begin{pmatrix} -\frac{16}{3} & 2 & 3 \\ 2 & -\frac{13}{3} & 1 \\ 3 & 1 & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and divide by the entry which is largest, 3.870 7, to get

$$\mathbf{u}_2 = \begin{pmatrix} .699\,25 \\ .493\,89 \\ 1.0 \end{pmatrix}$$

Now solve

$$\begin{pmatrix} -\frac{16}{3} & 2 & 3 \\ 2 & -\frac{13}{3} & 1 \\ 3 & 1 & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .699\,25 \\ .493\,89 \\ 1.0 \end{pmatrix}$$

and divide by the largest entry, 2.997 9 to get

$$\mathbf{u}_3 = \begin{pmatrix} .714\,73 \\ .522\,63 \\ 1.0 \end{pmatrix}$$

Now solve

$$\begin{pmatrix} -\frac{16}{3} & 2 & 3 \\ 2 & -\frac{13}{3} & 1 \\ 3 & 1 & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .714\,73 \\ .522\,63 \\ 1.0 \end{pmatrix}$$

and divide by the largest entry, 3.045 4, to get

$$\mathbf{u}_4 = \begin{pmatrix} .713\,7 \\ .520\,56 \\ 1.0 \end{pmatrix}$$

Solve

$$\begin{pmatrix} -\frac{16}{3} & 2 & 3 \\ 2 & -\frac{13}{3} & 1 \\ 3 & 1 & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .713\,7 \\ .520\,56 \\ 1.0 \end{pmatrix}$$

and divide by the largest entry, 3.042 1 to get

$$\mathbf{u}_5 = \begin{pmatrix} .713\,78 \\ .520\,73 \\ 1.0 \end{pmatrix}$$

You can see these scaling factors are not changing much. The predicted eigenvalue is then about

$$\frac{1}{3.042\,1} + \frac{19}{3} = 6.662\,1.$$

How close is this?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} .713\,78 \\ .520\,73 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 4.755\,2 \\ 3.469 \\ 6.662\,1 \end{pmatrix}$$

while

$$6.662\,1\begin{pmatrix} .713\,78 \\ .520\,73 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 4.755\,3 \\ 3.469\,2 \\ 6.662\,1 \end{pmatrix}.$$

You see that for practical purposes, this has found the eigenvalue and an eigenvector.

15.2 The QR Algorithm

15.2.1 Basic Properties And Definition

Recall the theorem about the QR factorization in Theorem 5.7.5. It says that given an $n \times n$ real matrix A, there exists a real orthogonal matrix Q and an upper triangular matrix R such that $A = QR$ and that this factorization can be accomplished by a systematic procedure. One such procedure was given in proving this theorem.

There is also a way to generalize the QR factorization to the case where A is just a complex $n \times n$ matrix and Q is unitary while R is upper triangular with nonnegative entries on the main diagonal. Letting $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be the matrix with the $\mathbf{a}_j$ the columns, each a vector in $\mathbb{C}^n$, let Q_1 be a unitary matrix which maps $\mathbf{a}_1$ to $|\mathbf{a}_1|\,\mathbf{e}_1$ in the case that $\mathbf{a}_1 \neq \mathbf{0}$. If $\mathbf{a}_1 = \mathbf{0}$, let $Q_1 = I$. Why does such a unitary matrix exist? Let

$$\{\mathbf{a}_1/|\mathbf{a}_1|, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$$

be an orthonormal basis and let $Q_1\left(\frac{\mathbf{a}_1}{|\mathbf{a}_1|}\right) = \mathbf{e}_1, Q_1(\mathbf{u}_2) = \mathbf{e}_2$ etc. Extend Q_1 linearly. Then Q_1 preserves lengths so it is unitary by Lemma 13.6.1. Now

$$\begin{aligned} Q_1 A &= \begin{pmatrix} Q_1\mathbf{a}_1 & Q_1\mathbf{a}_2 & \cdots & Q_1\mathbf{a}_n \end{pmatrix} \\ &= \begin{pmatrix} |\mathbf{a}_1|\,\mathbf{e}_1 & Q_1\mathbf{a}_2 & \cdots & Q_1\mathbf{a}_n \end{pmatrix} \end{aligned}$$

which is a matrix of the form

$$\begin{pmatrix} |\mathbf{a}_1| & \mathbf{b} \\ \mathbf{0} & A_1 \end{pmatrix}$$

Now do the same thing for A_1 obtaining an $n-1 \times n-1$ unitary matrix Q_2' which when multiplied on the left of A_1 yields something of the form

$$\begin{pmatrix} a & \mathbf{b}_1 \\ \mathbf{0} & A_2 \end{pmatrix}$$

Then multiplying A on the left by the product

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2' \end{pmatrix} Q_1 \equiv Q_2 Q_1$$

yields a matrix which is upper triangular with respect to the first two columns. Continuing this way

$$Q_n Q_{n-1} \cdots Q_1 A = R$$

where R is upper triangular having all positive entries on the main diagonal. Then the desired unitary matrix is

$$Q = (Q_n Q_{n-1} \cdots Q_1)^*$$

The QR algorithm is described in the following definition.

Definition 15.2.1 *The QR algorithm is the following. In the description of this algorithm, Q is unitary and R is upper triangular having nonnegative entries on the main diagonal. Starting with A an $n \times n$ matrix, form*

$$A_0 \equiv A = Q_1 R_1 \tag{15.7}$$

Then

$$A_1 \equiv R_1 Q_1. \tag{15.8}$$

In general given

$$A_k = R_k Q_k, \tag{15.9}$$

obtain A_{k+1} by

$$A_k = Q_{k+1} R_{k+1},\ A_{k+1} = R_{k+1} Q_{k+1} \tag{15.10}$$

This algorithm was proposed by Francis in 1961. The sequence $\{A_k\}$ is the desired sequence of iterates. Now with the above definition of the algorithm, here are its properties. The next lemma shows each of the A_k is unitarily similar to A and the amazing thing about this algorithm is that it becomes increasingly easy to find the eigenvalues of the A_k.

Lemma 15.2.2 *Let A be an $n \times n$ matrix and let the Q_k and R_k be as described in the algorithm. Then each A_k is unitarily similar to A and denoting by $Q^{(k)}$ the product $Q_1 Q_2 \cdots Q_k$ and $R^{(k)}$ the product $R_k R_{k-1} \cdots R_1$, it follows that*

$$A^k = Q^{(k)} R^{(k)}$$

(The matrix on the left is A raised to the k^{th} power.)

$$A = Q^{(k)} A_k Q^{(k)*},\ A_k = Q^{(k)*} A Q^{(k)}.$$

Proof: From the algorithm, $R_{k+1} = A_{k+1} Q_{k+1}^*$ and so

$$A_k = Q_{k+1} R_{k+1} = Q_{k+1} A_{k+1} Q_{k+1}^*$$

Now iterating this, it follows

$$A_{k-1} = Q_k A_k Q_k^* = Q_k Q_{k+1} A_{k+1} Q_{k+1}^* Q_k^*$$

$$A_{k-2} = Q_{k-1} A_{k-1} Q_{k-1}^* = Q_{k-1} Q_k Q_{k+1} A_{k+1} Q_{k+1}^* Q_k^* Q_{k-1}^*$$

etc. Thus, after $k-2$ more iterations,

$$A = Q^{(k+1)} A_{k+1} Q^{(k+1)*}$$

The product of unitary matrices is unitary and so this proves the first claim of the lemma.

Now consider the part about A^k. From the algorithm, this is clearly true for $k = 1$. ($A^1 = QR$) Suppose then that

$$A^k = Q_1 Q_2 \cdots Q_k R_k R_{k-1} \cdots R_1$$

What was just shown indicated

$$A = Q_1 Q_2 \cdots Q_{k+1} A_{k+1} Q_{k+1}^* Q_k^* \cdots Q_1^*$$

and now from the algorithm, $A_{k+1} = R_{k+1} Q_{k+1}$ and so

$$A = Q_1 Q_2 \cdots Q_{k+1} R_{k+1} Q_{k+1} Q_{k+1}^* Q_k^* \cdots Q_1^*$$

Then

$$A^{k+1} = AA^k =$$

$$\overbrace{Q_1Q_2\cdots Q_{k+1}R_{k+1}Q_{k+1}Q_{k+1}^*Q_k^*\cdots Q_1^*}^{A}Q_1\cdots Q_kR_kR_{k-1}\cdots R_1$$

$$= Q_1Q_2\cdots Q_{k+1}R_{k+1}R_kR_{k-1}\cdots R_1 \equiv Q^{(k+1)}R^{(k+1)}$$

This proves the lemma.

Here is another very interesting lemma.

Lemma 15.2.3 *Suppose $Q^{(k)}, Q$ are unitary and R_k is upper triangular such that the diagonal entries on R_k are all positive and*

$$Q = \lim_{k\to\infty} Q^{(k)}R_k$$

Then

$$\lim_{k\to\infty} Q^{(k)} = Q, \ \lim_{k\to\infty} R_k = I.$$

Also the QR factorization of A is unique whenever A^{-1} exists.

Proof: Let

$$Q = (\mathbf{q}_1, \cdots, \mathbf{q}_n), \ Q^{(k)} = \left(\mathbf{q}_1^k, \cdots, \mathbf{q}_n^k\right)$$

where the $\mathbf{q}$ are the columns. Also denote by r_{ij}^k the ij^{th} entry of R_k. Thus

$$Q^{(k)}R_k = \left(\mathbf{q}_1^k, \cdots, \mathbf{q}_n^k\right)\begin{pmatrix} r_{11}^k & & * \\ & \ddots & \\ 0 & & r_{nn}^k \end{pmatrix}$$

It follows

$$r_{11}^k\mathbf{q}_1^k \to \mathbf{q}_1$$

and so

$$r_{11}^k = \left|r_{11}^k\mathbf{q}_1^k\right| \to 1$$

Therefore,

$$\mathbf{q}_1^k \to \mathbf{q}_1.$$

Next consider the second column.

$$r_{12}^k\mathbf{q}_1^k + r_{22}^k\mathbf{q}_2^k \to \mathbf{q}_2$$

Taking the inner product of both sides with $\mathbf{q}_1^k$ it follows

$$\lim_{k\to\infty} r_{12}^k = \lim_{k\to\infty}\left(\mathbf{q}_2\cdot\mathbf{q}_1^k\right) = (\mathbf{q}_2\cdot\mathbf{q}_1) = 0.$$

Therefore,

$$\lim_{k\to\infty} r_{22}^k\mathbf{q}_2^k = \mathbf{q}_2$$

and since $r_{22}^k > 0$, it follows as in the first part that $r_{22}^k \to 1$. Hence

$$\lim_{k\to\infty} \mathbf{q}_2^k = \mathbf{q}_2.$$

Continuing this way, it follows

$$\lim_{k\to\infty} r_{ij}^k = 0$$

for all $i \neq j$ and

$$\lim_{k\to\infty} r_{jj}^k = 1, \ \lim_{k\to\infty} \mathbf{q}_j^k = \mathbf{q}_j.$$

Thus $R_k \to I$ and $Q^{(k)} \to Q$. This proves the first part of the lemma.

The second part follows immediately. If $QR = Q'R' = A$ where A^{-1} exists, then

$$Q^*Q' = R\left(R'\right)^{-1}$$

and I need to show both sides of the above are equal to I. The left side of the above is unitary and the right side is upper triangular having positive entries on the diagonal. This is because the inverse of such an upper triangular matrix having positive entries on the main diagonal is still upper triangular having positive entries on the main diagonal and the product of two such upper triangular matrices gives another of the same form having positive entries on the main diagonal. Suppose then that $Q = R$ where Q is unitary and R is upper triangular having positive entries on the main diagonal. Let $Q_k = Q$ and $R_k = R$. It follows

$$IR_k \to R = Q$$

and so from the first part, $R_k \to I$ but $R_k = R$ and so $R = I$. Thus applying this to $Q^*Q' = R\left(R'\right)^{-1}$ yields both sides equal I. This proves the lemma.

15.2.2 The Case Of Real Eigenvalues

With these lemmas, it is possible to prove that for the QR algorithm and certain conditions, the sequence A_k converges pointwise to an upper triangular matrix having the eigenvalues of A down the diagonal. I will assume all the matrices are real here.

This convergence won't always happen. Consider for example the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

You can verify quickly that the algorithm will return this matrix for each k. The problem here is that, although the matrix has the two eigenvalues $-1, 1$, they have the same absolute value. The QR algorithm works in somewhat the same way as the power method, exploiting differences in the size of the eigenvalues.

If A has all real eigenvalues and you are interested in finding these eigenvalues along with the corresponding eigenvectors, you could always consider $A + \lambda I$ instead where λ is sufficiently large and positive that $A + \lambda I$ has all positive eigenvalues. (Recall Gerschgorin's theorem.) Then if μ is an eigenvalue of $A + \lambda I$ with

$$(A + \lambda I)\,\mathbf{x} = \mu\mathbf{x}$$

then

$$A\mathbf{x} = (\mu - \lambda)\,\mathbf{x}$$

so to find the eigenvalues of A you just subtract λ from the eigenvalues of $A + \lambda I$. Thus there is no loss of generality in assuming at the outset that the eigenvalues of A are all positive. Here is the theorem. It involves a technical condition which will often hold. The proof presented here follows [20] and is a special case of that presented in this reference.

Before giving the proof, note that the product of upper triangular matrices is upper triangular. If they both have positive entries on the main diagonal so will the product. Furthermore, the inverse of an upper triangular matrix is upper triangular. I will use these simple facts without much comment whenever convenient.

Theorem 15.2.4 *Let A be a real matrix having eigenvalues*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$$

and let

$$A = SDS^{-1} \tag{15.11}$$

where

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

and suppose S^{-1} has an LU factorization. Then the matrices A_k in the QR algorithm described above converge to an upper triangular matrix T' having the eigenvalues of A, $\lambda_1, \cdots, \lambda_n$ descending on the main diagonal. The matrices $Q^{(k)}$ converge to Q', an orthogonal matrix which equals Q except for possibly having some columns multiplied by -1 for Q the unitary part of the QR factorization of S,

$$S = QR,$$

and

$$\lim_{k\to\infty} A_k = T' = Q'^T A Q'$$

Proof: From Lemma 15.2.2

$$A^k = Q^{(k)} R^{(k)} = SD^k S^{-1} \tag{15.12}$$

Let $S = QR$ where this is just a QR factorization which is known to exist and let $S^{-1} = LU$ which is assumed to exist. Thus

$$Q^{(k)} R^{(k)} = QRD^k LU \tag{15.13}$$

and so

$$\begin{aligned} Q^{(k)} R^{(k)} &= QRD^k LU \\ &= QRD^k LD^{-k} D^k U \end{aligned}$$

That matrix in the middle, $D^k L D^{-k}$ satisfies

$$\left(D^k L D^{-k}\right)_{ij} = \lambda_i^k L_{ij} \lambda_j^{-k} \text{ for } j \le i,\ 0 \text{ if } j > i.$$

Thus for $j < i$ the expression converges to 0 because $\lambda_j > \lambda_i$ when this happens. When $i = j$ it reduces to 1. Thus the matrix in the middle is of the form

$$I + E_k$$

where $E_k \to 0$. Then it follows

$$A^k = Q^{(k)} R^{(k)} = QR\left(I + E_k\right) D^k U$$

$$\begin{aligned} &= Q\left(I + RE_k R^{-1}\right) RD^k U \\ &\equiv Q\left(I + F_k\right) RD^k U \end{aligned}$$

where $F_k \to 0$. Then let $I + F_k = Q_k R_k$ where this is another QR factorization. Then it reduces to

$$Q^{(k)} R^{(k)} = Q Q_k R_k R D^k U$$

This looks really interesting because by Lemma 15.2.3 $Q_k \to I$ and $R_k \to I$ because $Q_k R_k = (I + F_k) \to I$. So it follows QQ_k is an orthogonal matrix converging to Q while

$$R_k R D^k U \left(R^{(k)} \right)^{-1}$$

is upper triangular, being the product of upper triangular matrices. Unfortunately, it is not known that the diagonal entries of this matrix are nonnegative because of the U. Let Λ be just like the identity matrix but having some of the ones replaced with -1 in such a way that ΛU is an upper triangular matrix having positive diagonal entries. Note $\Lambda^2 = I$ and also Λ commutes with a diagonal matrix. Thus

$$\begin{aligned} Q^{(k)} R^{(k)} &= Q Q_k R_k R D^k \Lambda^2 U \\ &= Q Q_k R_k R \Lambda D^k (\Lambda U) \end{aligned}$$

At this point, one does some inspired massaging to write the above in the form

$$\begin{aligned} & Q Q_k \left(\Lambda D^k \right) \left[\left(\Lambda D^k \right)^{-1} R_k R \Lambda D^k \right] (\Lambda U) \\ = \; & Q \left(Q_k \Lambda \right) D^k \left[\left(\Lambda D^k \right)^{-1} R_k R \Lambda D^k \right] (\Lambda U) \\ = \; & Q \left(Q_k \Lambda \right) \overbrace{D^k \left[\left(\Lambda D^k \right)^{-1} R_k R \Lambda D^k \right] (\Lambda U)}^{\equiv G_k} \end{aligned}$$

Now I claim the middle matrix in $[\cdot]$ is upper triangular and has all positive entries on the diagonal. This is because it is an upper triangular matrix which is similar to the upper triangular matrix, $R_k R$ and so it has the same eigenvalues (diagonal entries) as $R_k R$. Thus the matrix $G_k \equiv D^k \left[\left(\Lambda D^k \right)^{-1} R_k R \Lambda D^k \right] (\Lambda U)$ is upper triangular and has all positive entries on the diagonal. Multipy on the right by G_k^{-1} to get

$$Q^{(k)} R^{(k)} G_k^{-1} = Q Q_k \Lambda \to Q'$$

where Q' is essentially equal to Q but might have some of the columns multiplied by -1. This is because $Q_k \to I$ and so $Q_k \Lambda \to \Lambda$. Now by Lemma 15.2.3, it follows

$$Q^{(k)} \to Q', \; R^{(k)} G_k^{-1} \to I.$$

It remains to verify A_k converges to an upper triangular matrix. Recall that from 15.12 and the definition below this ($S = QR$)

$$\begin{aligned} A &= S D S^{-1} = (QR) D (QR)^{-1} \\ &= Q R D R^{-1} Q^T = Q T Q^T \end{aligned}$$

Where T is an upper triangular matrix. This is because it is the product of upper triangular matrices R, D, R^{-1}. Thus

$$Q^T A Q = T.$$

If you replace Q with Q' in the above, it still results in an upper triangular matrix, T' having the same diagonal entries as T. This is because

$$T = Q^T A Q = (Q' \Lambda)^T A (Q' \Lambda) = \Lambda Q'^T A Q' \Lambda$$

and considering the ii^{th} entry yields

$$\left(Q^T AQ\right)_{ii} \equiv \sum_{j,k} \Lambda_{ij} \left(Q'^T AQ'\right)_{jk} \Lambda_{ki} = \Lambda_{ii}\Lambda_{ii} \left(Q'^T AQ'\right)_{ii} = \left(Q'^T AQ'\right)_{ii}$$

Recall from Lemma 15.2.2,

$$A_k = Q^{(k)T} AQ^{(k)}$$

Thus taking a limit and using the first part,

$$A_k = Q^{(k)T} AQ^{(k)} \to Q'^T AQ' = T'.$$

This proves the theorem.

An easy case is for A symmetric. Recall Corollary 7.4.13. By this corollary, there exists an orthogonal (real unitary) matrix Q such that

$$Q^T AQ = D$$

where D is diagonal having the eigenvalues on the main diagonal decreasing in size from the upper left corner to the lower right.

Corollary 15.2.5 *Let A be a real symmetric $n \times n$ matrix having eigenvalues*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$$

and let Q be defined by

$$QDQ^T = A,\ D = Q^T AQ, \tag{15.14}$$

where Q is orthogonal and D is a diagonal matrix having the eigenvalues on the main diagonal decreasing in size from the upper left corner to the lower right. Let Q^T have an LU factorization. Then in the QR algorithm, the matrices $Q^{(k)}$ converge to Q' where Q' is the same as Q except having some columns multiplied by (-1). Thus the columns of Q' are eigenvectors of A. The matrices A_k converge to D.

Proof: This follows from Theorem 15.2.4. Here $S = Q, S^{-1} = Q^T$. Thus

$$Q = S = QR$$

and $R = I$. By Theorem 15.2.4 and Lemma 15.2.2,

$$A_k = Q^{(k)T} AQ^{(k)} \to Q'^T AQ' = Q^T AQ = D.$$

because formula 15.14 is unaffected by replacing Q with Q'. This proves the corollary.

When using the QR algorithm, it is not necessary to check technical condition about S^{-1} having an LU factorization. The algorithm delivers a sequence of matrices which are similar to the original one. If that sequence converges to an upper triangular matrix, then the algorithm worked. Furthermore, the technical condition is sufficient but not necessary. The algorithm will work even without the technical condition.

Example 15.2.6 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

It is a symmetric matrix but other than that, I just pulled it out of the air. By Lemma 15.2.2 it follows $A_k = Q^{(k)T} A Q^{(k)}$. And so to get to the answer quickly I could have the computer raise A to a power and then take the QR factorization of what results to get the k^{th} iteration using the above formula. Lets pick $k = 10$.

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}^{10} = \begin{pmatrix} 4.227\,3 \times 10^7 & 2.595\,9 \times 10^7 & 1.861\,1 \times 10^7 \\ 2.595\,9 \times 10^7 & 1.607\,2 \times 10^7 & 1.150\,6 \times 10^7 \\ 1.861\,1 \times 10^7 & 1.150\,6 \times 10^7 & 8.239\,6 \times 10^6 \end{pmatrix}$$

Now take QR factorization of this. The computer will do that also.
This yields

$$\begin{pmatrix} .797\,85 & -.599\,12 & -6.694\,3 \times 10^{-2} \\ .489\,95 & .709\,12 & -.507\,06 \\ .351\,26 & .371\,76 & .859\,31 \end{pmatrix} \cdot \begin{pmatrix} 5.298\,3 \times 10^7 & 3.262\,7 \times 10^7 & 2.338 \times 10^7 \\ 0 & 1.217\,2 \times 10^5 & 71946. \\ 0 & 0 & 277.03 \end{pmatrix}$$

Next it follows

$$\begin{aligned} A_{10} &= \begin{pmatrix} .797\,85 & -.599\,12 & -6.694\,3 \times 10^{-2} \\ .489\,95 & .709\,12 & -.507\,06 \\ .351\,26 & .371\,76 & .859\,31 \end{pmatrix}^T \cdot \\ &\quad \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} .797\,85 & -.599\,12 & -6.694\,3 \times 10^{-2} \\ .489\,95 & .709\,12 & -.507\,06 \\ .351\,26 & .371\,76 & .859\,31 \end{pmatrix} \end{aligned}$$

and this equals

$$\begin{pmatrix} 6.057\,1 & 3.698 \times 10^{-3} & 3.434\,6 \times 10^{-5} \\ 3.698 \times 10^{-3} & 3.200\,8 & -4.064\,3 \times 10^{-4} \\ 3.434\,6 \times 10^{-5} & -4.064\,3 \times 10^{-4} & -.257\,9 \end{pmatrix}$$

By Gerschgorin's theorem, the eigenvalues are pretty close to the diagonal entries of the above matrix. Note I didn't use the theorem, just Lemma 15.2.2 and Gerschgorin's theorem to verify the eigenvalues are close to the above numbers. The eigenvectors are close to

$$\begin{pmatrix} .797\,85 \\ .489\,95 \\ .351\,26 \end{pmatrix}, \begin{pmatrix} -.599\,12 \\ .709\,12 \\ .371\,76 \end{pmatrix}, \begin{pmatrix} -6.694\,3 \times 10^{-2} \\ -.507\,06 \\ .859\,31 \end{pmatrix}$$

Lets check one of these.

$$\begin{aligned} &\left(\left(\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} - 6.057\,1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) \begin{pmatrix} .797\,85 \\ .489\,95 \\ .351\,26 \end{pmatrix} \\ = &\begin{pmatrix} -2.197\,2 \times 10^{-3} \\ 2.543\,9 \times 10^{-3} \\ 1.393\,1 \times 10^{-3} \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Now lets see how well the smallest approximate eigenvalue and eigenvector works.

$$\left(\left(\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} - (-.257\,9) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) \begin{pmatrix} -6.694\,3 \times 10^{-2} \\ -.507\,06 \\ .859\,31 \end{pmatrix}$$

$$\begin{pmatrix} 2.704 \times 10^{-4} \\ -2.7377 \times 10^{-4} \\ -1.3695 \times 10^{-4} \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For practical purposes, this has found the eigenvalues and eigenvectors.

15.2.3 The QR Algorithm In The General Case

In the case where A has distinct positive eigenvalues it was shown above that under reasonable conditions related to a certain matrix having an LU factorization the QR algorithm produces a sequence of matrices $\{A_k\}$ which converges to an upper triangular matrix. What if A is just an $n \times n$ matrix having possibly complex eigenvalues but A is nondefective? What happens with the QR algorithm in this case? The short answer to this question is that the A_k of the algorithm **typically cannot converge**. However, this does not mean the algorithm is not useful in finding eigenvalues. It turns out the sequence of matrices $\{A_k\}$ have the appearance of a block upper triangular matrix for large k in the sense that the entries below the blocks on the main diagonal are small. Then looking at these blocks gives a way to approximate the eigenvalues. An important example of the concept of a block triangular matrix is the real Schur form for a matrix discussed in Theorem 7.4.6 but the concept as described here allows for any size block centered on the diagonal.

First it is important to note a simple fact about unitary diagonal matrices. In what follows Λ will denote a unitary matrix which is also a diagonal matrix. These matrices are just the identity matrix with some of the ones replaced with a number of the form $e^{i\theta}$ for some θ. The important property of multiplication of any matrix by Λ on either side is that it leaves all the zero entries the same and also preserves the absolute values of the other entries. Thus a block triangular matrix multiplied by Λ on either side is still block triangular. If the matrix is close to being block triangular this property of being close to a block triangular matrix is also preserved by multiplying on either side by Λ. Other patterns depending only on the size of the absolute value occuring in the matrix are also preserved by multiplying on either side by Λ. In other words, in looking for a pattern in a matrix, multiplication by Λ is irrelevant.

Now let A be an $n \times n$ matrix having real or complex entries. By Lemma 15.2.2 and the assumption that A is nondefective, there exists an invertible S,

$$A^k = Q^{(k)}R^{(k)} = SD^kS^{-1} \tag{15.15}$$

where

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

and by rearranging the columns of S, D can be made such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

Assume S^{-1} has an LU factorization. Then

$$A^k = SD^kLU = SD^kLD^{-k}D^kU.$$

Consider the matrix in the middle, D^kLD^{-k}. The ij^{th} entry is of the form

$$\left(D^kLD^{-k}\right)_{ij} = \begin{cases} \lambda_i^k L_{ij}\lambda_j^{-k} \text{ if } j < i \\ 1 \text{ if } i = j \\ 0 \text{ if } j > i \end{cases}$$

and these all converge to 0 whenever $|\lambda_i| < |\lambda_j|$. Thus

$$D^k L D^{-k} = (L_k + E_k)$$

where L_k is a lower triangular matrix which has all ones down the diagonal and some subdiagonal terms of the form

$$\lambda_i^k L_{ij} \lambda_j^{-k} \tag{15.16}$$

for which $|\lambda_i| = |\lambda_j|$ while $E_k \to 0$. (Note the entries of L_k are all bounded independent of k but some may fail to converge.) Then

$$Q^{(k)} R^{(k)} = S\left(L_k + E_k\right) D^k U$$

Let

$$SL_k = Q_k R_k \tag{15.17}$$

where this is the QR factorization of SL_k. Then

$$\begin{aligned} Q^{(k)} R^{(k)} &= \left(Q_k R_k + S E_k\right) D^k U \\ &= Q_k \left(I + Q_k^* S E_k R_k^{-1}\right) R_k D^k U \\ &= Q_k \left(I + F_k\right) R_k D^k U \end{aligned}$$

where $F_k \to 0$. Let $I + F_k = Q_k' R_k'$. Then

$$Q^{(k)} R^{(k)} = Q_k Q_k' R_k' R_k D^k U$$

By Lemma 15.2.3

$$Q_k' \to I \text{ and } R_k' \to I. \tag{15.18}$$

Now let Λ_k be a diagonal unitary matrix which has the property that

$$\Lambda_k^* D^k U$$

is an upper triangular matrix which has all the diagonal entries positive. Then

$$Q^{(k)} R^{(k)} = Q_k Q_k' \Lambda_k \left(\Lambda_k^* R_k' R_k \Lambda_k\right) \Lambda_k^* D^k U$$

That matrix in the middle has all positive diagonal entries because it is itself an upper triangular matrix, being the product of such, and is similar to the matrix $R_k' R_k$ which is upper triangular with positive diagonal entries. By Lemma 15.2.3 again, this time using the uniqueness assertion,

$$Q^{(k)} = Q_k Q_k' \Lambda_k, \; R^{(k)} = \left(\Lambda_k^* R_k' R_k \Lambda_k\right) \Lambda_k^* D^k U$$

Note the term $Q_k Q_k' \Lambda_k$ must be real because the algorithm gives all $Q^{(k)}$ as real matrices. By 15.18 it follows that for k large enough

$$Q^{(k)} \approx Q_k \Lambda_k$$

where $\approx$ means the two matrices are close. Recall

$$A_k = Q^{(k)T} A Q^{(k)}$$

and so for large k,

$$A_k \approx \left(Q_k \Lambda_k\right)^* A \left(Q_k \Lambda_k\right) = \Lambda_k^* Q_k^* A Q_k \Lambda_k$$

As noted above, the form of $\Lambda_k^* Q_k^* A Q_k \Lambda_k$ in terms of which entries are large and small is not affected by the presence of Λ_k and Λ_k^*. Thus, in considering what form this is in, it suffices to consider $Q_k^* A Q_k$.

This could get pretty complicated but I will consider the case where if

$$\text{if } |\lambda_i| = |\lambda_{i+1}|, \text{ then } |\lambda_{i+2}| < |\lambda_{i+1}|. \tag{15.19}$$

This is typical of the situation where the eigenvalues are all distinct and the matrix A is real so the eigenvalues occur as conjugate pairs. Then in this case, L_k above is lower triangular with some nonzero terms on the diagonal right below the main diagonal but zeros everywhere else. Thus maybe

$$(L_k)_{s+1,s} \neq 0$$

Recall 15.17 which implies

$$Q_k = S L_k R_k^{-1} \tag{15.20}$$

where R_k^{-1} is upper triangular. Also recall that from the definition of S in 15.15,

$$S^{-1} A S = D$$

and so the columns of S are eigenvectors of A, the i^{th} being an eigenvector for λ_i. Now from the form of L_k, it follows $L_k R_k^{-1}$ is a block upper triangular matrix denoted by T_B and so $Q_k = S T_B$. It follows from the above construction in 15.16 and the given assumption on the sizes of the eigenvalues, there are finitely many 2×2 blocks centered on the main diagonal along with possibly some diagonal entries. Therefore, for large k the matrix

$$A_k = Q^{(k)T} A Q^{(k)}$$

is approximately of the same form as that of

$$Q_k^* A Q_k = T_B^{-1} S^{-1} A S T_B = T_B^{-1} D T_B$$

which is a block upper triangular matrix. As explained above, multiplication by the various diagonal unitary matrices does not affect this form. Therefore, for large k, A_k is approximately a block upper triangular matrix.

How would this change if the above assumption on the size of the eigenvalues were relaxed but the matrix was still nondefective with appropriate matrices having an LU factorization as above? It would mean the blocks on the diagonal would be larger. This immediately makes the problem more cumbersome to deal with. However, in the case that the eigenvalues of A are distinct, the above situation really is typical of what occurs and in any case can be quickly reduced to this case.

To see this, suppose condition 15.19 is violated and $\lambda_j, \cdots, \lambda_{j+p}$ are complex eigenvalues having nonzero imaginary parts such that each has the same absolute value but they are all distinct. Then let $\mu > 0$ and consider the matrix $A + \mu I$. Thus the corresponding eigenvalues of $A + \mu I$ are $\lambda_j + \mu, \cdots, \lambda_{j+p} + \mu$. A short computation shows shows $|\lambda_j + \mu|, \cdots, |\lambda_{j+p} + \mu|$ are all distinct and so the above situation of 15.19 is obtained. Of course, if there are repeated eigenvalues, it may not be possible to reduce to the case above and you would end up with large blocks on the main diagonal which could be difficult to deal with.

So how do you identify the eigenvalues? You know A_k and behold that it is close to a block upper triangular matrix T_B'. You know A_k is also similar to A. Therefore, T_B' has eigenvalues which are close to the eigenvalues of A_k and hence those of A provided k is sufficiently large. See Theorem 7.9.2 which depends on complex analysis. Thus you find the eigenvalues of this block triangular matrix T_B' and assert that these are good approximations

of the eigenvalues of A_k and hence to those of A. How do you find the eigenvalues of a block triangular matrix? This is easy from Lemma 7.4.5. Say

$$T'_B = \begin{pmatrix} B_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & B_m \end{pmatrix}$$

Then forming $\lambda I - T'_B$ and taking the determinant, it follows from Lemma 7.4.5 this equals

$$\prod_{j=1}^{m} \det(\lambda I_j - B_j)$$

and so all you have to do is take the union of the eigenvalues for each B_j. In the case emphasized here this is very easy because these blocks are just 2×2 matrices.

How do you identify approximate eigenvectors from this? First try to find the approximate eigenvectors for A_k. Pick an approximate eigenvalue λ, an exact eigenvalue for T'_B. Then find $\mathbf{v}$ solving $T'_B\mathbf{v} = \lambda\mathbf{v}$. It follows since T'_B is close to A_k that

$$A_k\mathbf{v} \approx \lambda\mathbf{v}$$

and so

$$Q^{(k)}AQ^{(k)T}\mathbf{v} = A_k\mathbf{v} \approx \lambda\mathbf{v}$$

Hence

$$AQ^{(k)T}\mathbf{v} \approx \lambda Q^{(k)T}\mathbf{v}$$

and so $Q^{(k)T}\mathbf{v}$ is an approximation to the eigenvector which goes with the eigenvalue of A which is close to λ.

Example 15.2.7 *Here is a matrix.*

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

It happens that the eigenvalues of this matrix are $1, 1+i, 1-i$. *Lets apply the QR algorithm as if the eigenvalues were not known.*

Applying the QR algorithm to this matrix yields the following sequence of matrices.

$$A_1 = \begin{pmatrix} 1.2353 & 1.9412 & 4.3657 \\ -.39215 & 1.5425 & 5.3886 \times 10^{-2} \\ -.16169 & -.18864 & .22222 \end{pmatrix}$$

$$\vdots$$

$$A_{12} = \begin{pmatrix} 9.1772 \times 10^{-2} & .63089 & -2.0398 \\ -2.8556 & 1.9082 & -3.1043 \\ 1.0786 \times 10^{-2} & 3.4614 \times 10^{-4} & 1.0 \end{pmatrix}$$

At this point the bottom two terms on the left part of the bottom row are both very small so it appears the real eigenvalue is near 1.0. The complex eigenvalues are obtained from solving

$$\det\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 9.1772 \times 10^{-2} & .63089 \\ -2.8556 & 1.9082 \end{pmatrix}\right) = 0$$

This yields

$$\lambda = 1.0 - .98828i,\ 1.0 + .98828i$$

Example 15.2.8 *The equation $x^4+x^3+4x^2+x-2=0$ has exactly two real solutions. You can see this by graphing it. However, the rational root theorem from algebra shows neither of these solutions are rational. Also, graphing it does not yield any information about the complex solutions. Lets use the QR algorithm to approximate all the solutions, real and complex.*

A matrix whose characteristic polynomial is the given polynomial is

$$\begin{pmatrix} -1 & -4 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Using the QR algorithm yields the following sequence of iterates for A_k

$$A_1 = \begin{pmatrix} .999\,99 & -2.592\,7 & -1.758\,8 & -1.297\,8 \\ 2.121\,3 & -1.777\,8 & -1.604\,2 & -.994\,15 \\ 0 & .342\,46 & -.327\,49 & -.917\,99 \\ 0 & 0 & -.446\,59 & .105\,26 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} .999\,99 & -2.592\,7 & -1.758\,8 & -1.297\,8 \\ 2.121\,3 & -1.777\,8 & -1.604\,2 & -.994\,15 \\ 0 & .342\,46 & -.327\,49 & -.917\,99 \\ 0 & 0 & -.446\,59 & .105\,26 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1.454\,5 & -3.668\,9 & -2.055\,2 & -.133\,78 \\ 1.468\,7 & .849\,18 & .846\,7 & -2.314\,2\times10^{-2} \\ 0 & -.143\,2 & -.977\,31 & -.110\,15 \\ 0 & 0 & .508\,84 & .582\,68 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} .808\,57 & -3.846 & -1.516\,2 & -.933\,61 \\ 1.431\,3 & -1.480\,1 & -.666\,05 & -.507\,15 \\ 0 & -7.432\,9\times10^{-2} & -.729\,76 & -.833\,11 \\ 0 & 0 & -.223\,58 & .401\,33 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} .226\,23 & -4.131\,8 & -1.869 & -.867\,34 \\ 1.078\,7 & -.919\,86 & -.416\,05 & -.254\,97 \\ 0 & 1.360\,5\times10^{-2} & -.799\,59 & -.699\,88 \\ 0 & 0 & -.104\,16 & .493\,34 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} -1.484\,9 & -1.440\,4 & -.834\,07 & -.323\,31 \\ 3.772\,9 & .797\,42 & 1.825\,7 & .564\,67 \\ 0 & -2.867\,2\times10^{-3} & -.886\,97 & -.524\,26 \\ 0 & 0 & .0\,742\,8 & .574\,58 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} -.303\,61 & -4.243\,3 & -1.932 & -.807\,95 \\ .975\,47 & -.384\,22 & -.100\,53 & -.104\,24 \\ 0 & -2.433\,6\times10^{-3} & -.839\,08 & -.642\,43 \\ 0 & 0 & -4.414\,7\times10^{-2} & .526\,98 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} .550\,18 & -1.241\,5 & -.484\,18 & -.115\,27 \\ 3.977\,6 & -1.239\,1 & -1.914 & -.703\,08 \\ 0 & -4.902\,2\times10^{-4} & -.870\,25 & -.568\,36 \\ 0 & 0 & 2.946\,2\times10^{-2} & .559\,25 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} -.834\,12 & -4.168\,2 & -1.939 & -.778\,3 \\ 1.05 & .145\,14 & .217\,1 & 2.547\,4 \times 10^{-2} \\ 0 & 4.026\,4 \times 10^{-4} & -.850\,29 & -.616\,08 \\ 0 & 0 & -1.826\,3 \times 10^{-2} & .539\,39 \end{pmatrix}$$

Now this is similar to A and the eigenvalues are close to the eigenvalues obtained from the two blocks on the diagonal. Of course the lower left corner of the bottom block is vanishing but it is still fairly large so the eigenvalues are approximated by the solution to

$$\det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -.850\,29 & -.616\,08 \\ -1.826\,3 \times 10^{-2} & .539\,39 \end{pmatrix}\right) = 0$$

The solution to this is

$$\lambda = -.858\,34,\ .547\,44$$

and for the complex eigenvalues,

$$\det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -.834\,12 & -4.168\,2 \\ 1.05 & .145\,14 \end{pmatrix}\right) = 0$$

The solution is

$$\lambda = -.344\,49 - 2.033\,9i,\ -.344\,49 + 2.033\,9i$$

How close are the complex eigenvalues just obtained to giving a solution to the original equation? Try $-.344\,49 + 2.033\,9i$. When this is plugged in it yields

$$-.00\,12 + 2.006\,8 \times 10^{-4}i$$

which is pretty close to 0. The real eigenvalues are also very close to the corresponding real solutions to the original equation.

It seems like most of the attention to the QR algorithm has to do with finding ways to get it to "converge" faster. Great and marvelous are the clever tricks which have been proposed to do this but my intent is to present the basic ideas, not to go in to the numerous refinements of this algorithm. However, there is one thing which is usually done. It involves reducing to the case of an upper Hessenberg matrix. Let A be an invertible $n \times n$ matrix. Let Q_1' be a unitary matrix

$$Q_1'\begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} \sqrt{\sum_{j=2}^{n} |a_{j1}|^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \equiv \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The vector Q_1' is multiplying is just the bottom $n-1$ entries of the first column of A. Then let Q_1 be

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_1' \end{pmatrix}$$

It follows

$$Q_1 A Q_1^* = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_1' \end{pmatrix} A Q_1^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a & & & \\ \vdots & & A_1' & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_1'^* \end{pmatrix}$$

$$= \begin{pmatrix} * & * & \cdots & * \\ a & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

Now let Q_2' be the $n-2 \times n-2$ matrix which does to the first column of A_1 the same sort of thing that the $n-1 \times n-1$ matrix Q_1' did to the first column of A. Let

$$Q_2 \equiv \begin{pmatrix} I & 0 \\ 0 & Q_2' \end{pmatrix}$$

where I is the 2×2 identity. Then applying block multiplication,

$$Q_2 Q_1 A Q_1^* Q_2^* = \begin{pmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ 0 & * & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{pmatrix}$$

where A_2 is now an $n-2 \times n-2$ matrix. Continuing this way you eventually get a unitary matrix Q which is a product of those discussed above such that

$$QAQ^T = \begin{pmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ 0 & * & * & & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & & * & * \end{pmatrix}$$

This matrix equals zero below the subdiagonal. It is called an upper Hessenberg matrix.

It happens that in the QR algorithm if A_k is upper Hessenberg, so is A_{k+1}. To see this, note that the matrix is upper Hessenberg means that $A_{ij} = 0$ whenever $i - j \geq 2$.

$$A_{k+1} = R_k Q_k$$

where $A_k = Q_k R_k$. Therefore as shown before,

$$A_{k+1} = R_k A_k R_k^{-1}$$

Let the ij^{th} entry of A_k be a_{ij}^k. Then if $i - j \geq 2$

$$a_{ij}^{k+1} = \sum_{p=i}^{n} \sum_{q=1}^{j} r_{ip} a_{pq}^k r_{qj}^{-1}$$

It is given that $a_{pq}^k = 0$ whenever $p - q \geq 2$. However, from the above sum,

$$p - q \geq i - j \geq 2$$

and so the sum equals 0.

Since upper Hessenberg matrices stay that way in the algorithm and it is closer to being upper triangular, it is reasonable to suppose the QR algorithm will yield good results more quickly for this upper Hessenberg matrix than for the original matrix. This would be especially true if the matrix is good sized.

15.3 Exercises

1. In Example 15.1.10 an eigenvalue was found correct to several decimal places along with an eigenvector. Find the other eigenvalues along with their eigenvectors.

2. Find the eigenvalues and eigenvectors of the matrix, $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically.
   In this case the exact eigenvalues are $\pm\sqrt{3}, 6$. Compare with the exact answers.

3. Find the eigenvalues and eigenvectors of the matrix, $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically.
   The exact eigenvalues are $2, 4+\sqrt{15}, 4-\sqrt{15}$. Compare your numerical results with the exact values. Is it much fun to compute the exact eigenvectors?

4. Find the eigenvalues and eigenvectors of the matrix, $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically.
   I don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

5. Find the eigenvalues and eigenvectors of the matrix, $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically.
   I don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

6. Consider the matrix, $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{pmatrix}$ and the vector $(1,1,1)^T$. Find the shortest distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

7. Consider the matrix, $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & 5 \end{pmatrix}$ and the vector $(1,1,1)^T$. Find the shortest distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

8. Consider the matrix, $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix}$ and the vector $(1,1,1)^T$. Find the shortest distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

9. Using Gerschgorin's theorem, find upper and lower bounds for the eigenvalues of $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix}$.

10. Generalize the proof of the convergence of the QR algorithm to cover the case where
    $$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0.$$
    It is just a technical generalization which involves a little more fussing in the proof.

11. If you have access to a computer algebra system, try the QR algorithm on some situations where the technical condition about S^{-1} having an LU factorization does not hold. What happens to the diagonal entries in the limit of the A_k? This can be accounted for by using the PLU factorization.

12. Suppose $\lambda_j, \cdots, \lambda_{j+p}$ are complex eigenvalues having nonzero imaginary parts such that each has the same absolute value but they are all distinct. Then let $\mu > 0$ and consider the matrix $A + \mu I$. Verify the corresponding eigenvalues of $A + \mu I$ are $\lambda_j + \mu, \cdots, \lambda_{j+p} + \mu$ and that $|\lambda_j + \mu|, \cdots, |\lambda_{j+p} + \mu|$ are distinct.

13. Tell how to find a matrix whose characteristic polynomial is a given polynomial. This is called a companion matrix.

14. Suppose A is a real symmetric matrix and the technique of reducing to an upper Hessenberg matrix is followed. Show the resulting upper Hessenberg matrix is actually equal to 0 on the top as well as the bottom.

Some Interesting Topics

A.1 Positive Matrices

Earlier theorems about Markov matrices were presented. These were matrices in which all the entries were nonnegative and either the columns or the rows added to 1. It turns out that many of the theorems presented can be generalized to positive matrices. When this is done, the resulting theory is mainly due to Perron and Frobenius. I will give an introduction to this theory here following Karlin and Taylor [13].

Definition A.1.1 *For A a matrix or vector, the notation, $A >> 0$ will mean every entry of A is positive. By $A > 0$ is meant that every entry is nonnegative and at least one is positive. By $A \geq 0$ is meant that every entry is nonnegative. Thus the matrix or vector consisting only of zeros is ≥ 0. An expression like $A >> B$ will mean $A - B >> 0$ with similar modifications for $>$ and $\geq$.*

For the sake of this section only, define the following for $\mathbf{x} = (x_1, \cdots, x_n)^T$, a vector.

$$|\mathbf{x}| \equiv (|x_1|, \cdots, |x_n|)^T.$$

Thus $|\mathbf{x}|$ is the vector which results by replacing each entry of $\mathbf{x}$ with its absolute value[1]*. Also define for $\mathbf{x} \in \mathbb{C}^n$,*

$$||\mathbf{x}||_1 \equiv \sum_k |x_k|.$$

Lemma A.1.2 *Let $A >> 0$ and let $\mathbf{x} > \mathbf{0}$. Then $A\mathbf{x} >> \mathbf{0}$.*

Proof: $(A\mathbf{x})_i = \sum_j A_{ij} x_j > 0$ because all the $A_{ij} > 0$ and at least one $x_j > 0$.

Lemma A.1.3 *Let $A >> 0$. Define*

$$S \equiv \{\lambda : A\mathbf{x} > \lambda\mathbf{x} \text{ for some } \mathbf{x} >> \mathbf{0}\},$$

and let

$$K \equiv \{\mathbf{x} \geq \mathbf{0} \text{ such that } ||\mathbf{x}||_1 = 1\}.$$

Now define

$$S_1 \equiv \{\lambda : A\mathbf{x} \geq \lambda\mathbf{x} \text{ for some } \mathbf{x} \in K\}.$$

Then

$$\sup(S) = \sup(S_1).$$

[1] This notation is just about the most abominable thing imaginable. However, it saves space in the presentation of this theory of positive matrices and avoids the use of new symbols. Please forget about it when you leave this section.

Proof: Let $\lambda \in S$. Then there exists $\mathbf{x} >> \mathbf{0}$ such that $A\mathbf{x} > \lambda\mathbf{x}$. Consider $\mathbf{y} \equiv \mathbf{x}/\left|\left|\mathbf{x}\right|\right|_1$. Then $\left|\left|\mathbf{y}\right|\right|_1 = 1$ and $A\mathbf{y} > \lambda\mathbf{y}$. Therefore, $\lambda \in S_1$ and so $S \subseteq S_1$. Therefore, $\sup(S) \leq \sup(S_1)$.

Now let $\lambda \in S_1$. Then there exists $\mathbf{x} \geq \mathbf{0}$ such that $\left|\left|\mathbf{x}\right|\right|_1 = 1$ so $\mathbf{x} > \mathbf{0}$ and $A\mathbf{x} > \lambda\mathbf{x}$. Letting $\mathbf{y} \equiv A\mathbf{x}$, it follows from Lemma A.1.2 that $A\mathbf{y} >> \lambda\mathbf{y}$ and $\mathbf{y} >> \mathbf{0}$. Thus $\lambda \in S$ and so $S_1 \subseteq S$ which shows that $\sup(S_1) \leq \sup(S)$. This proves the lemma.

This lemma is significant because the set, $\{\mathbf{x} \geq \mathbf{0} \text{ such that } \left|\left|\mathbf{x}\right|\right|_1 = 1\} \equiv K$ is a compact set in $\mathbb{R}^n$. Define

$$\lambda_0 \equiv \sup(S) = \sup(S_1). \tag{1.1}$$

The following theorem is due to Perron.

Theorem A.1.4 *Let $A >> 0$ be an $n \times n$ matrix and let λ_0 be given in 1.1. Then*

1. *$\lambda_0 > 0$ and there exists $\mathbf{x}_0 >> \mathbf{0}$ such that $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$ so λ_0 is an eigenvalue for A.*
2. *If $A\mathbf{x} = \mu\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$, and $\mu \neq \lambda_0$. Then $|\mu| < \lambda_0$.*
3. *The eigenspace for λ_0 has dimension 1.*

Proof: To see $\lambda_0 > 0$, consider the vector, $\mathbf{e} \equiv (1, \cdots, 1)^T$. Then

$$(A\mathbf{e})_i = \sum_j A_{ij} > 0$$

and so λ_0 is at least as large as

$$\min_i \sum_j A_{ij}.$$

Let $\{\lambda_k\}$ be an increasing sequence of numbers from S_1 converging to λ_0. Letting $\mathbf{x}_k$ be the vector from K which occurs in the definition of S_1, these vectors are in a compact set. Therefore, there exists a subsequence, still denoted by $\mathbf{x}_k$ such that $\mathbf{x}_k \to \mathbf{x}_0 \in K$ and $\lambda_k \to \lambda_0$. Then passing to the limit,

$$A\mathbf{x}_0 \geq \lambda_0\mathbf{x}_0, \ \mathbf{x}_0 > \mathbf{0}.$$

If $A\mathbf{x}_0 > \lambda_0\mathbf{x}_0$, then letting $\mathbf{y} \equiv A\mathbf{x}_0$, it follows from Lemma A.1.2 that $A\mathbf{y} >> \lambda_0\mathbf{y}$ and $\mathbf{y} >> \mathbf{0}$. But this contradicts the definition of λ_0 as the supremum of the elements of S because since $A\mathbf{y} >> \lambda_0\mathbf{y}$, it follows $A\mathbf{y} >> (\lambda_0 + \varepsilon)\mathbf{y}$ for ε a small positive number. Therefore, $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$. It remains to verify that $\mathbf{x}_0 >> \mathbf{0}$. But this follows immediately from

$$0 < \sum_j A_{ij}x_{0j} = (A\mathbf{x}_0)_i = \lambda_0 x_{0i}.$$

This proves 1.

Next suppose $A\mathbf{x} = \mu\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$ and $\mu \neq \lambda_0$. Then $|A\mathbf{x}| = |\mu|\,|\mathbf{x}|$. But this implies $A|\mathbf{x}| \geq |\mu|\,|\mathbf{x}|$. (See the above abominable definition of $|\mathbf{x}|$.)

Case 1: $|\mathbf{x}| \neq \mathbf{x}$ and $|\mathbf{x}| \neq -\mathbf{x}$.

In this case, $A|\mathbf{x}| > |A\mathbf{x}| = |\mu|\,|\mathbf{x}|$ and letting $\mathbf{y} = A|\mathbf{x}|$, it follows $\mathbf{y} >> \mathbf{0}$ and $A\mathbf{y} >> |\mu|\,\mathbf{y}$ which shows $A\mathbf{y} >> (|\mu| + \varepsilon)\mathbf{y}$ for sufficiently small positive ε and verifies $|\mu| < \lambda_0$.

Case 2: $|\mathbf{x}| = \mathbf{x}$ or $|\mathbf{x}| = -\mathbf{x}$

In this case, the entries of $\mathbf{x}$ are all real and have the same sign. Therefore, $A|\mathbf{x}| = |A\mathbf{x}| = |\mu|\,|\mathbf{x}|$. Now let $\mathbf{y} \equiv |\mathbf{x}|/\left|\left|\mathbf{x}\right|\right|_1$. Then $A\mathbf{y} = |\mu|\,\mathbf{y}$ and so $|\mu| \in S_1$ showing that

$|\mu| \leq \lambda_0$. But also, the fact the entries of $\mathbf{x}$ all have the same sign shows $\mu = |\mu|$ and so $\mu \in S_1$. Since $\mu \neq \lambda_0$, it must be that $\mu = |\mu| < \lambda_0$. This proves 2.

It remains to verify 3. Suppose then that $A\mathbf{y} = \lambda_0\mathbf{y}$ and for all scalars, $\alpha, \alpha\mathbf{x}_0 \neq \mathbf{y}$. Then

$$A\,\mathrm{Re}\,\mathbf{y} = \lambda_0\,\mathrm{Re}\,\mathbf{y},\ A\,\mathrm{Im}\,\mathbf{y} = \lambda_0\,\mathrm{Im}\,\mathbf{y}.$$

If $\mathrm{Re}\,\mathbf{y} = \alpha_1\mathbf{x}_0$ and $\mathrm{Im}\,\mathbf{y} = \alpha_2\mathbf{x}_0$ for real numbers, α_i,then $\mathbf{y} = (\alpha_1 + i\alpha_2)\,\mathbf{x}_0$ and it is assumed this does not happen. Therefore, either

$$t\,\mathrm{Re}\,\mathbf{y} \neq \mathbf{x}_0 \text{ for all } t \in \mathbb{R}$$

or

$$t\,\mathrm{Im}\,\mathbf{y} \neq \mathbf{x}_0 \text{ for all } t \in \mathbb{R}.$$

Assume the first holds. Then varying $t \in \mathbb{R}$, there exists a value of t such that $\mathbf{x}_0 + t\,\mathrm{Re}\,\mathbf{y} > \mathbf{0}$ but it is not the case that $\mathbf{x}_0 + t\,\mathrm{Re}\,\mathbf{y} >> 0$. Then $A\,(\mathbf{x}_0 + t\,\mathrm{Re}\,\mathbf{y}) >> 0$ by Lemma A.1.2. But this implies $\lambda_0\,(\mathbf{x}_0 + t\,\mathrm{Re}\,\mathbf{y}) >> 0$ which is a contradiction. Hence there exist real numbers, α_1 and α_2 such that $\mathrm{Re}\,\mathbf{y} = \alpha_1\mathbf{x}_0$ and $\mathrm{Im}\,\mathbf{y} = \alpha_2\mathbf{x}_0$ showing that $\mathbf{y} = (\alpha_1 + i\alpha_2)\,\mathbf{x}_0$. This proves 3.

It is possible to obtain a simple corollary to the above theorem.

Corollary A.1.5 *If $A > 0$ and $A^m >> 0$ for some $m \in \mathbb{N}$, then all the conclusions of the above theorem hold.*

Proof: There exists $\mu_0 > 0$ such that $A^m\mathbf{y}_0 = \mu_0\mathbf{y}_0$ for $\mathbf{y}_0 >> 0$ by Theorem A.1.4 and

$$\mu_0 = \sup\{\mu : A^m\mathbf{x} \geq \mu\mathbf{x} \text{ for some } \mathbf{x} \in K\}.$$

Let $\lambda_0^m = \mu_0$. Then

$$(A - \lambda_0 I)\left(A^{m-1} + \lambda_0 A^{m-2} + \cdots + \lambda_0^{m-1} I\right)\mathbf{y}_0 = (A^m - \lambda_0^m I)\,\mathbf{y}_0 = \mathbf{0}$$

and so letting $\mathbf{x}_0 \equiv \left(A^{m-1} + \lambda_0 A^{m-2} + \cdots + \lambda_0^{m-1} I\right)\mathbf{y}_0$, it follows $\mathbf{x}_0 >> 0$ and $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$.

Suppose now that $A\mathbf{x} = \mu\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$ and $\mu \neq \lambda_0$. Suppose $|\mu| \geq \lambda_0$. Multiplying both sides by A, it follows $A^m\mathbf{x} = \mu^m\mathbf{x}$ and $|\mu^m| = |\mu|^m \geq \lambda_0^m = \mu_0$ and so from Theorem A.1.4, since $|\mu^m| \geq \mu_0$, and μ^m is an eigenvalue of A^m, it follows that $\mu^m = \mu_0$. But by Theorem A.1.4 again, this implies $\mathbf{x} = c\mathbf{y}_0$ for some scalar, c and hence $A\mathbf{y}_0 = \mu\mathbf{y}_0$. Since $\mathbf{y}_0 >> \mathbf{0}$, it follows $\mu \geq 0$ and so $\mu = \lambda_0$, a contradiction. Therefore, $|\mu| < \lambda_0$.

Finally, if $A\mathbf{x} = \lambda_0\mathbf{x}$, then $A^m\mathbf{x} = \lambda_0^m\mathbf{x}$ and so $\mathbf{x} = c\mathbf{y}_0$ for some scalar, c. Consequently,

$$\begin{aligned}\left(A^{m-1} + \lambda_0 A^{m-2} + \cdots + \lambda_0^{m-1} I\right)\mathbf{x} &= c\left(A^{m-1} + \lambda_0 A^{m-2} + \cdots + \lambda_0^{m-1} I\right)\mathbf{y}_0 \\ &= c\mathbf{x}_0.\end{aligned}$$

Hence

$$m\lambda_0^{m-1}\mathbf{x} = c\mathbf{x}_0$$

which shows the dimension of the eigenspace for λ_0 is one. This proves the corollary.

The following corollary is an extremely interesting convergence result involving the powers of positive matrices.

Corollary A.1.6 *Let $A > 0$ and $A^m >> 0$ for some $m \in \mathbb{N}$. Then for λ_0 given in 1.1, there exists a rank one matrix, P such that $\lim_{m\to\infty}\left|\left|\left(\frac{A}{\lambda_0}\right)^m - P\right|\right| = 0$.*

Proof: Considering A^T, and the fact that A and A^T have the same eigenvalues, Corollary A.1.5 implies the existence of a vector, $\mathbf{v} >> \mathbf{0}$ such that

$$A^T\mathbf{v} = \lambda_0\mathbf{v}.$$

Also let $\mathbf{x}_0$ denote the vector such that $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$ with $\mathbf{x}_0 >> \mathbf{0}$. First note that $\mathbf{x}_0^T\mathbf{v} > 0$ because both these vectors have all entries positive. Therefore, $\mathbf{v}$ may be scaled such that

$$\mathbf{v}^T\mathbf{x}_0 = \mathbf{x}_0^T\mathbf{v} = 1. \tag{1.2}$$

Define

$$P \equiv \mathbf{x}_0\mathbf{v}^T.$$

Thanks to 1.2,

$$\frac{A}{\lambda_0}P = \mathbf{x}_0\mathbf{v}^T = P,\ P\left(\frac{A}{\lambda_0}\right) = \mathbf{x}_0\mathbf{v}^T\left(\frac{A}{\lambda_0}\right) = \mathbf{x}_0\mathbf{v}^T = P, \tag{1.3}$$

and

$$P^2 = \mathbf{x}_0\mathbf{v}^T\mathbf{x}_0\mathbf{v}^T = \mathbf{v}^T\mathbf{x}_0 = P. \tag{1.4}$$

Therefore,

$$\begin{aligned}\left(\frac{A}{\lambda_0} - P\right)^2 &= \left(\frac{A}{\lambda_0}\right)^2 - 2\left(\frac{A}{\lambda_0}\right)P + P^2 \\ &= \left(\frac{A}{\lambda_0}\right)^2 - P.\end{aligned}$$

Continuing this way, using 1.3 repeatedly, it follows

$$\left(\left(\frac{A}{\lambda_0}\right) - P\right)^m = \left(\frac{A}{\lambda_0}\right)^m - P. \tag{1.5}$$

The eigenvalues of $\left(\frac{A}{\lambda_0}\right) - P$ are of interest because it is powers of this matrix which determine the convergence of $\left(\frac{A}{\lambda_0}\right)^m$ to P. Therefore, let μ be a nonzero eigenvalue of this matrix. Thus

$$\left(\left(\frac{A}{\lambda_0}\right) - P\right)\mathbf{x} = \mu\mathbf{x} \tag{1.6}$$

for $\mathbf{x} \neq \mathbf{0}$, and $\mu \neq 0$. Applying P to both sides and using the second formula of 1.3 yields

$$\mathbf{0} = (P - P)\,\mathbf{x} = \left(P\left(\frac{A}{\lambda_0}\right) - P^2\right)\mathbf{x} = \mu P\mathbf{x}.$$

But since $P\mathbf{x} = \mathbf{0}$, it follows from 1.6 that

$$A\mathbf{x} = \lambda_0\mu\mathbf{x}$$

which implies $\lambda_0\mu$ is an eigenvalue of A. Therefore, by Corollary A.1.5 it follows that either $\lambda_0\mu = \lambda_0$ in which case $\mu = 1$, or $\lambda_0\,|\mu| < \lambda_0$ which implies $|\mu| < 1$. But if $\mu = 1$, then $\mathbf{x}$ is a multiple of $\mathbf{x}_0$ and 1.6 would yield

$$\left(\left(\frac{A}{\lambda_0}\right) - P\right)\mathbf{x}_0 = \mathbf{x}_0$$

which says $\mathbf{x}_0 - \mathbf{x}_0\mathbf{v}^T\mathbf{x}_0 = \mathbf{x}_0$ and so by 1.2, $\mathbf{x}_0 = \mathbf{0}$ contrary to the property that $\mathbf{x}_0 >> \mathbf{0}$. Therefore, $|\mu| < 1$ and so this has shown that the absolute values of all eigenvalues of $\left(\frac{A}{\lambda_0}\right) - P$ are less than 1. By Gelfand's theorem, Theorem 14.3.3, it follows

$$\left|\left|\left(\left(\frac{A}{\lambda_0}\right) - P\right)^m\right|\right|^{1/m} < r < 1$$

whenever m is large enough. Now by 1.5 this yields

$$\left|\left|\left(\frac{A}{\lambda_0}\right)^m - P\right|\right| = \left|\left|\left(\left(\frac{A}{\lambda_0}\right) - P\right)^m\right|\right| \leq r^m$$

whenever m is large enough. It follows

$$\lim_{m\to\infty} \left|\left|\left(\frac{A}{\lambda_0}\right)^m - P\right|\right| = 0$$

as claimed.

What about the case when $A > 0$ but maybe it is not the case that $A >> 0$? As before,

$$K \equiv \{\mathbf{x} \geq \mathbf{0} \text{ such that } ||\mathbf{x}||_1 = 1\}.$$

Now define

$$S_1 \equiv \{\lambda : A\mathbf{x} \geq \lambda\mathbf{x} \text{ for some } \mathbf{x} \in K\}$$

and

$$\lambda_0 \equiv \sup(S_1) \tag{1.7}$$

Theorem A.1.7 *Let $A > 0$ and let λ_0 be defined in 1.7. Then there exists $\mathbf{x}_0 > \mathbf{0}$ such that $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$.*

Proof: Let E consist of the matrix which has a one in every entry. Then from Theorem A.1.4 it follows there exists $\mathbf{x}_\delta >> \mathbf{0}$, $||\mathbf{x}_\delta||_1 = 1$, such that $(A + \delta E)\mathbf{x}_\delta = \lambda_{0\delta}\mathbf{x}_\delta$ where

$$\lambda_{0\delta} \equiv \sup\{\lambda : (A + \delta E)\mathbf{x} \geq \lambda\mathbf{x} \text{ for some } \mathbf{x} \in K\}.$$

Now if $\alpha < \delta$

$$\{\lambda : (A + \alpha E)\mathbf{x} \geq \boldsymbol{\lambda}\mathbf{x} \text{ for some } \mathbf{x} \in K\} \subseteq$$
$$\{\lambda : (A + \delta E)\mathbf{x} \geq \boldsymbol{\lambda}\mathbf{x} \text{ for some } \mathbf{x} \in K\}$$

and so $\lambda_{0\delta} \geq \lambda_{0\alpha}$ because $\lambda_{0\delta}$ is the sup of the second set and $\lambda_{0\alpha}$ is the sup of the first. It follows the limit, $\lambda_1 \equiv \lim_{\delta\to 0+} \lambda_{0\delta}$ exists. Taking a subsequence and using the compactness of K, there exists a subsequence, still denoted by δ such that as $\delta \to 0$, $\mathbf{x}_\delta \to \mathbf{x} \in K$. Therefore,

$$A\mathbf{x} = \lambda_1\mathbf{x}$$

and so, in particular, $A\mathbf{x} \geq \lambda_1\mathbf{x}$ and so $\lambda_1 \leq \lambda_0$. But also, if $\lambda \leq \lambda_0$,

$$\lambda\mathbf{x} \leq A\mathbf{x} < (A + \delta E)\mathbf{x}$$

showing that $\lambda_{0\delta} \geq \lambda$ for all such λ. But then $\lambda_{0\delta} \geq \lambda_0$ also. Hence $\lambda_1 \geq \lambda_0$, showing these two numbers are the same. Hence $A\mathbf{x} = \lambda_0\mathbf{x}$ and this proves the theorem.

If $A^m >> 0$ for some m and $A > 0$, it follows that the dimension of the eigenspace for λ_0 is one and that the absolute value of every other eigenvalue of A is less than λ_0. If it is only assumed that $A > 0$, not necessarily $>> 0$, this is no longer true. However, there is something which is very interesting which can be said. First here is an interesting lemma.

Lemma A.1.8 *Let M be a matrix of the form*

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

or

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is an $r \times r$ matrix and C is an $(n-r) \times (n-r)$ matrix. Then $\det(M) = \det(A)\det(B)$ and $\sigma(M) = \sigma(A) \cup \sigma(C)$.

Proof: To verify the claim about the determinants, note

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & C \end{pmatrix}$$

Therefore,

$$\det \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \det \begin{pmatrix} I & 0 \\ B & C \end{pmatrix}.$$

But it is clear from the method of Laplace expansion that

$$\det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \det A$$

and from the multilinear properties of the determinant and row operations that

$$\det \begin{pmatrix} I & 0 \\ B & C \end{pmatrix} = \det \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} = \det C.$$

The case where M is upper block triangular is similar.

This immediately implies $\sigma(M) = \sigma(A) \cup \sigma(C)$.

Theorem A.1.9 *Let $A > 0$ and let λ_0 be given in 1.7. If λ is an eigenvalue for A such that $|\lambda| = \lambda_0$, then λ/λ_0 is a root of unity. Thus $(\lambda/\lambda_0)^m = 1$ for some $m \in \mathbb{N}$.*

Proof: Applying Theorem A.1.7 to A^T, there exists $\mathbf{v} > \mathbf{0}$ such that $A^T\mathbf{v} = \lambda_0\mathbf{v}$. In the first part of the argument it is assumed $\mathbf{v} >> \mathbf{0}$. Now suppose $A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{0}$ and that $|\lambda| = \lambda_0$. Then

$$A|\mathbf{x}| \geq |\lambda||\mathbf{x}| = \lambda_0|\mathbf{x}|$$

and it follows that if $A|\mathbf{x}| > |\lambda||\mathbf{x}|$, then since $\mathbf{v} >> \mathbf{0}$,

$$\lambda_0(\mathbf{v}, |\mathbf{x}|) < (\mathbf{v}, A|\mathbf{x}|) = \left(A^T\mathbf{v}, |\mathbf{x}|\right) = \lambda_0(\mathbf{v}, |\mathbf{x}|),$$

a contradiction. Therefore,

$$A|\mathbf{x}| = \lambda_0|\mathbf{x}|. \tag{1.8}$$

It follows that

$$\left|\sum_j A_{ij}x_j\right| = \lambda_0|\mathbf{x}_i| = \sum_j A_{ij}|x_j|$$

and so the complex numbers,

$$A_{ij}x_j,\ A_{ik}x_k$$

must have the same argument for every k, j because equality holds in the triangle inequality. Therefore, there exists a complex number, μ_i such that

$$A_{ij}x_j = \mu_i A_{ij} |x_j| \tag{1.9}$$

and so, letting $r \in \mathbb{N}$,

$$A_{ij}x_j\mu_j^r = \mu_i A_{ij} |x_j| \mu_j^r.$$

Summing on j yields

$$\sum_j A_{ij}x_j\mu_j^r = \mu_i \sum_j A_{ij} |x_j| \mu_j^r. \tag{1.10}$$

Also, summing 1.9 on j and using that λ is an eigenvalue for $\mathbf{x}$, it follows from 1.8 that

$$\lambda x_i = \sum_j A_{ij}x_j = \mu_i \sum_j A_{ij} |x_j| = \mu_i \lambda_0 |x_i| . \tag{1.11}$$

From 1.10 and 1.11,

$$\begin{aligned}
\sum_j A_{ij}x_j\mu_j^r &= \mu_i \sum_j A_{ij} |x_j| \mu_j^r \\
&= \mu_i \sum_j A_{ij} \overbrace{\mu_j |x_j|}^{\text{see 1.11}} \mu_j^{r-1} \\
&= \mu_i \sum_j A_{ij} \left(\frac{\lambda}{\lambda_0}\right) x_j\mu_j^{r-1} \\
&= \mu_i \left(\frac{\lambda}{\lambda_0}\right) \sum_j A_{ij}x_j\mu_j^{r-1}
\end{aligned}$$

Now from 1.10 with r replaced by $r-1$, this equals

$$\begin{aligned}
\mu_i^2 \left(\frac{\lambda}{\lambda_0}\right) \sum_j A_{ij} |x_j| \mu_j^{r-1} &= \mu_i^2 \left(\frac{\lambda}{\lambda_0}\right) \sum_j A_{ij}\mu_j |x_j| \mu_j^{r-2} \\
&= \mu_i^2 \left(\frac{\lambda}{\lambda_0}\right)^2 \sum_j A_{ij}x_j\mu_j^{r-2}.
\end{aligned}$$

Continuing this way,

$$\sum_j A_{ij}x_j\mu_j^r = \mu_i^k \left(\frac{\lambda}{\lambda_0}\right)^k \sum_j A_{ij}x_j\mu_j^{r-k}$$

and eventually, this shows

$$\begin{aligned}
\sum_j A_{ij}x_j\mu_j^r &= \mu_i^r \left(\frac{\lambda}{\lambda_0}\right)^r \sum_j A_{ij}x_j \\
&= \left(\frac{\lambda}{\lambda_0}\right)^r \lambda (x_i\mu_i^r)
\end{aligned}$$

and this says $\left(\frac{\lambda}{\lambda_0}\right)^{r+1}$ is an eigenvalue for $\left(\frac{A}{\lambda_0}\right)$ with the eigenvector being $(x_1\mu_1^r, \cdots, x_n\mu_n^r)^T$. Now recall that $r \in \mathbb{N}$ was arbitrary and so this has shown that $\left(\frac{\lambda}{\lambda_0}\right)^2, \left(\frac{\lambda}{\lambda_0}\right)^3, \left(\frac{\lambda}{\lambda_0}\right)^4, \cdots$

are each eigenvalues of $\left(\frac{A}{\lambda_0}\right)$ which has only finitely many and hence this sequence must repeat. Therefore, $\left(\frac{\lambda}{\lambda_0}\right)$ is a root of unity as claimed. This proves the theorem in the case that $\mathbf{v} >> \mathbf{0}$.

Now it is necessary to consider the case where $\mathbf{v} > \mathbf{0}$ but it is not the case that $\mathbf{v} >> \mathbf{0}$. Then in this case, there exists a permutation matrix, P such that

$$P\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix} \equiv \mathbf{v}_1$$

Then

$$\lambda_0 \mathbf{v} = A^T \mathbf{v} = A^T P \mathbf{v}_1.$$

Therefore,

$$\lambda_0 \mathbf{v}_1 = P A^T P \mathbf{v}_1 = G \mathbf{v}_1$$

Now $P^2 = I$ because it is a permuation matrix. Therefore, the matrix, $G \equiv PA^TP$ and A are similar. Consequently, they have the same eigenvalues and it suffices from now on to consider the matrix, G rather than A. Then

$$\lambda_0 \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix}$$

where M_1 is $r \times r$ and M_4 is $(n-r) \times (n-r)$. It follows from block multiplication and the assumption that A and hence G are > 0 that

$$G = \begin{pmatrix} A' & B \\ 0 & C \end{pmatrix}.$$

Now let λ be an eigenvalue of G such that $|\lambda| = \lambda_0$. Then from Lemma A.1.8, either $\lambda \in \sigma(A')$ or $\lambda \in \sigma(C)$. Suppose without loss of generality that $\lambda \in \sigma(A')$. Since $A' > 0$ it has a largest positive eigenvalue, λ_0' which is obtained from 1.7. Thus $\lambda_0' \leq \lambda_0$ but λ being an eigenvalue of A', has its absolute value bounded by λ_0' and so $\lambda_0 = |\lambda| \leq \lambda_0' \leq \lambda_0$ showing that $\lambda_0 \in \sigma(A')$. Now if there exists $\mathbf{v} >> \mathbf{0}$ such that $A'^T \mathbf{v} = \lambda_0 \mathbf{v}$, then the first part of this proof applies to the matrix, A and so (λ/λ_0) is a root of unity. If such a vector, $\mathbf{v}$ does not exist, then let A' play the role of A in the above argument and reduce to the consideration of

$$G' \equiv \begin{pmatrix} A'' & B' \\ 0 & C' \end{pmatrix}$$

where G' is similar to A' and $\lambda, \lambda_0 \in \sigma(A'')$. Stop if $A''^T \mathbf{v} = \lambda_0 \mathbf{v}$ for some $\mathbf{v} >> \mathbf{0}$. Otherwise, decompose A'' similar to the above and add another prime. Continuing this way you must eventually obtain the situation where $\left(A'^{\cdots\prime}\right)^T \mathbf{v} = \lambda_0 \mathbf{v}$ for some $\mathbf{v} >> \mathbf{0}$. Indeed, this happens no later than when $A'^{\cdots\prime}$ is a 1×1 matrix. This proves the theorem.

A.2 Functions Of Matrices

The existence of the Jordan form also makes it possible to define various functions of matrices. Suppose

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \tag{1.12}$$

for all $|\lambda| < R$. There is a formula for $f(A) \equiv \sum_{n=0}^{\infty} a_n A^n$ which makes sense whenever $\rho(A) < R$. Thus you can speak of $\sin(A)$ or e^A for A an $n \times n$ matrix. To begin with, define

$$f_P(\lambda) \equiv \sum_{n=0}^{P} a_n \lambda^n$$

so for $k < P$

$$\begin{aligned} f_P^{(k)}(\lambda) &= \sum_{n=k}^{P} a_n n \cdots (n-k+1) \lambda^{n-k} \\ &= \sum_{n=k}^{P} a_n \binom{n}{k} k! \lambda^{n-k}. \end{aligned} \tag{1.13}$$

Thus

$$\frac{f_P^{(k)}(\lambda)}{k!} = \sum_{n=k}^{P} a_n \binom{n}{k} \lambda^{n-k} \tag{1.14}$$

To begin with consider $f(J_m(\lambda))$ where $J_m(\lambda)$ is an $m \times m$ Jordan block. Thus $J_m(\lambda) = D + N$ where $N^m = 0$ and N commutes with D. Therefore, letting $P > m$

$$\begin{aligned} \sum_{n=0}^{P} a_n J_m(\lambda)^n &= \sum_{n=0}^{P} a_n \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k \\ &= \sum_{k=0}^{P} \sum_{n=k}^{P} a_n \binom{n}{k} D^{n-k} N^k \\ &= \sum_{k=0}^{m-1} N^k \sum_{n=k}^{P} a_n \binom{n}{k} D^{n-k}. \end{aligned} \tag{1.15}$$

From 1.14 this equals

$$\sum_{k=0}^{m-1} N^k \operatorname{diag}\left(\frac{f_P^{(k)}(\lambda)}{k!}, \cdots, \frac{f_P^{(k)}(\lambda)}{k!} \right) \tag{1.16}$$

where for $k = 0, \cdots, m-1$, define $\operatorname{diag}_k(a_1, \cdots, a_{m-k})$ the $m \times m$ matrix which equals zero everywhere except on the k^{th} super diagonal where this diagonal is filled with the numbers, $\{a_1, \cdots, a_{m-k}\}$ from the upper left to the lower right. With no subscript, it is just the diagonal matrices having the indicated entries. Thus in 4×4 matrices, $\operatorname{diag}_2(1,2)$ would be the matrix,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then from 1.16 and 1.13,

$$\sum_{n=0}^{P} a_n J_m(\lambda)^n = \sum_{k=0}^{m-1} \operatorname{diag}_k \left(\frac{f_P^{(k)}(\lambda)}{k!}, \cdots, \frac{f_P^{(k)}(\lambda)}{k!} \right).$$

Therefore, $\sum_{n=0}^{P} a_n J_m(\lambda)^n =$

$$\begin{pmatrix} f_P(\lambda) & \frac{f_P'(\lambda)}{1!} & \frac{f_P^{(2)}(\lambda)}{2!} & \cdots & \frac{f_P^{(m-1)}(\lambda)}{(m-1)!} \\ & f_P(\lambda) & \frac{f_P'(\lambda)}{1!} & \ddots & \vdots \\ & & f_P(\lambda) & \ddots & \frac{f_P^{(2)}(\lambda)}{2!} \\ & & & \ddots & \frac{f_P'(\lambda)}{1!} \\ 0 & & & & f_P(\lambda) \end{pmatrix} \tag{1.17}$$

Now let A be an $n \times n$ matrix with $\rho(A) < R$ where R is given above. Then the Jordan form of A is of the form

$$J = \begin{pmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & J_r \end{pmatrix} \tag{1.18}$$

where $J_k = J_{m_k}(\lambda_k)$ is an $m_k \times m_k$ Jordan block and $A = S^{-1}JS$. Then, letting $P > m_k$ for all k,

$$\sum_{n=0}^{P} a_n A^n = S^{-1} \sum_{n=0}^{P} a_n J^n S,$$

and because of block multiplication of matrices,

$$\sum_{n=0}^{P} a_n J^n = \begin{pmatrix} \sum_{n=0}^{P} a_n J_1^n & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & \sum_{n=0}^{P} a_n J_r^n \end{pmatrix}$$

and from 1.17 $\sum_{n=0}^{P} a_n J_k^n$ converges as $P \to \infty$ to the $m_k \times m_k$ matrix,

$$\begin{pmatrix} f(\lambda_k) & \frac{f'(\lambda_k)}{1!} & \frac{f^{(2)}(\lambda_k)}{2!} & \cdots & \frac{f^{(m-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & f(\lambda_k) & \frac{f'(\lambda_k)}{1!} & \ddots & \vdots \\ 0 & 0 & f(\lambda_k) & \ddots & \frac{f^{(2)}(\lambda_k)}{2!} \\ \vdots & & \ddots & \ddots & \frac{f'(\lambda_k)}{1!} \\ 0 & 0 & \cdots & 0 & f(\lambda_k) \end{pmatrix} \tag{1.19}$$

There is no convergence problem because $|\lambda| < R$ for all $\lambda \in \sigma(A)$. This has proved the following theorem.

Theorem A.2.1 *Let f be given by 1.12 and suppose $\rho(A) < R$ where R is the radius of convergence of the power series in 1.12. Then the series,*

$$\sum_{k=0}^{\infty} a_n A^n \tag{1.20}$$

converges in the space $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ with respect to any of the norms on this space and furthermore,

$$\sum_{k=0}^{\infty} a_n A^n = S^{-1} \begin{pmatrix} \sum_{n=0}^{\infty} a_n J_1^n & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & \sum_{n=0}^{\infty} a_n J_r^n \end{pmatrix} S$$

where $\sum_{n=0}^{\infty} a_n J_k^n$ is an $m_k \times m_k$ matrix of the form given in 1.19 where $A = S^{-1}JS$ and the Jordan form of A, J is given by 1.18. Therefore, you can define $f(A)$ by the series in 1.20.

Here is a simple example.

Example A.2.2 *Find* $\sin(A)$ *where* $A = \begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & 2 & 1 & 4 \end{pmatrix}$.

In this case, the Jordan canonical form of the matrix is not too hard to find.

$$\begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 & -1 \\ 1 & -4 & -2 & -1 \\ 0 & 0 & -2 & 1 \\ -1 & 4 & 4 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{8} & 0 & -\frac{1}{8} \\ 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then from the above theorem $\sin(J)$ is given by

$$\sin \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \sin 4 & 0 & 0 & 0 \\ 0 & \sin 2 & \cos 2 & \frac{-\sin 2}{2} \\ 0 & 0 & \sin 2 & \cos 2 \\ 0 & 0 & 0 & \sin 2 \end{pmatrix}.$$

Therefore, $\sin(A) =$

$$\begin{pmatrix} 2 & 0 & -2 & -1 \\ 1 & -4 & -2 & -1 \\ 0 & 0 & -2 & 1 \\ -1 & 4 & 4 & 2 \end{pmatrix} \begin{pmatrix} \sin 4 & 0 & 0 & 0 \\ 0 & \sin 2 & \cos 2 & \frac{-\sin 2}{2} \\ 0 & 0 & \sin 2 & \cos 2 \\ 0 & 0 & 0 & \sin 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{8} & 0 & -\frac{1}{8} \\ 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \sin 4 & \sin 4 - \sin 2 - \cos 2 & -\cos 2 & \sin 4 - \sin 2 - \cos 2 \\ \frac{1}{2}\sin 4 - \frac{1}{2}\sin 2 & \frac{1}{2}\sin 4 + \frac{3}{2}\sin 2 - 2\cos 2 & \sin 2 & \frac{1}{2}\sin 4 + \frac{1}{2}\sin 2 - 2\cos 2 \\ 0 & -\cos 2 & \sin 2 - \cos 2 & -\cos 2 \\ -\frac{1}{2}\sin 4 + \frac{1}{2}\sin 2 & -\frac{1}{2}\sin 4 - \frac{1}{2}\sin 2 + 3\cos 2 & \cos 2 - \sin 2 & -\frac{1}{2}\sin 4 + \frac{1}{2}\sin 2 + 3\cos 2 \end{pmatrix}.$$

Perhaps this isn't the first thing you would think of. Of course the ability to get this nice closed form description of $\sin(A)$ was dependent on being able to find the Jordan form along with a similarity transformation which will yield the Jordan form.

The following corollary is known as the spectral mapping theorem.

Corollary A.2.3 *Let A be an $n \times n$ matrix and let $\rho(A) < R$ where for $|\lambda| < R$,*

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n.$$

Then $f(A)$ is also an $n \times n$ matrix and furthermore, $\sigma(f(A)) = f(\sigma(A))$. Thus the eigenvalues of $f(A)$ are exactly the numbers $f(\lambda)$ where λ is an eigenvalue of A. Furthermore, the algebraic multiplicity of $f(\lambda)$ coincides with the algebraic multiplicity of λ.

All of these things can be generalized to linear transformations defined on infinite dimensional spaces and when this is done the main tool is the Dunford integral along with the methods of complex analysis. It is good to see it done for finite dimensional situations first because it gives an idea of what is possible. Actually, some of the most interesting functions in applications do not come in the above form as a power series expanded about 0. One example of this situation has already been encountered in the proof of the right polar decomposition with the square root of an Hermitian transformation which had all nonnegative eigenvalues. Another example is that of taking the positive part of an Hermitian matrix. This is important in some physical models where something may depend on the positive part of the strain which is a symmetric real matrix. Obviously there is no way to consider this as a power series expanded about 0 because the function $f(r) = r^+$ is not even differentiable at 0. Therefore, a totally different approach must be considered. First the notion of a positive part is defined.

Definition A.2.4 *Let A be an Hermitian matrix. Thus it suffices to consider A as an element of $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ according to the usual notion of matrix multiplication. Then there exists an orthonormal basis of eigenvectors, $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ such that*

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \otimes \mathbf{u}_j,$$

for λ_j the eigenvalues of A, all real. Define

$$A^+ \equiv \sum_{j=1}^{n} \lambda_j^+ \mathbf{u}_j \otimes \mathbf{u}_j$$

where $\lambda^+ \equiv \frac{|\lambda| + \lambda}{2}$.

This gives us a nice definition of what is meant but it turns out to be very important in the applications to determine how this function depends on the choice of symmetric matrix, A. The following addresses this question.

Theorem A.2.5 *If A, B be Hermitian matrices, then for $|\cdot|$ the Frobenius norm,*

$$\left|A^+ - B^+\right| \leq |A - B|.$$

Proof: Let $A = \sum_i \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$ and let $B = \sum_j \mu_j \mathbf{w}_j \otimes \mathbf{w}_j$ where $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_j\}$ are orthonormal bases of eigenvectors.

$$\left|A^+ - B^+\right|^2 = \text{trace}\left(\sum_i \lambda_i^+ \mathbf{v}_i \otimes \mathbf{v}_i - \sum_j \mu_j^+ \mathbf{w}_j \otimes \mathbf{w}_j\right)^2 =$$

$$\text{trace}\left[\sum_i \left(\lambda_i^+\right)^2 \mathbf{v}_i \otimes \mathbf{v}_i + \sum_j \left(\mu_j^+\right)^2 \mathbf{w}_j \otimes \mathbf{w}_j\right.$$

$$\left. - \sum_{i,j} \lambda_i^+ \mu_j^+ \left(\mathbf{w}_j, \mathbf{v}_i\right) \mathbf{v}_i \otimes \mathbf{w}_j - \sum_{i,j} \lambda_i^+ \mu_j^+ \left(\mathbf{v}_i, \mathbf{w}_j\right) \mathbf{w}_j \otimes \mathbf{v}_i\right]$$

Since the trace of $\mathbf{v}_i \otimes \mathbf{w}_j$ is $\left(\mathbf{v}_i, \mathbf{w}_j\right)$, a fact which follows from $\left(\mathbf{v}_i, \mathbf{w}_j\right)$ being the only possibly nonzero eigenvalue,

$$= \sum_i \left(\lambda_i^+\right)^2 + \sum_j \left(\mu_j^+\right)^2 - 2 \sum_{i,j} \lambda_i^+ \mu_j^+ \left|\left(\mathbf{v}_i, \mathbf{w}_j\right)\right|^2. \tag{1.21}$$

Since these are orthonormal bases,

$$\sum_i |(\mathbf{v}_i, \mathbf{w}_j)|^2 = 1 = \sum_j |(\mathbf{v}_i, \mathbf{w}_j)|^2$$

and so 1.21 equals

$$= \sum_i \sum_j \left(\left(\lambda_i^+\right)^2 + \left(\mu_j^+\right)^2 - 2\lambda_i^+ \mu_j^+ \right) |(\mathbf{v}_i, \mathbf{w}_j)|^2 .$$

Similarly,

$$|A - B|^2 = \sum_i \sum_j \left((\lambda_i)^2 + \left(\mu_j\right)^2 - 2\lambda_i \mu_j \right) |(\mathbf{v}_i, \mathbf{w}_j)|^2 .$$

Now it is easy to check that $(\lambda_i)^2 + \left(\mu_j\right)^2 - 2\lambda_i\mu_j \geq \left(\lambda_i^+\right)^2 + \left(\mu_j^+\right)^2 - 2\lambda_i^+\mu_j^+$ and so this proves the theorem.

Applications To Differential Equations

B.1 Theory Of Ordinary Differntial Equations

Here I will present fundamental existence and uniqueness theorems for initial value problems for the differential equation,

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Suppose that $\mathbf{f} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following two conditions.

$$|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{x}_1)| \leq K |\mathbf{x} - \mathbf{x}_1|, \tag{2.1}$$

$$\mathbf{f} \text{ is continuous.} \tag{2.2}$$

The first of these conditions is known as a Lipschitz condition.

Lemma B.1.1 *Suppose $\mathbf{x} : [a,b] \to \mathbb{R}^n$ is a continuous function and $c \in [a,b]$. Then $\mathbf{x}$ is a solution to the initial value problem,*

$$\mathbf{x}' = \mathbf{f}(t,\mathbf{x}), \ \mathbf{x}(c) = \mathbf{x}_0 \tag{2.3}$$

if and only if $\mathbf{x}$ is a solution to the integral equation,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}(s))\, ds. \tag{2.4}$$

Proof: If $\mathbf{x}$ solves 2.4, then since $\mathbf{f}$ is continuous, we may apply the fundamental theorem of calculus to differentiate both sides and obtain $\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t))$. Also, letting $t = c$ on both sides, gives $\mathbf{x}(c) = \mathbf{x}_0$. Conversely, if $\mathbf{x}$ is a solution of the initial value problem, we may integrate both sides from c to t to see that $\mathbf{x}$ solves 2.4. This proves the lemma.

Theorem B.1.2 *Let $\mathbf{f}$ satisfy 2.1 and 2.2. Then there exists a unique solution to the initial value problem, 2.3 on the interval $[a,b]$.*

Proof: Let $||\mathbf{x}||_\lambda \equiv \sup\left\{e^{\lambda t} |\mathbf{x}(t)| : t \in [a,b]\right\}$. Then this norm is equivalent to the usual norm on $BC([a,b], \mathbb{F}^n)$ described in Example 14.6.2. This means that for $||\cdot||$ the norm given there, there exist constants δ and Δ such that

$$||\mathbf{x}||_\lambda \delta \leq ||\mathbf{x}|| \leq \Delta ||\mathbf{x}||$$

for all $\mathbf{x} \in BC([a,b],\mathbb{F}^n)$. In fact, you can take $\delta \equiv e^{\lambda a}$ and $\Delta \equiv e^{\lambda b}$ in case $\lambda > 0$ with the two reversed in case $\lambda < 0$. Thus $BC([a,b],\mathbb{F}^n)$ is a Banach space with this norm, $||\cdot||_\lambda$. Then let $F : BC([a,b],\mathbb{F}^n) \to BC([a,b],\mathbb{F}^n)$ be defined by

$$F\mathbf{x}(t) \equiv \mathbf{x}_0 + \int_c^t \mathbf{f}(s,\mathbf{x}(s))\,ds.$$

Let $\lambda < 0$. It follows

$$\begin{aligned} e^{\lambda t}|F\mathbf{x}(t) - F\mathbf{y}(t)| &\leq \left| e^{\lambda t}\int_c^t |\mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{y}(s))|\,ds \right| \\ &\leq \left| \int_c^t Ke^{\lambda(t-s)}|\mathbf{x}(s) - \mathbf{y}(s)|\,e^{\lambda s}ds \right| \\ &\leq ||\mathbf{x}-\mathbf{y}||_\lambda \int_a^t Ke^{\lambda(t-s)}ds \\ &\leq ||\mathbf{x}-\mathbf{y}||_\lambda \frac{K}{|\lambda|} \end{aligned}$$

and therefore,

$$||F\mathbf{x} - F\mathbf{y}||_\lambda \leq ||\mathbf{x}-\mathbf{y}||\frac{K}{|\lambda|}.$$

If $|\lambda|$ is chosen larger than K, this implies F is a contraction mapping on $BC([a,b],\mathbb{F}^n)$. Therefore, there exists a unique fixed point. With Lemma B.1.1 this proves the theorem.

B.2 Linear Systems

As an example of the above theorem, consider for $t \in [a,b]$ the system

$$\mathbf{x}' = A(t)\mathbf{x}(t) + \mathbf{g}(t),\ \mathbf{x}(c) = \mathbf{x}_0 \tag{2.5}$$

where $A(t)$ is an $n \times n$ matrix whose entries are continuous functions of $t, (a_{ij}(t))$ and $\mathbf{g}(t)$ is a vector whose components are continuous functions of t satisfies the conditions of Theorem B.1.2 with $\mathbf{f}(t,\mathbf{x}) = A(t)\mathbf{x} + \mathbf{g}(t)$. To see this, let $\mathbf{x} = (x_1,\cdots,x_n)^T$ and $\mathbf{x}_1 = (x_{11},\cdots x_{1n})^T$. Then letting $M = \max\{|a_{ij}(t)| : t \in [a,b], i,j \leq n\}$,

$$|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{x}_1)| = |A(t)(\mathbf{x}-\mathbf{x}_1)|$$

$$\begin{aligned} &= \left| \left(\sum_{i=1}^n \left| \sum_{j=1}^n a_{ij}(t)(x_j - x_{1j}) \right|^2 \right)^{1/2} \right| \\ &\leq M \left| \left(\sum_{i=1}^n \left(\sum_{j=1}^n |x_j - x_{1j}| \right)^2 \right)^{1/2} \right| \\ &\leq M \left| \left(\sum_{i=1}^n n \sum_{j=1}^n |x_j - x_{1j}|^2 \right)^{1/2} \right| \\ &= Mn \left(\sum_{j=1}^n |x_j - x_{1j}|^2 \right)^{1/2} = Mn|\mathbf{x}-\mathbf{x}_1|. \end{aligned}$$

Therefore, let $K = Mn$. This proves

Theorem B.2.1 *Let $A(t)$ be a continuous $n \times n$ matrix and let $\mathbf{g}(t)$ be a continuous vector for $t \in [a,b]$ and let $c \in [a,b]$ and $\mathbf{x}_0 \in \mathbb{F}^n$. Then there exists a unique solution to 2.5 valid for $t \in [a,b]$.*

This includes more examples of linear equations than are typically encountered in an entire differential equations course.

B.3 Local Solutions

Lemma B.3.1 *Let $D(\mathbf{x}_0, r) \equiv \{\mathbf{x} \in \mathbb{F}^n : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and suppose U is an open set containing $D(\mathbf{x}_0, r)$ such that $\mathbf{f} : U \to \mathbb{F}^n$ is $C^1(U)$. (Recall this means all partial derivatives of $\mathbf{f}$ exist and are continuous.) Then for $K = Mn$, where M denotes the maximum of $\left|\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{z})\right|$ for $\mathbf{z} \in D(\mathbf{x}_0, r)$, it follows that for all $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$,*

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|.$$

Proof: Let $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$ and consider the line segment joining these two points, $\mathbf{x} + t(\mathbf{y} - \mathbf{x})$ for $t \in [0,1]$. Letting $\mathbf{h}(t) = \mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$ for $t \in [0,1]$, then

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \mathbf{h}(1) - \mathbf{h}(0) = \int_0^1 \mathbf{h}'(t)\, dt.$$

Also, by the chain rule,

$$\mathbf{h}'(t) = \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i).$$

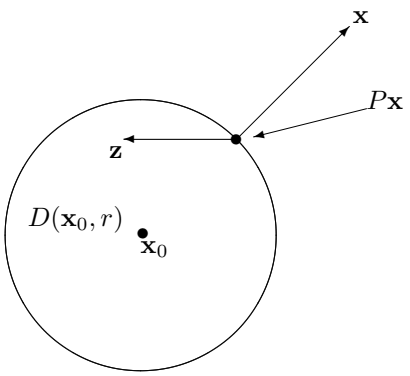
Therefore,

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| =$$

$$\begin{aligned} & \left| \int_0^1 \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i)\, dt \right| \\ \leq & \int_0^1 \sum_{i=1}^{n} \left| \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \right| |y_i - x_i|\, dt \\ \leq & M \sum_{i=1}^{n} |y_i - x_i| \leq Mn |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

This proves the lemma.

Now consider the map, P which maps all of $\mathbb{R}^n$ to $D(\mathbf{x}_0, r)$ given as follows. For $\mathbf{x} \in D(\mathbf{x}_0, r)$, $P\mathbf{x} = \mathbf{x}$. For $\mathbf{x} \notin D(\mathbf{x}_0, r)$, $P\mathbf{x}$ will be the closest point in $D(\mathbf{x}_0, r)$ to $\mathbf{x}$. Such a closest point exists because $D(\mathbf{x}_0, r)$ is a closed and bounded set. Taking $f(\mathbf{y}) \equiv |\mathbf{y} - \mathbf{x}|$, it follows f is a continuous function defined on $D(\mathbf{x}_0, r)$ which must achieve its minimum value by the extreme value theorem from calculus.



Lemma B.3.2 *For any pair of points,* $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, $|P\mathbf{x} - P\mathbf{y}| \leq |\mathbf{x} - \mathbf{y}|$.

Proof: The above picture suggests the geometry of what is going on. Letting $\mathbf{z} \in D(\mathbf{x}_0, r)$, it follows that for all $t \in [0,1]$,

$$|\mathbf{x} - P\mathbf{x}|^2 \leq |\mathbf{x} - (P\mathbf{x} + t(\mathbf{z} - P\mathbf{x}))|^2$$

$$= |\mathbf{x} - P\mathbf{x}|^2 + 2t \operatorname{Re}((\mathbf{x} - P\mathbf{x}) \cdot (P\mathbf{x} - \mathbf{z})) + t^2 |\mathbf{z} - P\mathbf{x}|^2$$

Hence

$$2t \operatorname{Re}((\mathbf{x} - P\mathbf{x}) \cdot (P\mathbf{x} - \mathbf{z})) + t^2 |\mathbf{z} - P\mathbf{x}|^2 \geq 0$$

and this can only happen if

$$\operatorname{Re}((\mathbf{x} - P\mathbf{x}) \cdot (P\mathbf{x} - \mathbf{z})) \geq 0.$$

Therefore,

$$\begin{aligned} \operatorname{Re}((\mathbf{x} - P\mathbf{x}) \cdot (P\mathbf{x} - P\mathbf{y})) &\geq 0 \\ \operatorname{Re}((\mathbf{y} - P\mathbf{y}) \cdot (P\mathbf{y} - P\mathbf{x})) &\geq 0 \end{aligned}$$

and so

$$\operatorname{Re}(\mathbf{x} - P\mathbf{x} - (\mathbf{y} - P\mathbf{y})) \cdot (P\mathbf{x} - P\mathbf{y}) \geq 0$$

which implies

$$\operatorname{Re}(\mathbf{x} - \mathbf{y}) \cdot (P\mathbf{x} - P\mathbf{y}) \geq |P\mathbf{x} - P\mathbf{y}|^2$$

Then using the Cauchy Schwarz inequality it follows

$$|\mathbf{x} - \mathbf{y}| \geq |P\mathbf{x} - P\mathbf{y}|.$$

This proves the lemma.

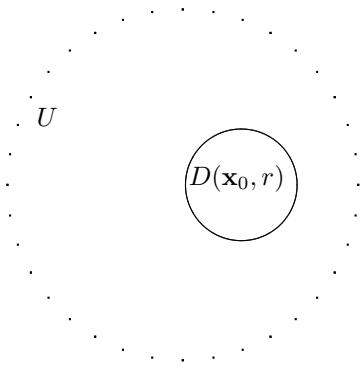
With this here is the local existence and uniqueness theorem.

Theorem B.3.3 *Let* $[a,b]$ *be a closed interval and let* U *be an open subset of* $\mathbb{F}^n$. *Let* $\mathbf{f} : [a,b] \times U \to \mathbb{F}^n$ *be continuous and suppose that for each* $t \in [a,b]$, *the map* $\mathbf{x} \to \frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x})$ *is continuous. Also let* $\mathbf{x}_0 \in U$ *and* $c \in [a,b]$. *Then there exists an interval,* $I \subseteq [a,b]$ *such that* $c \in I$ *and there exists a unique solution to the initial value problem,*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),\ \mathbf{x}(c) = \mathbf{x}_0 \tag{2.6}$$

valid for $t \in I$.

Proof: Consider the following picture.



The large dotted circle represents U and the little solid circle represents $D(\mathbf{x}_0, r)$ as indicated. Here r is so small that $D(\mathbf{x}_0, r)$ is contained in U as shown. Now let P denote the projection map defined above. Consider the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, P\mathbf{x}),\ \mathbf{x}(c) = \mathbf{x}_0. \tag{2.7}$$

From Lemma B.3.1 and the continuity of $\mathbf{x} \to \frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x})$, there exists a constant, K such that if $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$, then $|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$ for all $t \in [a, b]$. Therefore, by Lemma B.3.2

$$|\mathbf{f}(t, P\mathbf{x}) - \mathbf{f}(t, P\mathbf{y})| \leq K |P\mathbf{x} - P\mathbf{y}| \leq K |\mathbf{x} - \mathbf{y}|.$$

It follows from Theorem B.1.2 that 2.7 has a unique solution valid for $t \in [a, b]$. Since $\mathbf{x}$ is continuous, it follows that there exists an interval, I containing c such that for $t \in I$, $\mathbf{x}(t) \in D(\mathbf{x}_0, r)$. Therefore, for these values of t, $\mathbf{f}(t, P\mathbf{x}) = \mathbf{f}(t, \mathbf{x})$ and so there is a unique solution to 2.6 on I. This proves the theorem.

Now suppose $\mathbf{f}$ has the property that for every $R > 0$ there exists a constant, K_R such that for all $\mathbf{x}, \mathbf{x}_1 \in \overline{B(0, R)}$,

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x}_1)| \leq K_R |\mathbf{x} - \mathbf{x}_1|. \tag{2.8}$$

Corollary B.3.4 *Let $\mathbf{f}$ satisfy 2.8 and suppose also that $(t, \mathbf{x}) \to \mathbf{f}(t, \mathbf{x})$ is continuous. Suppose now that $\mathbf{x}_0$ is given and there exists an estimate of the form $|\mathbf{x}(t)| < R$ for all $t \in [0, T)$ where $T \leq \infty$ on the local solution to*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0. \tag{2.9}$$

Then there exists a unique solution to the initial value problem, 2.9 valid on $[0, T)$.

Proof: Replace $\mathbf{f}(t, \mathbf{x})$ with $\mathbf{f}(t, P\mathbf{x})$ where P is the projection onto $\overline{B(0, R)}$. Then by Theorem B.1.2 there exists a unique solution to the system

$$\mathbf{x}' = \mathbf{f}(t, P\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0$$

valid on $[0, T_1]$ for every $T_1 < T$. Therefore, the above system has a unique solution on $[0, T)$ and from the estimate, $P\mathbf{x} = \mathbf{x}$. This proves the corollary.

B.4 First Order Linear Systems

Here is a discussion of linear systems of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$$

where A is a constant $n \times n$ matrix and $\mathbf{f}$ is a vector valued function having all entries continuous. Of course the existence theory is a very special case of the general considerations above but I will give a self contained presentation based on elementary first order scalar differential equations and linear algebra.

Definition B.4.1 *Suppose $t \to M(t)$ is a matrix valued function of t. Thus $M(t) = (m_{ij}(t))$. Then define*

$$M'(t) \equiv \left(m'_{ij}(t)\right).$$

In words, the derivative of $M(t)$ is the matrix whose entries consist of the derivatives of the entries of $M(t)$. Integrals of matrices are defined the same way. Thus

$$\int_a^b M(t)\, di \equiv \left(\int_a^b m_{ij}(t)\, dt\right).$$

In words, the integral of $M(t)$ is the matrix obtained by replacing each entry of $M(t)$ by the integral of that entry.

With this definition, it is easy to prove the following theorem.

Theorem B.4.2 *Suppose $M(t)$ and $N(t)$ are matrices for which $M(t)N(t)$ makes sense. Then if $M'(t)$ and $N'(t)$ both exist, it follows that*

$$(M(t)N(t))' = M'(t)N(t) + M(t)N'(t).$$

Proof:

$$\begin{aligned}
\left((M(t)N(t))'\right)_{ij} &\equiv \left((M(t)N(t))_{ij}\right)' \\
&= \left(\sum_k M(t)_{ik} N(t)_{kj}\right)' \\
&= \sum_k (M(t)_{ik})' N(t)_{kj} + M(t)_{ik}\left(N(t)_{kj}\right)' \\
&\equiv \sum_k \left(M(t)'\right)_{ik} N(t)_{kj} + M(t)_{ik}\left(N(t)'\right)_{kj} \\
&\equiv \left(M'(t)N(t) + M(t)N'(t)\right)_{ij}
\end{aligned}$$

and this proves the theorem.

In the study of differential equations, one of the most important theorems is Gronwall's inequality which is next.

Theorem B.4.3 *Suppose $u(t) \geq 0$ and for all $t \in [0,T]$,*

$$u(t) \leq u_0 + \int_0^t Ku(s)\, ds. \tag{2.10}$$

where K is some constant. Then

$$u(t) \leq u_0 e^{Kt}. \tag{2.11}$$

Proof: Let $w(t) = \int_0^t u(s)\, ds$. Then using the fundamental theorem of calculus, 2.10 $w(t)$ satisfies the following.

$$u(t) - Kw(t) = w'(t) - Kw(t) \leq u_0,\ w(0) = 0. \tag{2.12}$$

Multiply both sides of this inequality by e^{-Kt} and using the product rule and the chain rule,

$$e^{-Kt}\left(w'(t)-Kw(t)\right)=\frac{d}{dt}\left(e^{-Kt}w(t)\right)\leq u_0e^{-Kt}.$$

Integrating this from 0 to t,

$$e^{-Kt}w(t)\leq u_0\int_0^t e^{-Ks}ds=u_0\left(-\frac{e^{-tK}-1}{K}\right).$$

Now multiply through by e^{Kt} to obtain

$$w(t)\leq u_0\left(-\frac{e^{-tK}-1}{K}\right)e^{Kt}=-\frac{u_0}{K}+\frac{u_0}{K}e^{tK}.$$

Therefore, 2.12 implies

$$u(t)\leq u_0+K\left(-\frac{u_0}{K}+\frac{u_0}{K}e^{tK}\right)=u_0e^{Kt}.$$

This proves the theorem.

With Gronwall's inequality, here is a theorem on uniqueness of solutions to the initial value problem,

$$\mathbf{x}'=A\mathbf{x}+\mathbf{f}(t),\ \mathbf{x}(a)=\mathbf{x}_a, \tag{2.13}$$

in which A is an $n\times n$ matrix and $\mathbf{f}$ is a continuous function having values in $\mathbb{C}^n$.

Theorem B.4.4 *Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy 2.13. Then* $\mathbf{x}(t)=\mathbf{y}(t)$ *for all* t.

Proof: Let $\mathbf{z}(t)=\mathbf{x}(t+a)-\mathbf{y}(t+a)$. Then for $t\geq 0$,

$$\mathbf{z}'=A\mathbf{z},\ \mathbf{z}(0)=\mathbf{0}. \tag{2.14}$$

Note that for $K=\max\{|a_{ij}|\}$, where $A=(a_{ij})$,

$$|(A\mathbf{z},\mathbf{z})|=\left|\sum_{ij}a_{ij}z_j\overline{z_i}\right|\leq K\sum_{ij}|z_i|\,|z_j|$$

$$\leq K\sum_{ij}\left(\frac{|z_i|^2}{2}+\frac{|z_j|^2}{2}\right)=nK|\mathbf{z}|^2.$$

(For x and y real numbers, $xy\leq\frac{x^2}{2}+\frac{y^2}{2}$ because this is equivalent to saying $(x-y)^2\geq 0$.) Similarly,

$$|(\mathbf{z},A\mathbf{z})|\leq nK|\mathbf{z}|^2$$

Thus,

$$|(\mathbf{z},A\mathbf{z})|,|(A\mathbf{z},\mathbf{z})|\leq nK|\mathbf{z}|^2. \tag{2.15}$$

Now multiplying 2.14 by $\mathbf{z}$ and observing that

$$\frac{d}{dt}\left(|\mathbf{z}|^2\right)=(\mathbf{z}',\mathbf{z})+(\mathbf{z},\mathbf{z}')=(A\mathbf{z},\mathbf{z})+(\mathbf{z},A\mathbf{z}),$$

it follows from 2.15 and the observation that $\mathbf{z}(0)=0$,

$$|\mathbf{z}(t)|^2\leq\int_0^t 2nK|\mathbf{z}(s)|^2\,ds$$

and so by Gronwall's inequality, $|\mathbf{z}(t)|^2 = 0$ for all $t \geq 0$. Thus,

$$\mathbf{x}(t) = \mathbf{y}(t)$$

for all $t \geq a$.

Now let $\mathbf{w}(t) = \mathbf{x}(a-t) - \mathbf{y}(a-t)$ for $t \geq 0$. Then $\mathbf{w}'(t) = (-A)\mathbf{w}(t)$ and you can repeat the argument which was just given to conclude that $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \leq a$. This proves the theorem.

Definition B.4.5 *Let A be an $n \times n$ matrix. We say $\Phi(t)$ is a fundamental matrix for A if*

$$\Phi'(t) = A\Phi(t),\ \Phi(0) = I, \tag{2.16}$$

and $\Phi(t)^{-1}$ exists for all $t \in \mathbb{R}$.

Why should anyone care about a fundamental matrix? The reason is that such a matrix valued function makes possible a convenient description of the solution of the initial value problem,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(0) = \mathbf{x}_0, \tag{2.17}$$

on the interval, $[0,T]$. First consider the special case where $n = 1$. This is the first order linear differential equation,

$$r' = \lambda r + g,\ r(0) = r_0, \tag{2.18}$$

where g is a continuous scalar valued function. First consider the case where $g = 0$.

Lemma B.4.6 *There exists a unique solution to the initial value problem,*

$$r' = \lambda r,\ r(0) = 1, \tag{2.19}$$

and the solution for $\lambda = a + ib$ is given by

$$r(t) = e^{at}(\cos bt + i \sin bt). \tag{2.20}$$

This solution to the initial value problem is denoted as $e^{\lambda t}$. (If λ is real, $e^{\lambda t}$ as defined here reduces to the usual exponential function so there is no contradiction between this and earlier notation seen in Calculus.)

Proof: From the uniqueness theorem presented above, Theorem B.4.4, applied to the case where $n = 1$, there can be no more than one solution to the initial value problem, 2.19. Therefore, it only remains to verify 2.20 is a solution to 2.19. However, this is an easy calculus exercise. This proves the Lemma.

Note the differential equation in 2.19 says

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}. \tag{2.21}$$

With this lemma, it becomes possible to easily solve the case in which $g \neq 0$.

Theorem B.4.7 *There exists a unique solution to 2.18 and this solution is given by the formula,*

$$r(t) = e^{\lambda t} r_0 + e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds. \tag{2.22}$$

Proof: By the uniqueness theorem, Theorem B.4.4, there is no more than one solution. It only remains to verify that 2.22 is a solution. But $r(0) = e^{\lambda 0} r_0 + \int_0^0 e^{-\lambda s} g(s)\, ds = r_0$ and so the initial condition is satisfied. Next differentiate this expression to verify the differential equation is also satisfied. Using 2.21, the product rule and the fundamental theorem of calculus,

$$\begin{aligned} r'(t) &= \lambda e^{\lambda t} r_0 + \lambda e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds + e^{\lambda t} e^{-\lambda t} g(t) \\ &= \lambda r(t) + g(t). \end{aligned}$$

This proves the Theorem.

Now consider the question of finding a fundamental matrix for A. When this is done, it will be easy to give a formula for the general solution to 2.17 known as the variation of constants formula, arguably the most important result in differential equations.

The next theorem gives a formula for the fundamental matrix 2.16. It is known as Putzer's method [1],[15].

Theorem B.4.8 *Let A be an $n \times n$ matrix whose eigenvalues are $\{\lambda_1, \cdots, \lambda_n\}$. Define*

$$P_k(A) \equiv \prod_{m=1}^{k} (A - \lambda_m I),\ P_0(A) \equiv I,$$

and let the scalar valued functions, $r_k(t)$ be defined as the solutions to the following initial value problem

$$\begin{pmatrix} r_0'(t) \\ r_1'(t) \\ r_2'(t) \\ \vdots \\ r_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_1 r_1(t) + r_0(t) \\ \lambda_2 r_2(t) + r_1(t) \\ \vdots \\ \lambda_n r_n(t) + r_{n-1}(t) \end{pmatrix}, \quad \begin{pmatrix} r_0(0) \\ r_1(0) \\ r_2(0) \\ \vdots \\ r_n(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note the system amounts to a list of single first order linear differential equations. Now define

$$\Phi(t) \equiv \sum_{k=0}^{n-1} r_{k+1}(t)\, P_k(A).$$

Then

$$\Phi'(t) = A\Phi(t),\ \Phi(0) = I. \tag{2.23}$$

Furthermore, if $\Phi(t)$ is a solution to 2.23 for all t, then it follows $\Phi(t)^{-1}$ exists for all t and $\Phi(t)$ is the unique fundamental matrix for A.

Proof: The first part of this follows from a computation. First note that by the Cayley Hamilton theorem, $P_n(A) = 0$. Now for the computation:

$$\Phi'(t) = \sum_{k=0}^{n-1} r_{k+1}'(t)\, P_k(A) = \sum_{k=0}^{n-1} (\lambda_{k+1} r_{k+1}(t) + r_k(t))\, P_k(A) =$$

$$\sum_{k=0}^{n-1} \lambda_{k+1} r_{k+1}(t)\, P_k(A) + \sum_{k=0}^{n-1} r_k(t)\, P_k(A) = \sum_{k=0}^{n-1} (\lambda_{k+1} I - A)\, r_{k+1}(t)\, P_k(A) +$$

$$\sum_{k=0}^{n-1} r_k(t) P_k(A) + \sum_{k=0}^{n-1} A r_{k+1}(t) P_k(A)$$

$$= -\sum_{k=0}^{n-1} r_{k+1}(t) P_{k+1}(A) + \sum_{k=0}^{n-1} r_k(t) P_k(A) + A\sum_{k=0}^{n-1} r_{k+1}(t) P_k(A). \tag{2.24}$$

Now using $r_0(t) = 0$, the first term equals

$$\begin{aligned} -\sum_{k=1}^{n} r_k(t) P_k(A) &= -\sum_{k=1}^{n-1} r_k(t) P_k(A) \\ &= -\sum_{k=0}^{n-1} r_k(t) P_k(A) \end{aligned}$$

and so 2.24 reduces to

$$A\sum_{k=0}^{n-1} r_{k+1}(t) P_k(A) = A\Phi(t).$$

This shows $\Phi'(t) = A\Phi(t)$. That $\Phi(0) = 0$ follows from

$$\Phi(0) = \sum_{k=0}^{n-1} r_{k+1}(0) P_k(A) = r_1(0) P_0 = I.$$

It remains to verify that if 2.23 holds, then $\Phi(t)^{-1}$ exists for all t. To do so, consider $\mathbf{v} \neq \mathbf{0}$ and suppose for some t_0, $\Phi(t_0)\mathbf{v} = \mathbf{0}$. Let $\mathbf{x}(t) \equiv \Phi(t_0 + t)\mathbf{v}$. Then

$$\mathbf{x}'(t) = A\Phi(t_0 + t)\mathbf{v} = A\mathbf{x}(t),\ \mathbf{x}(0) = \Phi(t_0)\mathbf{v} = \mathbf{0}.$$

But also $\mathbf{z}(t) \equiv \mathbf{0}$ also satisfies

$$\mathbf{z}'(t) = A\mathbf{z}(t),\ \mathbf{z}(0) = \mathbf{0},$$

and so by the theorem on uniqueness, it must be the case that $\mathbf{z}(t) = \mathbf{x}(t)$ for all t, showing that $\Phi(t + t_0)\mathbf{v} = \mathbf{0}$ for all t, and in particular for $t = -t_0$. Therefore,

$$\Phi(-t_0 + t_0)\mathbf{v} = I\mathbf{v} = \mathbf{0}$$

and so $\mathbf{v} = \mathbf{0}$, a contradiction. It follows that $\Phi(t)$ must be one to one for all t and so, $\Phi(t)^{-1}$ exists for all t.

It only remains to verify the solution to 2.23 is unique. Suppose Ψ is another fundamental matrix solving 2.23. Then letting $\mathbf{v}$ be an arbitrary vector,

$$\mathbf{z}(t) \equiv \Phi(t)\mathbf{v},\ \mathbf{y}(t) \equiv \Psi(t)\mathbf{v}$$

both solve the initial value problem,

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{v},$$

and so by the uniqueness theorem, $\mathbf{z}(t) = \mathbf{y}(t)$ for all t showing that $\Phi(t)\mathbf{v} = \Psi(t)\mathbf{v}$ for all t. Since $\mathbf{v}$ is arbitrary, this shows that $\Phi(t) = \Psi(t)$ for every t. This proves the theorem.

It is useful to consider the differential equations for the r_k for $k \geq 1$. As noted above, $r_0(t) = 0$ and $r_1(t) = e^{\lambda_1 t}$.

$$r'_{k+1} = \lambda_{k+1} r_{k+1} + r_k,\ r_{k+1}(0) = 0.$$

Thus

$$r_{k+1}(t) = \int_0^t e^{\lambda_{k+1}(t-s)} r_k(s)\, ds.$$

Therefore,

$$r_2(t) = \int_0^t e^{\lambda_2(t-s)} e^{\lambda_1 s} ds = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{-\lambda_2 + \lambda_1}$$

assuming $\lambda_1 \neq \lambda_2$.

Sometimes people define a fundamental matrix to be a matrix, $\Phi(t)$ such that $\Phi'(t) = A\Phi(t)$ and $\det(\Phi(t)) \neq 0$ for all t. Thus this avoids the initial condition, $\Phi(0) = I$. The next proposition has to do with this situation.

Proposition B.4.9 *Suppose A is an $n \times n$ matrix and suppose $\Phi(t)$ is an $n \times n$ matrix for each $t \in \mathbb{R}$ with the property that*

$$\Phi'(t) = A\Phi(t). \tag{2.25}$$

Then either $\Phi(t)^{-1}$ exists for all $t \in \mathbb{R}$ or $\Phi(t)^{-1}$ fails to exist for all $t \in \mathbb{R}$.

Proof: Suppose $\Phi(0)^{-1}$ exists and 2.25 holds. Let $\Psi(t) \equiv \Phi(t)\Phi(0)^{-1}$. Then $\Psi(0) = I$ and

$$\Psi'(t) = \Phi'(t)\Phi(0)^{-1} = A\Phi(t)\Phi(0)^{-1} = A\Psi(t)$$

so by Theorem B.4.8, $\Psi(t)^{-1}$ exists for all t. Therefore, $\Phi(t)^{-1}$ also exists for all t.

Next suppose $\Phi(0)^{-1}$ does not exist. I need to show $\Phi(t)^{-1}$ does not exist for any t. Suppose then that $\Phi(t_0)^{-1}$ does exist. Then let$\Psi(t) \equiv \Phi(t_0 + t)\Phi(t_0)^{-1}$. Then $\Psi(0) = I$ and $\Psi' = A\Psi$ so by Theorem B.4.8 it follows $\Psi(t)^{-1}$ exists for all t and so for all $t, \Phi(t + t_0)^{-1}$ must also exist, even for $t = -t_0$ which implies $\Phi(0)^{-1}$ exists after all. This proves the proposition.

The conclusion of this proposition is usually referred to as the Wronskian alternative and another way to say it is that if 2.25 holds, then either $\det(\Phi(t)) = 0$ for all t or $\det(\Phi(t))$ is never equal to 0. The Wronskian is the usual name of the function, $t \to \det(\Phi(t))$.

The following theorem gives the variation of constants formula,.

Theorem B.4.10 *Let $\mathbf{f}$ be continuous on $[0, T]$ and let A be an $n \times n$ matrix and $\mathbf{x}_0$ a vector in $\mathbb{C}^n$. Then there exists a unique solution to 2.17, $\mathbf{x}$, given by the variation of constants formula,*

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{f}(s)\, ds \tag{2.26}$$

for $\Phi(t)$ the fundamental matrix for A. Also, $\Phi(t)^{-1} = \Phi(-t)$ and $\Phi(t+s) = \Phi(t)\Phi(s)$ for all t, s and the above variation of constants formula can also be written as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t-s)\mathbf{f}(s)\, ds \tag{2.27}$$

$$= \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(s)\mathbf{f}(t-s)\, ds \tag{2.28}$$

Proof: From the uniqueness theorem there is at most one solution to 2.17. Therefore, if 2.26 solves 2.17, the theorem is proved. The verification that the given formula works is identical with the verification that the scalar formula given in Theorem B.4.7 solves the initial value problem given there. $\Phi(s)^{-1}$ is continuous because of the formula for the inverse of a matrix in terms of the transpose of the cofactor matrix. Therefore, the integrand in

2.26 is continuous and the fundamental theorem of calculus applies. To verify the formula for the inverse, fix s and consider $\mathbf{x}(t) = \Phi(s+t)\mathbf{v}$, and $\mathbf{y}(t) = \Phi(t)\Phi(s)\mathbf{v}$. Then

$$\mathbf{x}'(t) = A\Phi(t+s)\mathbf{v} = A\mathbf{x}(t),\ \mathbf{x}(0) = \Phi(s)\mathbf{v}$$

$$\mathbf{y}'(t) = A\Phi(t)\Phi(s)\mathbf{v} = A\mathbf{y}(t),\ \mathbf{y}(0) = \Phi(s)\mathbf{v}.$$

By the uniqueness theorem, $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. Since s and $\mathbf{v}$ are arbitrary, this shows $\Phi(t+s) = \Phi(t)\Phi(s)$ for all t, s. Letting $s = -t$ and using $\Phi(0) = I$ verifies $\Phi(t)^{-1} = \Phi(-t)$.

Next, note that this also implies $\Phi(t-s)\Phi(s) = \Phi(t)$ and so $\Phi(t-s) = \Phi(t)\Phi(s)^{-1}$. Therefore, this yields 2.27 and then 2.28follows from changing the variable. This proves the theorem.

If $\Phi' = A\Phi$ and $\Phi(t)^{-1}$ exists for all t, you should verify that the solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Phi(t-t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t-s)\mathbf{f}(s)\,ds.$$

Theorem B.4.10 is general enough to include all constant coefficient linear differential equations or any order. Thus it includes as a special case the main topics of an entire elementary differential equations class. This is illustrated in the following example. One can reduce an arbitrary linear differential equation to a first order system and then apply the above theory to solve the problem. The next example is a differential equation of damped vibration.

Example B.4.11 *The differential equation is* $y'' + 2y' + 2y = \cos t$ *and initial conditions,* $y(0) = 1$ *and* $y'(0) = 0$.

To solve this equation, let $x_1 = y$ and $x_2 = x_1' = y'$. Then, writing this in terms of these new variables, yields the following system.

$$\begin{aligned} x_2' + 2x_2 + 2x_1 &= \cos t \\ x_1' &= x_2 \end{aligned}$$

This system can be written in the above form as

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} x_2 \\ -2x_2 - 2x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}. \end{aligned}$$

and the initial condition is of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now $P_0(A) \equiv I$. The eigenvalues are $-1+i, -1-i$ and so

$$\begin{aligned} P_1(A) &= \left(\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} - (-1+i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}. \end{aligned}$$

Recall $r_0(t) \equiv 0$ and $r_1(t) = e^{(-1+i)t}$. Then

$$r_2' = (-1-i)\, r_2 + e^{(-1+i)t},\ r_2(0) = 0$$

and so

$$r_2(t) = \frac{e^{(-1+i)t} - e^{(-1-i)t}}{2i} = e^{-t}\sin(t)$$

Putzer's method yields the fundamental matrix as

$$\begin{aligned}\Phi(t) &= e^{(-1+i)t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{-t}\sin(t)\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos(t)-\sin(t)) \end{pmatrix}\end{aligned}$$

From variation of constants formula the desired solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}(\cos(t)-\sin(t)) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+\int_0^t \begin{pmatrix} e^{-s}(\cos(s)+\sin(s)) & e^{-s}\sin s \\ -2e^{-s}\sin s & e^{-s}(\cos(s)-\sin(s)) \end{pmatrix}\begin{pmatrix} 0 \\ \cos(t-s) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) \\ -2e^{-t}\sin t \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-s}\sin(s)\cos(t-s) \\ e^{-s}(\cos s - \sin s)\cos(t-s) \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) \\ -2e^{-t}\sin t \end{pmatrix} + \begin{pmatrix} -\frac{1}{5}(\cos t)\, e^{-t} - \frac{3}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t \\ -\frac{2}{5}(\cos t)\, e^{-t} + \frac{4}{5}e^{-t}\sin t + \frac{2}{5}\cos t - \frac{1}{5}\sin t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5}(\cos t)\, e^{-t} + \frac{2}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t \\ -\frac{6}{5}e^{-t}\sin t - \frac{2}{5}(\cos t)\, e^{-t} + \frac{2}{5}\cos t - \frac{1}{5}\sin t \end{pmatrix}$$

Thus $y(t) = x_1(t) = \frac{4}{5}(\cos t)\, e^{-t} + \frac{2}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t$.

B.5 Geometric Theory Of Autonomous Systems

Here a sufficient condition is given for stability of a first order system. First of all, here is a fundamental estimate for the entries of a fundamental matrix.

Lemma B.5.1 *Let the functions, r_k be given in the statement of Theorem B.4.8 and suppose that A is an $n \times n$ matrix whose eigenvalues are $\{\lambda_1, \cdots, \lambda_n\}$. Suppose that these eigenvalues are ordered such that*

$$\mathrm{Re}(\lambda_1) \le \mathrm{Re}(\lambda_2) \le \cdots \le \mathrm{Re}(\lambda_n) < 0.$$

Then if $0 > -\delta > \mathrm{Re}(\lambda_n)$ is given, there exists a constant, C such that for each $k = 0, 1, \cdots, n$,

$$|r_k(t)| \le Ce^{-\delta t} \tag{2.29}$$

for all $t > 0$.

Proof: This is obvious for $r_0(t)$ because it is identically equal to 0. From the definition of the r_k,

$$r_1' = \lambda_1 r_1, r_1(0) = 1$$

and so

$$r_1(t) = e^{\lambda_1 t}$$

which implies

$$|r_1(t)| \leq e^{\mathrm{Re}(\lambda_1)t}.$$

Suppose for some $m \geq 1$ there exists a constant, C_m such that

$$|r_k(t)| \leq C_m t^m e^{\mathrm{Re}(\lambda_m)t}$$

for all $k \leq m$ for all $t > 0$. Then

$$r'_{m+1}(t) = \lambda_{m+1} r_{m+1}(t) + r_m(t),\ r_{m+1}(0) = 0$$

and so

$$r_{m+1}(t) = e^{\lambda_{m+1}t} \int_0^t e^{-\lambda_{m+1}s} r_m(s)\, ds.$$

Then by the induction hypothesis,

$$\begin{aligned} |r_{m+1}(t)| &\leq e^{\mathrm{Re}(\lambda_{m+1})t} \int_0^t \left|e^{-\lambda_{m+1}s}\right| C_m s^m e^{\mathrm{Re}(\lambda_m)s} ds \\ &\leq e^{\mathrm{Re}(\lambda_{m+1})t} \int_0^t s^m C_m e^{-\mathrm{Re}(\lambda_{m+1})s} e^{\mathrm{Re}(\lambda_m)s} ds \\ &\leq e^{\mathrm{Re}(\lambda_{m+1})t} \int_0^t s^m C_m ds = \frac{C_m}{m+1} t^{m+1} e^{\mathrm{Re}(\lambda_{m+1})t} \end{aligned}$$

It follows by induction there exists a constant, C such that for all $k \leq n$,

$$|r_k(t)| \leq C t^n e^{\mathrm{Re}(\lambda_n)t}$$

and this obviously implies the conclusion of the lemma.

The proof of the above lemma yields the following corollary.

Corollary B.5.2 *Let the functions, r_k be given in the statement of Theorem B.4.8 and suppose that A is an $n \times n$ matrix whose eigenvalues are $\{\lambda_1, \cdots, \lambda_n\}$. Suppose that these eigenvalues are ordered such that*

$$\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq \cdots \leq \mathrm{Re}(\lambda_n).$$

Then there exists a constant C such that for all $k \leq m$

$$|r_k(t)| \leq C t^m e^{\mathrm{Re}(\lambda_m)t}.$$

With the lemma, the following sloppy estimate is available for a fundamental matrix.

Theorem B.5.3 *Let A be an $n \times n$ matrix and let $\Phi(t)$ be the fundamental matrix for A. That is,*

$$\Phi'(t) = A\Phi(t),\ \Phi(0) = I.$$

Suppose also the eigenvalues of A are $\{\lambda_1, \cdots, \lambda_n\}$ where these eigenvalues are ordered such that

$$\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq \cdots \leq \mathrm{Re}(\lambda_n) < 0.$$

Then if $0 > -\delta > \mathrm{Re}(\lambda_n)$, is given, there exists a constant, C such that

$$\left|\Phi(t)_{ij}\right| \leq C e^{-\delta t}$$

for all $t > 0$. Also

$$|\Phi(t)\mathbf{x}| \leq C n^{3/2} e^{-\delta t} |\mathbf{x}|. \tag{2.30}$$

Proof: Let

$$M \equiv \max\left\{\left|P_k\left(A\right)_{ij}\right| \text{ for all } i,j,k\right\}.$$

Then from Putzer's formula for $\Phi\left(t\right)$ and Lemma B.5.1, there exists a constant, C such that

$$\left|\Phi\left(t\right)_{ij}\right| \leq \sum_{k=0}^{n-1} Ce^{-\delta t}M.$$

Let the new C be given by nCM. This proves the theorem.

Next,

$$\begin{aligned}
\left|\Phi\left(t\right)\mathbf{x}\right|^2 &\equiv \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\Phi_{ij}\left(t\right)x_j\right)^2 \\
&\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|\Phi_{ij}\left(t\right)\right|\left|x_j\right|\right)^2 \\
&\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}Ce^{-\delta t}\left|\mathbf{x}\right|\right)^2 \\
&= C^2e^{-2\delta t}\sum_{i=1}^{n}\left(n\left|\mathbf{x}\right|\right)^2 = C^2e^{-2\delta t}n^3\left|\mathbf{x}\right|^2
\end{aligned}$$

This proves 2.30 and completes the proof.

Definition B.5.4 *Let $\mathbf{f}: U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^n$ such that $\mathbf{a} \in U$ and $\mathbf{f}\left(\mathbf{a}\right) = \mathbf{0}$. A point, $\mathbf{a}$ where $\mathbf{f}\left(\mathbf{a}\right) = \mathbf{0}$ is called an equilibrium point. Then $\mathbf{a}$ is assymptotically stable if for any $\varepsilon > 0$ there exists $r > 0$ such that whenever $\left|\mathbf{x}_0 - \mathbf{a}\right| < r$ and $\mathbf{x}\left(t\right)$ the solution to the initial value problem,*

$$\mathbf{x}' = \mathbf{f}\left(\mathbf{x}\right),\ \mathbf{x}\left(0\right) = \mathbf{x}_0,$$

it follows

$$\lim_{t\to\infty}\mathbf{x}\left(t\right) = \mathbf{a},\ \left|\mathbf{x}\left(t\right) - \mathbf{a}\right| < \varepsilon$$

A differential equation of the form $\mathbf{x}' = \mathbf{f}\left(\mathbf{x}\right)$ is called autonomous as opposed to a nonautonomous equation of the form $\mathbf{x}' = \mathbf{f}\left(t,\mathbf{x}\right)$. The equilibrium point $\mathbf{a}$ is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\left|\mathbf{x}_0 - \mathbf{a}\right| < \delta$, then if $\mathbf{x}$ is the solution of

$$\mathbf{x}' = \mathbf{f}\left(\mathbf{x}\right),\ \mathbf{x}\left(0\right) = \mathbf{x}_0, \tag{2.31}$$

then $\left|\mathbf{x}\left(t\right) - \mathbf{a}\right| < \varepsilon$ for all $t > 0$.

Obviously assymptotic stability implies stability.

An ordinary differential equation is called almost linear if it is of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}\left(\mathbf{x}\right)$$

where A is an $n \times n$ matrix and

$$\lim_{\mathbf{x}\to\mathbf{0}}\frac{\mathbf{g}\left(\mathbf{x}\right)}{\left|\mathbf{x}\right|} = \mathbf{0}.$$

Now the stability of an equilibrium point of an autonomous system,

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

can always be reduced to the consideration of the stability of $\mathbf{0}$ for an almost linear system. Here is why. If you are considering the equilibrium point, $\mathbf{a}$ for $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, you could define a new variable, $\mathbf{y}$ by

$$\mathbf{a} + \mathbf{y} = \mathbf{x}.$$

Then assymptotic stability would involve $|\mathbf{y}(t)| < \varepsilon$ and $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ while stability would only require $|\mathbf{y}(t)| < \varepsilon$. Then since $\mathbf{a}$ is an equilibrium point, $\mathbf{y}$ solves the following initial value problem.

$$\mathbf{y}' = \mathbf{f}(\mathbf{a} + \mathbf{y}) - \mathbf{f}(\mathbf{a}),\ \mathbf{y}(0) = \mathbf{y}_0,$$

where $\mathbf{y}_0 = \mathbf{x}_0 - \mathbf{a}$.

Let $A = D\mathbf{f}(\mathbf{a})$. Then from the definition of the derivative of a function,

$$\mathbf{y}' = A\mathbf{y} + \mathbf{g}(\mathbf{y}),\ \mathbf{y}(0) = \mathbf{y}_0 \tag{2.32}$$

where

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{y})}{|\mathbf{y}|} = \mathbf{0}.$$

Thus there is never any loss of generality in considering only the equilibrium point $\mathbf{0}$ for an almost linear system.[1] Therefore, from now on I will only consider the case of almost linear systems and the equilibrium point $\mathbf{0}$.

Theorem B.5.5 *Consider the almost linear system of equations,*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}) \tag{2.33}$$

where

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{x})}{|\mathbf{x}|} = \mathbf{0}$$

and $\mathbf{g}$ *is a* C^1 *function. Suppose that for all* λ *an eigenvalue of* A, $\operatorname{Re}\lambda < 0$. *Then* $\mathbf{0}$ *is asymptotically stable.*

Proof: By Theorem B.5.3 there exist constants $\delta > 0$ and K such that for $\Phi(t)$ the fundamental matrix for A,

$$|\Phi(t)\mathbf{x}| \le Ke^{-\delta t}|\mathbf{x}|.$$

Let $\varepsilon > 0$ be given and let r be small enough that $Kr < \varepsilon$ and for $|\mathbf{x}| < (K+1)r, |\mathbf{g}(\mathbf{x})| < \eta|\mathbf{x}|$ where η is so small that $K\eta < \delta$, and let $|\mathbf{y}_0| < r$. Then by the variation of constants formula, the solution to 2.33, at least for small t satisfies

$$\mathbf{y}(t) = \Phi(t)\mathbf{y}_0 + \int_0^t \Phi(t-s)\mathbf{g}(\mathbf{y}(s))\,ds.$$

The following estimate holds.

$$\begin{aligned} |\mathbf{y}(t)| &\le Ke^{-\delta t}|\mathbf{y}_0| + \int_0^t Ke^{-\delta(t-s)}\eta|\mathbf{y}(s)|\,ds \\ &< Ke^{-\delta t}r + \int_0^t Ke^{-\delta(t-s)}\eta|\mathbf{y}(s)|\,ds. \end{aligned}$$

[1] This is no longer true when you study partial differential equations as ordinary differential equations in infinite dimensional spaces.

Therefore,

$$e^{\delta t}\left|\mathbf{y}\left(t\right)\right| < Kr + \int_0^t K\eta e^{\delta s}\left|\mathbf{y}\left(s\right)\right| ds.$$

By Gronwall's inequality,

$$e^{\delta t}\left|\mathbf{y}\left(t\right)\right| < Kre^{K\eta t}$$

and so

$$\left|\mathbf{y}\left(t\right)\right| < Kre^{(K\eta-\delta)t} < \varepsilon e^{(K\eta-\delta)t}$$

Therefore, $\left|\mathbf{y}\left(t\right)\right| < Kr < \varepsilon$ for all t and so from Corollary B.3.4, the solution to 2.33 exists for all $t \geq 0$ and since $K\eta - \delta < 0$,

$$\lim_{t\to\infty}\left|\mathbf{y}\left(t\right)\right| = 0.$$

This proves the theorem.

B.6 General Geometric Theory

Here I will consider the case where the matrix, A has both postive and negative eigenvalues. First here is a useful lemma.

Lemma B.6.1 *Suppose A is an $n \times n$ matrix and there exists $\delta > 0$ such that*

$$0 < \delta < \mathrm{Re}\left(\lambda_1\right) \leq \cdots \leq \mathrm{Re}\left(\lambda_n\right)$$

where $\{\lambda_1, \cdots, \lambda_n\}$ are the eigenvalues of A, with possibly some repeated. Then there exists a constant, C such that for all $t < 0$,

$$\left|\Phi\left(t\right)\mathbf{x}\right| \leq Ce^{\delta t}\left|\mathbf{x}\right|$$

Proof: I want an estimate on the solutions to the system

$$\Phi'\left(t\right) = A\Phi\left(t\right),\ \Phi\left(0\right) = I.$$

for $t < 0$. Let $s = -t$ and let $\Psi\left(s\right) = \Phi\left(t\right)$. Then writing this in terms of Ψ,

$$\Psi'\left(s\right) = -A\Psi\left(s\right),\ \Psi\left(0\right) = I.$$

Now the eigenvalues of $-A$ have real parts less than $-\delta$ because these eigenvalues are obtained from the eigenvalues of A by multiplying by -1. Then by Theorem B.5.3 there exists a constant, C such that for any $\mathbf{x}$,

$$\left|\Psi\left(s\right)\mathbf{x}\right| \leq Ce^{-\delta s}\left|\mathbf{x}\right|.$$

Therefore, from the definition of Ψ,

$$\left|\Phi\left(t\right)\mathbf{x}\right| \leq Ce^{\delta t}\left|\mathbf{x}\right|.$$

This proves the lemma.

Here is another essential lemma which is found in Coddington and Levinson [5]

Lemma B.6.2 *Let $p_j(t)$ be polynomials with complex coefficients and let*

$$f(t) = \sum_{j=1}^{m} p_j(t) e^{\lambda_j t}$$

where $m \geq 1, \lambda_j \neq \lambda_k$ for $j \neq k$, and none of the $p_j(t)$ vanish identically. Let

$$\sigma = \max(\mathrm{Re}(\lambda_1), \cdots, \mathrm{Re}(\lambda_m)).$$

Then there exists a positive number, r and arbitrarily large positive values of t such that

$$e^{-\sigma t} |f(t)| > r.$$

In particular, $|f(t)|$ is unbounded.

Proof: Suppose the largest exponent of any of the p_j is M and let $\lambda_j = a_j + ib_j$. First assume each $a_j = 0$. This is convenient because $\sigma = 0$ in this case and the largest of the $\mathrm{Re}(\lambda_j)$ occurs in every λ_j.

Then arranging the above sum as a sum of decreasing powers of t,

$$f(t) = t^M f_M(t) + \cdots + t f_1(t) + f_0(t).$$

Then

$$t^{-M} f(t) = f_M(t) + O\left(\frac{1}{t}\right)$$

where the last term means that $tO\left(\frac{1}{t}\right)$ is bounded. Then

$$f_M(t) = \sum_{j=1}^{m} c_j e^{ib_j t}$$

It can't be the case that all the c_j are equal to 0 because then M would not be the highest power exponent. Suppose $c_k \neq 0$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T t^{-M} f(t) e^{-ib_k t} dt = \sum_{j=1}^{m} c_j \frac{1}{T} \int_0^T e^{i(b_j - b_k)t} dt = c_k \neq 0.$$

Letting $r = |c_k/2|$, it follows $\left|t^{-M} f(t) e^{-ib_k t}\right| > r$ for arbitrarily large values of t. Thus it is also true that $|f(t)| > r$ for arbitrarily large values of t.

Next consider the general case in which σ is given above. Thus

$$e^{-\sigma t} f(t) = \sum_{j:a_j=\sigma} p_j(t) e^{b_j t} + g(t)$$

where $\lim_{t\to\infty} g(t) = 0$, $g(t)$ being of the form $\sum_s p_s(t) e^{(a_s - \sigma + ib_s)t}$ where $a_s - \sigma < 0$. Then this reduces to the case above in which $\sigma = 0$. Therefore, there exists $r > 0$ such that

$$\left|e^{-\sigma t} f(t)\right| > r$$

for arbitrarily large values of t. This proves the lemma.

Next here is a Banach space which will be useful.

Lemma B.6.3 *For $\gamma > 0$, let*

$$E_\gamma = \left\{ \mathbf{x} \in BC\left([0,\infty), \mathbb{F}^n\right) : t \to e^{\gamma t}\mathbf{x}\left(t\right) \text{ is also in } BC\left([0,\infty), \mathbb{F}^n\right) \right\}$$

and let the norm be given by

$$||\mathbf{x}||_\gamma \equiv \sup\left\{ \left| e^{\gamma t}\mathbf{x}\left(t\right) \right| : t \in [0,\infty) \right\}$$

Then E_γ is a Banach space.

Proof: Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in E_γ. Then since $BC\left([0,\infty), \mathbb{F}^n\right)$ is a Banach space, there exists $\mathbf{y} \in BC\left([0,\infty), \mathbb{F}^n\right)$ such that $e^{\gamma t}\mathbf{x}_k\left(t\right)$ converges uniformly on $[0,\infty)$ to $\mathbf{y}\left(t\right)$. Therefore $e^{-\gamma t}e^{\gamma t}\mathbf{x}_k\left(t\right) = \mathbf{x}_k\left(t\right)$ converges uniformly to $e^{-\gamma t}\mathbf{y}\left(t\right)$ on $[0,\infty)$. Define $\mathbf{x}\left(t\right) \equiv e^{-\gamma t}\mathbf{y}\left(t\right)$. Then $\mathbf{y}\left(t\right) = e^{\gamma t}\mathbf{x}\left(t\right)$ and by definition,

$$||\mathbf{x}_k - \mathbf{x}||_\gamma \to 0.$$

This proves the lemma.

B.7 The Stable Manifold

Here assume

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix} \tag{2.34}$$

where A_- and A_+ are square matrices of size $k \times k$ and $(n-k) \times (n-k)$ respectively. Also assume A_- has eigenvalues whose real parts are all less than $-\alpha$ while A_+ has eigenvalues whose real parts are all larger than α. Assume also that each of A_- and A_+ is upper triangular.

Also, I will use the following convention. For $\mathbf{v} \in \mathbb{F}^n$,

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_- \\ \mathbf{v}_+ \end{pmatrix}$$

where $\mathbf{v}_-$ consists of the first k entries of $\mathbf{v}$.

Then from Theorem B.5.3 and Lemma B.6.1 the following lemma is obtained.

Lemma B.7.1 *Let A be of the form given in 2.34 as explained above and let $\Phi_+\left(t\right)$ and $\Phi_-\left(t\right)$ be the fundamental matrices corresponding to A_+ and A_- respectively. Then there exist positive constants, α and γ such that*

$$\left|\Phi_+\left(t\right)\mathbf{y}\right| \leq Ce^{\alpha t} \text{ for all } t < 0 \tag{2.35}$$

$$\left|\Phi_-\left(t\right)\mathbf{y}\right| \leq Ce^{-(\alpha+\gamma)t} \text{ for all } t > 0. \tag{2.36}$$

Also for any nonzero $\mathbf{x} \in \mathbb{C}^{n-k}$,

$$\left|\Phi_+\left(t\right)\mathbf{x}\right| \text{ is unbounded.} \tag{2.37}$$

Proof: The first two claims have been established already. It suffices to pick α and γ such that $-\left(\alpha+\gamma\right)$ is larger than all eigenvalues of A_- and α is smaller than all eigenvalues of A_+. It remains to verify 2.37. From the Putzer formula for $\Phi_+\left(t\right)$,

$$\Phi_+\left(t\right)\mathbf{x} = \sum_{k=0}^{n-1} r_{k+1}\left(t\right) P_k\left(A\right)\mathbf{x}$$

where $P_0(A) \equiv I$. Now each r_k is a polynomial (possibly a constant) times an exponential. This follows easily from the definition of the r_k as solutions of the differential equations

$$r'_{k+1} = \lambda_{k+1} r_{k+1} + r_k.$$

Now by assumption the eigenvalues have positive real parts so

$$\sigma \equiv \max(\mathrm{Re}(\lambda_1), \cdots, \mathrm{Re}(\lambda_{n-k})) > 0.$$

It can also be assumed

$$\mathrm{Re}(\lambda_1) \geq \cdots \geq \mathrm{Re}(\lambda_{n-k})$$

By Lemma B.6.2 it follows $|\Phi_+(t)\mathbf{x}|$ is unbounded. This follows because

$$\Phi_+(t)\mathbf{x} = r_1(t)\mathbf{x} + \sum_{k=1}^{n-1} r_{k+1}(t)\mathbf{y}_k,\ r_1(t) = e^{\lambda_1 t}.$$

Since $\mathbf{x} \neq \mathbf{0}$, it has a nonzero entry, say $x_m \neq 0$. Consider the m^{th} entry of the vector $\Phi_+(t)\mathbf{x}$. By this Lemma the m^{th} entry is unbounded and this is all it takes for $\mathbf{x}(t)$ to be unbounded. This proves the lemma.

Lemma B.7.2 *Consider the initial value problem for the almost linear system*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{g}$ *is* C^1 *and* A *is of the special form*

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}$$

in which A_- *is a* $k \times k$ *matrix which has eigenvalues for which the real parts are all negative and* A_+ *is a* $(n-k) \times (n-k)$ *matrix for which the real parts of all the eigenvalues are positive. Then* $\mathbf{0}$ *is not stable. More precisely, there exists a set of points* $(\mathbf{a}_-, \boldsymbol{\psi}(\mathbf{a}_-))$ *for* $\mathbf{a}_-$ *small such that for* $\mathbf{x}_0$ *on this set,*

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}$$

and for $\mathbf{x}_0$ *not on this set, there exists a* $\delta > 0$ *such that* $|\mathbf{x}(t, \mathbf{x}_0)|$ *cannot remain less than* δ *for all positive* t.

Proof: Consider the initial value problem for the almost linear equation,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{a} = \begin{pmatrix} \mathbf{a}_- \\ \mathbf{a}_+ \end{pmatrix}.$$

Then by the variation of constants formula, a local solution has the form

$$\begin{aligned} \mathbf{x}(t, \mathbf{a}) &= \begin{pmatrix} \Phi_-(t) & 0 \\ 0 & \Phi_+(t) \end{pmatrix} \begin{pmatrix} \mathbf{a}_- \\ \mathbf{a}_+ \end{pmatrix} \\ &\quad + \int_0^t \begin{pmatrix} \Phi_-(t-s) & 0 \\ 0 & \Phi_+(t-s) \end{pmatrix} \mathbf{g}(\mathbf{x}(s, \mathbf{a}))\, ds \end{aligned} \tag{2.38}$$

Write $\mathbf{x}(t)$ for $\mathbf{x}(t,\mathbf{a})$ for short. Let $\varepsilon>0$ be given and suppose δ is such that if $|\mathbf{x}|<\delta$, then $|\mathbf{g}_{\pm}(\mathbf{x})|<\varepsilon|\mathbf{x}|$. Assume from now on that $|\mathbf{a}|<\delta$. Then suppose $|\mathbf{x}(t)|<\delta$ for all $t>0$. Writing 2.38 differently yields

$$\begin{aligned}\mathbf{x}(t,\mathbf{a}) &= \begin{pmatrix}\Phi_-(t) & 0\\ 0 & \Phi_+(t)\end{pmatrix}\begin{pmatrix}\mathbf{a}_-\\ \mathbf{a}_+\end{pmatrix}+\begin{pmatrix}\int_0^t \Phi_-(t-s)\,\mathbf{g}_-(\mathbf{x}(s,\mathbf{a}))\,ds\\ 0\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ \int_0^t \Phi_+(t-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix}\\ &=\begin{pmatrix}\Phi_-(t) & 0\\ 0 & \Phi_+(t)\end{pmatrix}\begin{pmatrix}\mathbf{a}_-\\ \mathbf{a}_+\end{pmatrix}+\begin{pmatrix}\int_0^t \Phi_-(t-s)\,\mathbf{g}_-(\mathbf{x}(s,\mathbf{a}))\,ds\\ 0\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ \int_0^\infty \Phi_+(t-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds-\int_t^\infty \Phi_+(t-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix}.\end{aligned}$$

These improper integrals converge thanks to the assumption that $\mathbf{x}$ is bounded and the estimates 2.35 and 2.36. Continuing the rewriting,

$$\begin{aligned}\begin{pmatrix}\mathbf{x}_-(t)\\ \mathbf{x}_+(t)\end{pmatrix} &= \begin{pmatrix}\left(\Phi_-(t)\,\mathbf{a}_-+\int_0^t \Phi_-(t-s)\,\mathbf{g}_-(\mathbf{x}(s,\mathbf{a}))\,ds\right)\\ \Phi_+(t)\left(\mathbf{a}_++\int_0^\infty \Phi_+(-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\right)\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ -\int_t^\infty \Phi_+(t-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix}.\end{aligned}$$

It follows from Lemma B.7.1 that if $|\mathbf{x}(t,\mathbf{a})|$ is bounded by δ as asserted, then it must be the case that $\mathbf{a}_++\int_0^\infty \Phi_+(-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds=\mathbf{0}$. Consequently, it must be the case that

$$\mathbf{x}(t)=\Phi(t)\begin{pmatrix}\mathbf{a}_-\\ \mathbf{0}\end{pmatrix}+\begin{pmatrix}\int_0^t \Phi_-(t-s)\,\mathbf{g}_-(\mathbf{x}(s,\mathbf{a}))\,ds\\ -\int_t^\infty \Phi_+(t-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix} \tag{2.39}$$

Letting $t\to 0$, this requires that for a solution to the initial value problem to exist and also satisfy $|\mathbf{x}(t)|<\delta$ for all $t>0$ it must be the case that

$$\mathbf{x}(0)=\begin{pmatrix}\mathbf{a}_-\\ -\int_0^\infty \Phi_+(-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix}$$

where $\mathbf{x}(t,\mathbf{a})$ is the solution of

$$\mathbf{x}'=A\mathbf{x}+\mathbf{g}(\mathbf{x}),\ \mathbf{x}(0)=\begin{pmatrix}\mathbf{a}_-\\ -\int_0^\infty \Phi_+(-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds\end{pmatrix}$$

This is because in 2.39, if $\mathbf{x}$ is bounded by δ then the reverse steps show $\mathbf{x}$ is a solution of the above differential equation and initial condition.

It follows if I can show that for all $\mathbf{a}_-$ sufficiently small and $\mathbf{a}=(\mathbf{a}_-,\mathbf{0})^T$, there exists a solution to 2.39 $\mathbf{x}(s,\mathbf{a})$ on $(0,\infty)$ for which $|\mathbf{x}(s,\mathbf{a})|<\delta$, then I can define

$$\boldsymbol{\psi}(\mathbf{a})\equiv-\int_0^\infty \Phi_+(-s)\,\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds$$

and conclude that $|\mathbf{x}(t,\mathbf{x}_0)|<\delta$ for all $t>0$ if and only if $\mathbf{x}_0=(\mathbf{a}_-,\boldsymbol{\psi}(\mathbf{a}_-))^T$ for some sufficiently small $\mathbf{a}_-$.

Let C,α,γ be the constants of Lemma B.7.1. Let η be a small positive number such that

$$\frac{C\eta}{\alpha}<\frac{1}{6}$$

Note that $\frac{\partial \mathbf{g}}{\partial x_i}(0) = \mathbf{0}$. Therefore, by Lemma B.3.1, there exists $\delta > 0$ such that if $|\mathbf{x}|, |\mathbf{y}| \leq \delta$, then

$$|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \eta |\mathbf{x} - \mathbf{y}|$$

and in particular,

$$|\mathbf{g}_{\pm}(\mathbf{x}) - \mathbf{g}_{\pm}(\mathbf{y})| < \eta |\mathbf{x} - \mathbf{y}| \tag{2.40}$$

because each $\frac{\partial \mathbf{g}}{\partial x_i}(\mathbf{x})$ is very small. In particular, this implies

$$|\mathbf{g}_{-}(\mathbf{x})| < \eta |\mathbf{x}|, |\mathbf{g}_{+}(\mathbf{x})| < \eta |\mathbf{x}|.$$

For $\mathbf{x} \in E_\gamma$ defined in Lemma B.6.3 and $|\mathbf{a}_-| < \frac{\delta}{2C}$,

$$F\mathbf{x}(t) \equiv \begin{pmatrix} \Phi_-(t)\mathbf{a}_- + \int_0^t \Phi_-(t-s)\mathbf{g}_-(\mathbf{x}(s))\,ds \\ -\int_t^\infty \Phi_+(t-s)\mathbf{g}_+(\mathbf{x}(s))\,ds \end{pmatrix}.$$

I need to find a fixed point of F. Letting $||\mathbf{x}||_\gamma < \delta$, and using the estimates of Lemma B.7.1,

$$\begin{aligned} e^{\gamma t}|F\mathbf{x}(t)| &\leq e^{\gamma t}|\Phi_-(t)\mathbf{a}_-| + e^{\gamma t}\int_0^t Ce^{-(\alpha+\gamma)(t-s)}\eta|\mathbf{x}(s)|\,ds \\ &\quad + e^{\gamma t}\int_t^\infty Ce^{\alpha(t-s)}\eta|\mathbf{x}(s)|\,ds \\ &\leq e^{\gamma t}C\frac{\delta}{2C}e^{-(\alpha+\gamma)t} + e^{\gamma t}||\mathbf{x}||_\gamma C\eta\int_0^t e^{-(\alpha+\gamma)(t-s)}e^{-\gamma s}ds \\ &\quad + e^{\gamma t}C\eta\int_t^\infty e^{\alpha(t-s)}e^{-\gamma s}ds\,||\mathbf{x}||_\gamma \\ &< \frac{\delta}{2} + \delta C\eta\int_0^t e^{-\alpha(t-s)}ds + C\eta\delta\int_t^\infty e^{(\alpha+\gamma)(t-s)}ds \\ &< \frac{\delta}{2} + \delta C\eta\frac{1}{\alpha} + \frac{\delta C\eta}{\alpha+\gamma} \leq \delta\left(\frac{1}{2} + \frac{C\eta}{\alpha}\right) < \frac{2\delta}{3}. \end{aligned}$$

Thus F maps every $\mathbf{x} \in E_\gamma$ having $||\mathbf{x}||_\gamma < \delta$ to $F\mathbf{x}$ where $||F\mathbf{x}||_\gamma \leq \frac{2\delta}{3}$.

Now let $\mathbf{x}, \mathbf{y} \in E_\gamma$ where $||\mathbf{x}||_\gamma, ||\mathbf{y}||_\gamma < \delta$. Then

$$\begin{aligned} e^{\gamma t}|F\mathbf{x}(t) - F\mathbf{y}(t)| &\leq e^{\gamma t}\int_0^t |\Phi_-(t-s)|\,\eta e^{-\gamma s}e^{\gamma s}|\mathbf{x}(s) - \mathbf{y}(s)|\,ds \\ &\quad + e^{\gamma t}\int_t^\infty |\Phi_+(t-s)|\,e^{-\gamma s}e^{\gamma s}\eta|\mathbf{x}(s) - \mathbf{y}(s)|\,ds \\ &\leq C\eta||\mathbf{x} - \mathbf{y}||_\gamma\left(\int_0^t e^{-\alpha(t-s)}ds\right) + \int_t^\infty e^{(\alpha+\gamma)(t-s)}ds \\ &\leq C\eta\left(\frac{1}{\alpha} + \frac{1}{\alpha+\gamma}\right)||\mathbf{x} - \mathbf{y}||_\gamma < \frac{2C\eta}{\alpha}||\mathbf{x} - \mathbf{y}||_\gamma < \frac{1}{3}||\mathbf{x} - \mathbf{y}||_\gamma. \end{aligned}$$

It follows from Lemma 14.6.4, for each $\mathbf{a}_-$ such that $|\mathbf{a}_-| < \frac{\delta}{2C}$, there exists a unique solution to 2.39 in E_γ.

As pointed out earlier, if

$$\boldsymbol{\psi}(\mathbf{a}) \equiv -\int_0^\infty \Phi_+(-s)\mathbf{g}_+(\mathbf{x}(s,\mathbf{a}))\,ds$$

then for $\mathbf{x}(t,\mathbf{x}_0)$ the solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0$$

has the property that if $\mathbf{x}_0$ is not of the form $\begin{pmatrix} \mathbf{a}_- \\ \boldsymbol{\psi}(\mathbf{a}_-) \end{pmatrix}$, then $|\mathbf{x}(t,\mathbf{x}_0)|$ cannot be less than δ for all $t > 0$.

On the other hand, if $\mathbf{x}_0 = \begin{pmatrix} \mathbf{a}_- \\ \boldsymbol{\psi}(\mathbf{a}_-) \end{pmatrix}$ for $|\mathbf{a}_-| < \frac{\delta}{2C}$, then $\mathbf{x}(t,\mathbf{x}_0)$,the solution to 2.39 is the unique solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0.$$

and it was shown that $||\mathbf{x}(\cdot,\mathbf{x}_0)||_\gamma < \delta$ and so in fact,

$$|\mathbf{x}(t,\mathbf{x}_0)| \leq \delta e^{-\gamma t}$$

showing that

$$\lim_{t\to\infty} \mathbf{x}(t,\mathbf{x}_0) = \mathbf{0}.$$

This proves the Lemma.

The following theorem is the main result. It involves a use of linear algebra and the above lemma.

Theorem B.7.3 *Consider the initial value problem for the almost linear system*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0$$

in which $\mathbf{g}$ *is* C^1 *and where at there are* $k < n$ *eigenvalues of* A *which have negative real parts and* $n-k$ *eigenvalues of* A *which have positive real parts. Then* $\mathbf{0}$ *is not stable. More precisely, there exists a set of points* $(\mathbf{a}, \boldsymbol{\psi}(\mathbf{a}))$ *for* $\mathbf{a}$ *small and in a* k *dimensional subspace such that for* $\mathbf{x}_0$ *on this set,*

$$\lim_{t\to\infty} \mathbf{x}(t,\mathbf{x}_0) = \mathbf{0}$$

and for $\mathbf{x}_0$ *not on this set, there exists a* $\delta > 0$ *such that* $|\mathbf{x}(t,\mathbf{x}_0)|$ *cannot remain less than* δ *for all positive* t.

Proof: This involves nothing more than a reduction to the situation of Lemma B.7.2. From Theorem 10.3.5 on Page 10.3.5 A is similar to a matrix of the form described in Lemma B.7.2. Thus $A = S^{-1}\begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix} S$. Letting $\mathbf{y} = S\mathbf{x}$, it follows

$$\mathbf{y}' = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix} \mathbf{y} + \mathbf{g}\left(S^{-1}\mathbf{y}\right)$$

Now $|\mathbf{x}| = \left|S^{-1}S\mathbf{x}\right| \leq \left|\left|S^{-1}\right|\right| |\mathbf{y}|$ and $|\mathbf{y}| = \left|SS^{-1}\mathbf{y}\right| \leq ||S||\,|\mathbf{x}|$. Therefore,

$$\frac{1}{||S||}|\mathbf{y}| \leq |\mathbf{x}| \leq \left|\left|S^{-1}\right|\right| |\mathbf{y}|.$$

It follows all conclusions of Lemma B.7.2 are valid for this theorem. This proves the theorem.

The set of points $(\mathbf{a}, \boldsymbol{\psi}(\mathbf{a}))$ for $\mathbf{a}$ small is called the stable manifold. Much more can be said about the stable manifold and you should look at a good differential equations book for this.

The Fundamental Theorem Of Algebra

The fundamental theorem of algebra states that every non constant polynomial having coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$. If $\mathbb{C}$ is replaced by $\mathbb{R}$, this is not true because of the example, $x^2+1=0$. This theorem is a very remarkable result and notwithstanding its title, all the best proofs of it depend on either analysis or topology. It was proved by Gauss in 1797 then proved with no loose ends by Argand in 1806 although others also worked on it. The proof given here follows Rudin [16]. See also Hardy [10] for another proof, more discussion and references. Recall De Moivre's theorem on Page 13 which is listed below for convenience.

Theorem C.0.4 *Let $r>0$ be given. Then if n is a positive integer,*

$$[r(\cos t+i\sin t)]^n = r^n(\cos nt+i\sin nt).$$

Now from this theorem, the following corollary on Page 1.3.5 is obtained.

Corollary C.0.5 *Let z be a non zero complex number and let k be a positive integer. Then there are always exactly k k^{th} roots of z in $\mathbb{C}$.*

Lemma C.0.6 *Let $a_k\in\mathbb{C}$ for $k=1,\cdots,n$ and let $p(z)\equiv\sum_{k=1}^n a_k z^k$. Then p is continuous.*

Proof:

$$|az^n-aw^n|\le|a||z-w|\left|z^{n-1}+z^{n-2}w+\cdots+w^{n-1}\right|.$$

Then for $|z-w|<1$, the triangle inequality implies $|w|<1+|z|$ and so if $|z-w|<1$,

$$|az^n-aw^n|\le|a||z-w|\,n(1+|z|)^n.$$

If $\varepsilon>0$ is given, let

$$\delta<\min\left(1,\frac{\varepsilon}{|a|\,n(1+|z|)^n}\right).$$

It follows from the above inequality that for $|z-w|<\delta$, $|az^n-aw^n|<\varepsilon$. The function of the lemma is just the sum of functions of this sort and so it follows that it is also continuous.

Theorem C.0.7 *(Fundamental theorem of Algebra) Let $p(z)$ be a nonconstant polynomial. Then there exists $z\in\mathbb{C}$ such that $p(z)=0$.*

Proof: Suppose not. Then

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

where $a_n \neq 0,\ n > 0$. Then

$$|p(z)| \geq |a_n| |z|^n - \sum_{k=0}^{n-1} |a_k| |z|^k$$

and so

$$\lim_{|z|\to\infty} |p(z)| = \infty. \tag{3.1}$$

Now let

$$\lambda \equiv \inf \{|p(z)| : z \in \mathbb{C}\}.$$

By 3.1, there exists an $R > 0$ such that if $|z| > R$, it follows that $|p(z)| > \lambda + 1$. Therefore,

$$\lambda \equiv \inf \{|p(z)| : z \in \mathbb{C}\} = \inf \{|p(z)| : |z| \leq R\}.$$

The set $\{z : |z| \leq R\}$ is a closed and bounded set and so this infimum is achieved at some point w with $|w| \leq R$. A contradiction is obtained if $|p(w)| = 0$ so assume $|p(w)| > 0$. Then consider

$$q(z) \equiv \frac{p(z+w)}{p(w)}.$$

It follows $q(z)$ is of the form

$$q(z) = 1 + c_k z^k + \cdots + c_n z^n$$

where $c_k \neq 0$, because $q(0) = 1$. It is also true that $|q(z)| \geq 1$ by the assumption that $|p(w)|$ is the smallest value of $|p(z)|$. Now let $\theta \in \mathbb{C}$ be a complex number with $|\theta| = 1$ and

$$\theta c_k w^k = -|w|^k |c_k|.$$

If

$$w \neq 0, \theta = \frac{-\left|w^k\right| |c_k|}{w^k c_k}$$

and if $w = 0$, $\theta = 1$ will work. Now let $\eta^k = \theta$ and let t be a small positive number.

$$q(t\eta w) \equiv 1 - t^k |w|^k |c_k| + \cdots + c_n t^n (\eta w)^n$$

which is of the form

$$1 - t^k |w|^k |c_k| + t^k (g(t,w))$$

where $\lim_{t\to 0} g(t,w) = 0$. Letting t be small enough,

$$|g(t,w)| < |w|^k |c_k| /2$$

and so for such t,

$$|q(t\eta w)| < 1 - t^k |w|^k |c_k| + t^k |w|^k |c_k| /2 < 1,$$

a contradiction to $|q(z)| \geq 1$. This proves the theorem.

Polynomials

D.1 Symmetric Polynomials In Many Variables

Polynomials are well understood for functions of a single variable. However, here I will consider polynomials which are functions of many variables. It turns out there are some very interesting things which can be said. I am following the presentation in [12] which does this in greater generality, replacing the field in the next definition with a ring.

Definition D.1.1 *For $\mathbb{F}$ a field, $\mathbb{F}[x_1, \cdots, x_n]$ consists of the polynomials in the variables $x_1, \cdots, x_n$. A typical polynomial of degree at most m is a function of the form*

$$\sum_{k_1+\cdots+k_n \leq m} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n},\ a_{k_1 \cdots k_n} \in \mathbb{F}$$

A polynomial is called homogeneous if for some $l \geq 0$ all the $a_{k_1 \cdots k_n}$ in the above equal 0 except for the ones for which $k_1 + \cdots + k_n = l$. A polynomial is symmetric if every permutation of the variables yields the same polynomial. Thus

$$x_1 x_2 x_4$$

is a symmetric polynomial in $\mathbb{F}[x_1, \cdots, x_n]$ for any $n \geq 4$. A monomial is a polynomial in which at most one of the $a_{k_1 \cdots k_n}$ is nonzero. As usual, a polynomial will be considered to equal 0 when all the coefficients are equal to 0. This is a matter of definition necessitated by the fact that there are polynomials which send every element of a field to 0 and yet the polynomial does not have zero coefficients. For example, $\lambda + \lambda^3$ for the field of scalars $\mathbb{Z}_2$, the integers mod 2. Similarly, two polynomials are equal when their corresponding coefficients are equal.

Of course, for reasonable fields like $\mathbb{R}$ or $\mathbb{Q}$, one can prove the condition on the vanishing of the coefficients by considering the polynomial as a function.

More generally, when one considers $\mathbb{F}$ to only be a commutative ring, you can consider polynomials with coefficients in the ring with the understanding that

$$\sum_{k_1 \cdots k_n} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n} = \sum_{k_1 \cdots k_n} b_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

means each $a_{k_1 \cdots k_n} = b_{k_1 \cdots k_n}$ and this will always be assumed in what follows.

With this definition, here is an important example.

Definition D.1.2 *The elementary symmetric polynomials p_k are $(-1)^k$ times the coefficients of the polynomial in x given by*

$$(x - x_1)(x - x_2) \cdots (x - x_n) \tag{4.1}$$

That is the above polynomial in x is given by

$$\sum_{k=0}^{n} (-1)^k p_k x^{n-k}$$

Thus

$$p_0 = 1, p_1 = \sum_{i=1}^{n} x_i, p_2 = \sum_{i<j} x_i x_j, p_3 = \sum_{i<j<k} x_i x_j x_k, \cdots, p_n = x_1 \cdots x_n.$$

Lemma D.1.3 *The elementary symmetric polynomials are symmetric as the name implies.*

Proof: They are symmetric because if you switch x_i and x_j in 4.1 the polynomial in x is unchanged which means the coefficients of the polynomial in x are unchanged. This proves the lemma.

One of the important things about these elementary symmetric polynomials is that every symmetric polynomial is a polynomial in these elementary symmetric polynomials.

Proposition D.1.4 *Let $f(x_1, \cdots, x_n)$ be a symmetric polynomial. Then there exist constants $a_{k_1 \cdots k_n}$ in the field such that*

$$f(x_1, \cdots, x_n) = \sum_{k_1 + \cdots + k_n \leq m} a_{k_1 \cdots k_n} p_1^{k_1} \cdots p_n^{k_n}$$

Proof: Given two monomials of degree m,

$$M \equiv a x_1^{k_1} \cdots x_n^{k_n}, \; N \equiv b x_1^{l_1} \cdots x_n^{l_n}$$

$M > N$, in words M is "higher" than N if the exponents are not all the same and that if the first discrepancy occurs at s, then $k_s > l_s$. In making this definition, it is understood that the exponents are read in the order of increasing subscripts on the variables. For example

$$x_1 x_2 x_3 > x_2 x_3 x_4$$

because the one on the left is $x_1 x_2 x_3 x_4^0$ and the one on the right is $x_1^0 x_2 x_3 x_4$ so in this case the first discrepancy is at 1 and $1 > 0$. Similarly

$$x_1 x_2^2 x_3 > x_1 x_2 x_3^2.$$

The first discrepancy is at $s = 2$ and the $2 > 1$. From this definition it is clear that if $M > N$ and if K is another monomial, then

$$KM > KN.$$

To see this, say

$$M = x_1^{k_1} \cdots x_n^{k_n}, \; N = x_1^{l_1} \cdots x_n^{l_n}$$

with the first discrepancy occuring at s. Then if

$$x_1^{r_1} \cdots x_n^{r_n}$$

is another monomial the respective products are

$$x_1^{k_1 + r_1} \cdots x_n^{k_n + r_n}, \; x_1^{l_1 + r_1} \cdots x_n^{l_n + r_n}$$

and it is still the case the first discrepancy occurs at s and $k_s + r_s > l_s + r_s$. Note also that if there are two monomials of the same degree, one must be higher than the other or else one is a scalar multiple of the other.

From this, it follows that if $M_i > N_i$ for $i = 1, 2$, then

$$M_1 M_2 > M_1 N_2 > N_1 N_2$$

Recall

$$p_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

and so the highest monomial of all these terms is $x_1 x_2 \cdots x_k$. It follows that the highest monomial in all the terms of

$$p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}$$

is

$$\begin{aligned} & x_1^{d_1} \left(x_1 x_2\right)^{d_2} \left(x_1 x_2 x_3\right)^{d_3} \cdots \left(x_1 \cdots x_n\right)^{d_n} \\ & = x_1^{d_1 + \cdots + d_r} x_2^{d_2 + \cdots + d_r} x_3^{d_3 + \cdots + d_r} \cdots x_n^{d_n} \end{aligned} \tag{4.2}$$

Now suppose f is a symmetric polynomial in $\mathbb{F}\left[x_1 \cdots x_n\right]$. Then it is the sum of homogeneous polynomials. These homogeneous polynomials must also be symmetric so it suffices to show that every symmetric homogeneous polynomial is a linear combination of the elementary symmetric polynomials. In other words, without loss of generality, assume f is homogeneous.

The homogeneous polynomial f is the sum of monomials say of degree r. Pick the highest one which has nonzero coefficient, $a x_1^{k_1} \cdots x_n^{k_n}$ where $\sum_j k_j = r$. Since f is symmetric, it has among its terms monomials of the form

$$a x_1^{k_{i_1}} \cdots x_n^{k_{i_n}}$$

where $k_{i_1} \cdots k_{i_n}$ is any permutation of $k_1 \cdots k_n$. It follows that $k_1 \geq k_2 \geq \cdots \geq k_n$ since otherwise, it could not be the highest monomial. Now it follows from 4.2 the highest monomial in all the terms of

$$p_1^{k_1 - k_2} p_2^{k_2 - k_3} \cdots p_n^{k_n} \tag{4.3}$$

is

$$x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

Consider the new symmetric polynomial

$$f - a p_1^{k_1 - k_2} p_2^{k_2 - k_3} \cdots p_n^{k_n}$$

which now is a smaller sum of monomials. You can apply the same argument to this and continuing this process, it follows f is indeed a polynomial in the elementary symmetric polynomials. Each time, you replace the highest monomial with a scalar times an expression like the one in 4.3 and subtract it off. This proves the proposition.

How many ways can you write a symmetric polynomial as a polynomial in the elementary symmetric polynomials? In other words, if

$$\sum_{k_1 + \cdots + k_n \leq m} a_{k_1 \cdots k_n} p_1^{k_1} \cdots p_n^{k_n} = 0,$$

does it follow each $a_{k_1 \cdots k_n} = 0$? The answer is yes. Recall that if each p_i is replaced with x_i in the above, the only way the expression is 0 is for each coefficient to equal 0. The following proposition is sometimes referred to by saying the $p_1^{k_1} \cdots p_n^{k_n}$ are algebraically independent.

Proposition D.1.5 *If*

$$\sum_{k_1,\cdots,k_n} a_{k_1\cdots k_n} p_1^{k_1}\cdots p_n^{k_n} = 0 \tag{4.4}$$

then each $a_{k_1\cdots k_n} = 0$, *the ordered lists* $(k_1,\cdots,k_n)$ *being distinct.*

Proof: The sum is a finite sum in the sense that all but finitely many of the coefficients equal 0, each $k_i \geq 0$. Consider one of the terms in the sum. By the same reasoning used in the previous proposition, the highest monomial in the sum making up $p_1^{k_1}\cdots p_n^{k_n}$ is $a_{k_1\cdots k_n}$ times

$$x_1^{k_1+\cdots+k_n} x_2^{k_2+\cdots+k_n} x_3^{k_3+\cdots+k_n}\cdots x_n^{k_n}$$

Let $l_i \equiv k_i+\cdots+k_n$. Then in terms of this, the highest monomial in the sum which yields $a_{k_1\cdots k_n} p_1^{k_1}\cdots p_n^{k_n}$ is $a_{k_1\cdots k_n}$ times

$$x_1^{l_1} x_2^{l_2}\cdots x_n^{l_n}$$

and there is only one of these highest monomials for each of these terms in 4.4. Could two different terms in $\sum_{k_1,\cdots,k_n} a_{k_1\cdots k_n} p_1^{k_1}\cdots p_n^{k_n}$ yield scalars times $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$ for the same sequence of l_i for the highest monomial? If this is so for $(k_1\cdots k_n)$ and $(k_1'\cdots k_n')$, then you would need

$$l_i = k_i+\cdots+k_n = k_i'+\cdots+k_n'$$

for each i and this requires that $k_n' = k_n$, then $k_{n-1}+k_n = k_{n-1}'+k_n'$ so $k_{n-1} = k_{n-1}'$ and continuing this way, it follows $k_i = k_i'$. Since it is assumed the $(k_1,\cdots,k_n)$ are distinct, it follows different terms of the sum in 4.4 lead to different $x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n}$ in the corresponding highest monomials. For each $(k_1\cdots k_n)$ let $(l_1\cdots l_n)$ be as defined above. Then writing in terms of the x_i variables, 4.4 is of the form

$$\sum_{k_1,\cdots,k_n} a_{k_1\cdots k_n} x_1^{l_1}\cdots x_n^{l_n} + \sum_{d_1,\cdots,d_n} b_{d_1\cdots d_n} x_1^{d_1}\cdots x_n^{d_n} = 0 \tag{4.5}$$

where the $x_1^{l_1}\cdots x_n^{l_n}$ are all distinct and the $x_1^{d_1}\cdots x_n^{d_n}$ are not highest of the monomials in some $a_{k_1\cdots k_n} p_1^{k_1}\cdots p_n^{k_n}$. Out of all the $x_1^{l_1}\cdots x_n^{l_n}$, consider the highest, $a'_{k_1'\cdots k_n'} x_1^{l_1'}\cdots x_n^{l_n'}$ with the property the coefficient is nonzero. Then as just explained, this $x_1^{l_1'}\cdots x_n^{l_n'}$ cannot be duplicated by any other in the first sum. It also cannot coincide with any of the $x_1^{d_1}\cdots x_n^{d_n}$ for if it did, then that $x_1^{d_1}\cdots x_n^{d_n}$ would be a multiple of a monomial from some $a_{k_1\cdots k_n} p_1^{k_1}\cdots p_n^{k_n}$ and the highest monomial from this term would then be higher than $a'_{k_1'\cdots k_n'} x_1^{l_1'}\cdots x_n^{l_n'}$ which is the highest. Therefore, combining such terms as it is possible to combine, one eventually has an expression of the form

$$\sum_{s_1,\cdots,s_n} b_{s_1\cdots s_n} x_1^{s_1}\cdots x_n^{s_n} = 0$$

where the $(s_1,\cdots,s_n)$ are all distinct yet not all $b_{s_1\cdots s_n} = 0$ because $a'_{k_1'\cdots k_n'} \neq 0$ and cannot be cancelled out by other terms in 4.5. This is a contradiction to the fact that a polynomial which equals 0 must have all the coefficients equal to 0. This proves the proposition.

The proof and conclusion of the above propositions is unchanged if $\mathbb{F}$ is only a commutative ring. A ring is like a field except it might not have 1 and multiplication might not be commutative and also not every nonzero element has a multiplicative inverse. A commutative ring is one for which multiplication is commutative. An interesting example of a commutative ring with unity which also has the property that if $ab = 0$ then one of a or b equals 0 is the integers. Rings like this are called integral domains. $\sum_{k_1\cdots k_n} a_{k_1\cdots k_n} x_1^{k_1}\cdots x_n^{k_n} = \sum_{k_1\cdots k_n} b_{k_1\cdots k_n} x_1^{k_1}\cdots x_n^{k_n}$

Thus you could denote by $\mathbb{F}[x_1,\cdots,x_n]$ as the polynomials having coefficients in a commutative ring, $\mathbb{F}$. This yields the following corollary.

Corollary D.1.6 *Let $f(x_1, \cdots, x_n)$ be a symmetric polynomial in $\mathbb{F}[x_1 \cdots x_n]$ where $\mathbb{F}$ is a commutative ring. Then there exist unique constants $a_{k_1 \cdots k_n}$ in $\mathbb{F}$ such that*

$$f(x_1, \cdots, x_n) = \sum_{k_1 + \cdots + k_n \leq m} a_{k_1 \cdots k_n} p_1^{k_1} \cdots p_n^{k_n}$$

Here is a very interesting result which I saw claimed in a paper by Steinberg and Redheffer on Lindemann's theorem which follows from the above corollary.

Theorem D.1.7 *Let $\alpha_1, \cdots, \alpha_n$ be roots of the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where each a_i is an integer. Then any symmetric polynomial in the quantities $a_n\alpha_1, \cdots, a_n\alpha_n$ having integer coefficients is also an integer. Also any symmetric polynomial in the quantities $\alpha_1, \cdots, \alpha_n$ having rational coefficients is a rational number.

Proof: Let $f(x_1, \cdots, x_n)$ be the symmetric polynomial. Thus

$$f(x_1, \cdots, x_n) \in \mathbb{Z}[x_1 \cdots x_n]$$

From Corollary D.1.6 it follows there are integers $a_{k_1 \cdots k_n}$ such that

$$f(x_1, \cdots, x_n) = \sum_{k_1 + \cdots + k_n \leq m} a_{k_1 \cdots k_n} p_1^{k_1} \cdots p_n^{k_n}$$

where the p_i are the elementary symmetric polynomials defined as the coefficients of

$$\prod_{j=1}^{n} (x - x_j)$$

Thus

$$\begin{aligned} & f(a_n\alpha_1, \cdots, a_n\alpha_n) \\ = & \sum_{k_1 + \cdots + k_n} a_{k_1 \cdots k_n} p_1^{k_1}(a_n\alpha_1, \cdots, a_n\alpha_n) \cdots p_n^{k_n}(a_n\alpha_1, \cdots, a_n\alpha_n) \end{aligned}$$

Now the given polynomial is of the form

$$a_n \prod_{j=1}^{n} (x - \alpha_j)$$

and so the coefficient of x^{n-k} is $p_k(\alpha_1, \cdots, \alpha_n)\, a_n = a_{n-k}$. Also

$$p_k(a_n\alpha_1, \cdots, a_n\alpha_n) = a_n^k p_k(\alpha_1, \cdots, \alpha_n) = a_n^k \frac{a_{n-k}}{a_n}$$

It follows

$$f(a_n\alpha_1, \cdots, a_n\alpha_n) = \sum_{k_1 + \cdots + k_n} a_{k_1 \cdots k_n} \left(a_n^1 \frac{a_{n-1}}{a_n}\right)^{k_1} \left(a_n^2 \frac{a_{n-2}}{a_n}\right)^{k_2} \cdots \left(a_n^n \frac{a_0}{a_n}\right)^{k_n}$$

which is an integer. To see the last claim follows from this, take the symmetric polynomial in $\alpha_1, \cdots, \alpha_n$ and multiply by the product of the denominators of the rational coefficients to get one which has integer coefficients. Then by the first part, each homogeneous term is just an integer divided by a_n raised to some power. This proves the theorem.

D.2 The Fundamental Theorem Of Algebra

This is devoted to a mostly algebraic proof of the fundamental theorem of algebra. It depends on the interesting results about symmetric polynomials which are presented above. I found it on the Wikipedia article about the fundamental theorem of algebra. You google "fundamental theorem of algebra" and go to the Wikipedia article. It gives several other proofs in addition to this one. According to this article, the first completely correct proof of this major theorem is due to Argand in 1806. Gauss and others did it earlier but their arguments had gaps in them.

You can't completely escape analysis when you prove this theorem. The necessary analysis is in the following lemma.

Lemma D.2.1 *Suppose* $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *where* n *is odd and the coefficients are real. Then* $p(x)$ *has a real root.*

Proof: This follows from the intermediate value theorem from calculus.

Next is an algebraic consideration. First recall some notation.

$$\prod_{i=1}^{m} a_i \equiv a_1 a_2 \cdots a_m$$

Recall a polynomial in $\{z_1, \cdots, z_n\}$ is symmetric only if it can be written as a sum of elementary symmetric polynomials raised to various powers multiplied by constants. This follows from Proposition D.1.4.

The following is the main part of the theorem. In fact this is one version of the fundamental theorem of algebra which people studied earlier in the 1700's.

Lemma D.2.2 *Let* $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *be a polynomial with real coefficients. Then it has a complex root.*

Proof: It is possible to write

$$n = 2^k m$$

where m is odd. If n is odd, $k = 0$. If n is even, keep dividing by 2 until you are left with an odd number. If $k = 0$ so that n is odd, it follows from Lemma D.2.1 that $p(x)$ has a real, hence complex root. The proof will be by induction on k, the case $k = 0$ being done. Suppose then that it works for $n = 2^l m$ where m is odd and $l \leq k-1$ and let $n = 2^k m$ where m is odd. Let $\{z_1, \cdots, z_n\}$ be the roots of the polynomial in the splitting field, the existence of this field being given by the above proposition. Then

$$p(x) = \prod_{j=1}^{n} (x - z_j) = \sum_{k=0}^{n} (-1)^k p_k x^k \tag{4.6}$$

where p_k is the k^{th} elementary symmetric polynomial. Note this shows

$$a_{n-k} = p_k (-1)^k . \tag{4.7}$$

There is another polynomial which has coefficients which are sums of real numbers times the p_k raised to various powers and it is

$$q_t(x) \equiv \prod_{1 \leq i < j \leq n} (x - (z_i + z_j + t z_i z_j)), \ t \in \mathbb{R}$$

I need to verify this is really the case for $q_t(x)$. When you switch any two of the z_i in $q_t(x)$ the polynomial does not change. For example, let $n=3$ when $q_t(x)$ is

$$(x-(z_1+z_2+tz_1z_2))(x-(z_1+z_3+tz_1z_3))(x-(z_2+z_3+tz_2z_3))$$

and you can observe the assertion about the polynomial is true when you switch two different z_i. Thus the coefficients of $q_t(x)$ must be symmetric polynomials in the z_i with real coefficients. Hence by Proposition D.1.4 these coefficients are real polynomials in terms of the elementary symmetric polynomials p_k. Thus by 4.7 the coefficients of $q_t(x)$ are real polynomials in terms of the a_k of the original polynomial. Recall these were all real. It follows, and this is what was wanted, that $q_t(x)$ has all real coefficients.

Note that the degree of $q_t(x)$ is $\begin{pmatrix} n \\ 2 \end{pmatrix}$ because there are this number of ways to pick $i<j$ out of $\{1,\cdots,n\}$. Now

$$\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2} = 2^{k-1}m\left(2^k m-1\right)$$

$$= 2^{k-1}\,(\text{odd})$$

and so by induction, for each $t\in\mathbb{R}, q_t(x)$ has a complex root.

There must exist $s\neq t$ such that for a single pair of indices i,j, with $i<j$,

$$(z_i+z_j+tz_iz_j),(z_i+z_j+sz_iz_j)$$

are both complex. Here is why. Let $A(i,j)$ denote those $t\in\mathbb{R}$ such that $(z_i+z_j+tz_iz_j)$ is complex. It was just shown that every $t\in\mathbb{R}$ must be in some $A(i,j)$. There are infinitely many $t\in\mathbb{R}$ and so some $A(i,j)$ contains two of them.

Now for that t,s,

$$\begin{aligned} z_i+z_j+tz_iz_j &= a \\ z_i+z_j+sz_iz_j &= b \end{aligned}$$

where $t\neq s$ and so by Cramer's rule,

$$z_i+z_j=\frac{\begin{vmatrix} a & t \\ b & s \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & s \end{vmatrix}}\in\mathbb{C}$$

and also

$$z_iz_j=\frac{\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & s \end{vmatrix}}\in\mathbb{C}$$

At this point, note that z_i, z_j are both solutions to the equation

$$x^2-(z_1+z_2)x+z_1z_2=0,$$

which from the above has complex coefficients. By the quadratic formula the z_i, z_j are both complex. Thus the original polynomial has a complex root. This proves the lemma.

With this lemma, it is easy to prove the fundamental theorem of algebra. The difference between the lemma and this theorem is that in the theorem, the coefficients are only assumed

to be complex. What this means is that if you have any polynomial with complex coefficients it has a complex root and so it is not irreducible. Hence the field extension is the same field. Another way to say this is that for **every** complex polynomial there exists a factorization into linear factors or in other words the splitting field for a complex polynomial is the field of complex numbers.

Theorem D.2.3 *Let $p(x) \equiv a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be any complex polynomial, $n \geq 1, a_n \neq 0$. Then it has a complex root. Furthermore, there exist complex numbers $z_1, \cdots, z_n$ such that*

$$p(x) = a_n \prod_{k=1}^{n} (x - z_k)$$

Proof: First suppose $a_n = 1$. Consider the polynomial

$$q(x) \equiv p(x)\,\overline{p(\overline{x})}$$

this is a polynomial and it has real coefficients. This is because it equals

$$\left(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) \cdot \left(x^n + \overline{a_{n-1}}x^{n-1} + \cdots + \overline{a_1}x + \overline{a_0}\right)$$

The x^{j+k} term of the above product is of the form

$$a_k x^k \overline{a_j} x^j + \overline{a_k} x^k a_j x^j = x^{k+j}\left(a_k \overline{a_j} + \overline{a_k} a_j\right)$$

and

$$a_k \overline{a_j} + \overline{a_k} a_j = a_k \overline{a_j} + \overline{a_k \overline{a_j}}$$

so it is of the form of a complex number added to its conjugate. Hence $q(x)$ has real coefficients as claimed. Therefore, by by Lemma D.2.2 it has a complex root z. Hence either $p(z) = 0$ or $p(\overline{z}) = 0$. Thus $p(x)$ has a complex root.

Next suppose $a_n \neq 0$. Then simply divide by it and get a polynomial in which $a_n = 1$. Denote this modified polynomial as $q(x)$. Then by what was just shown and the Euclidean algorithm, there exists $z_1 \in \mathbb{C}$ such that

$$q(x) = (x - z_1)\, q_1(x)$$

where $q_1(x)$ has complex coefficients. Now do the same thing for $q_1(x)$ to obtain

$$q(x) = (x - z_1)(x - z_2)\, q_2(x)$$

and continue this way. Thus

$$\frac{p(x)}{a_n} = \prod_{j=1}^{n} (x - z_j)$$

and this proves the theorem.

Obviously this is a harder proof than the other proof of the fundamental theorem of algebra presented earlier. However, this is a better proof. Consider the algebraic numbers $\mathbb{A}$ consisting of the real numbers which are roots of some polynomial having rational coefficients. By Theorem 8.3.28 they are a field. Now consider the field $\mathbb{A} + i\mathbb{A}$ with the usual conventions for complex arithmetic. You could repeat the above argument with small changes and conclude that every polynomial having coefficients in $\mathbb{A} + i\mathbb{A}$ has a root in $\mathbb{A} + i\mathbb{A}$. Recall from Problem 49 on Page 225 that $\mathbb{A}$ is countable and so this is also the case for $\mathbb{A} + i\mathbb{A}$. Thus this gives an algebraically complete field which is countable and so very different than $\mathbb{C}$. Of course there are other situations in which the above harder proof will work and yield interesting results.

D.3 Transcendental Numbers

Most numbers are like this. Here the algebraic numbers are those which are roots of a polynomial equation having rational numbers as coefficients. By the fundamental theorem of calculus, all these numbers are in $\mathbb{C}$. There are only countably many of these algebraic numbers, (Problem 49 on Page 225). Therefore, most numbers are transcendental. Nevertheless, it is very hard to prove that this or that number is transcendental. Probably the most famous theorem about this is the Lindemann Weierstrass theorem.

Theorem D.3.1 *Let the α_i be distinct nonzero algebraic numbers and let the a_i be nonzero algebraic numbers. Then*

$$\sum_{i=1}^{n} a_i e^{a_i} \neq 0$$

I am following the interesting Wikepedia article on this subject.You can also look at the book by Baker [3], Transcendental Number Theory, Cambridge University Press. There are also many other treatments which you can find on the web including an interesting article by Steinberg and Redheffer which appeared in about 1950.

The proof makes use of the following identity. For $f(x)$ a polynomial,

$$I(s) \equiv \int_0^s e^{s-x} f(x)\, dx = e^s \sum_{j=0}^{\deg(f)} f^{(j)}(0) - \sum_{j=0}^{\deg(f)} f^{(j)}(s). \tag{4.8}$$

where $f^{(j)}$ denotes the j^{th} derivative. In this formula, $s \in \mathbb{C}$ and the integral is defined in the natural way as

$$\int_0^1 s f(ts)\, e^{s-ts} dt \tag{4.9}$$

The identity follows from integration by parts.

$$\begin{aligned}
\int_0^1 s f(ts)\, e^{s-ts} dt &= s e^s \int_0^1 f(ts)\, e^{-ts} dt \\
&= s e^s \left[-\frac{e^{-ts}}{s} f(ts) \,|_0^1 + \int_0^1 \frac{e^{-ts}}{s} s f'(st)\, dt \right] \\
&= s e^s \left[\frac{1}{s} f(s) - \frac{e^{-s}}{s} f(0) + \int_0^1 e^{-ts} f'(st)\, dt \right] \\
&= f(0) - e^s f(s) + \int_0^1 s e^{s-ts} f'(st)\, dt \\
&\equiv f(0) - f(s)\, e^s + \int_0^s e^{s-x} f'(x)\, dx
\end{aligned}$$

Continuing this way establishes the identity.

Lemma D.3.2 *If K and c are nonzero integers, and $\beta_1, \cdots, \beta_m$ are the roots of a single polynomial with integer coefficients,*

$$Q(x) = v x^m + \cdots + u$$

where $v, u \neq 0$, then

$$K + c\left(e^{\beta_1} + \cdots + e^{\beta_m}\right) \neq 0.$$

Letting

$$f(x) = \frac{v^{(m-1)p} Q^p(x)\, x^{p-1}}{(p-1)!}$$

and $I(s)$ be defined in terms of $f(x)$ as above, it follows,

$$\lim_{p\to\infty} \sum_{i=1}^{m} I(\beta_i) = 0$$

and

$$\sum_{j=0}^{n} f^{(j)}(0) = v^{p(m-1)} u^p + m_1(p)\, p$$

$$\sum_{i=1}^{m}\sum_{j=0}^{n} f^{(j)}(\beta_i) = m_2(p)\, p$$

where $m_i(p)$ is some integer.

Proof: Let p be a prime number. Then consider the polynomial $f(x)$ of degree $n \equiv pm + p - 1$,

$$f(x) = \frac{v^{(m-1)p} Q^p(x)\, x^{p-1}}{(p-1)!}$$

From 4.8

$$c\sum_{i=1}^{m} I(\beta_i) = c\sum_{i=1}^{m}\left(e^{\beta_i}\sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(\beta_i)\right)$$

$$= \left(K + c\sum_{i=1}^{m} e^{\beta_i}\right)\sum_{j=0}^{n} f^{(j)}(0) - K\sum_{j=0}^{n} f^{(j)}(0) - c\sum_{i=1}^{m}\sum_{j=0}^{n} f^{(j)}(\beta_i) \tag{4.10}$$

Claim 1: $\lim_{p\to\infty} c\sum_{i=1}^{m} I(\beta_i) = 0$.

Proof: This follows right away from the definition of $I\left(\beta_j\right)$ and the definition of $f(x)$.

$$\left|I\left(\beta_j\right)\right| \le \int_0^1 \left|\beta_j f\left(t\beta_j\right) e^{\beta_j - t\beta_j}\right| dt$$

$$\le \int_0^1 \left|\frac{|v|^{(m-1)p}\left|Q\left(t\beta_j\right)\right|^p t^{p-1}\left|\beta_j\right|^{p-1}}{(p-1)!} dt\right|$$

which clearly converges to 0. This proves the claim.

The next thing to consider is the term on the end in 4.10,

$$K\sum_{j=0}^{n} f^{(j)}(0) + c\sum_{i=1}^{m}\sum_{j=0}^{n} f^{(j)}(\beta_i) \tag{4.11}$$

The idea is to show that for large enough p it is always an integer. When this is done, it can't happen that $K + c\sum_{i=1}^{m} e^{\beta_i} = 0$ because if this were so, you would have a very small number equal to an integer. Now

$$\begin{aligned} f(x) &= \frac{v^{(m-1)p}\left(\overbrace{v(x-\beta_1)(x-\beta_2)\cdots(x-\beta_m)}^{Q(x)}\right)^p x^{p-1}}{(p-1)!} \\ &= \frac{v^{mp}\left((x-\beta_1)(x-\beta_2)\cdots(x-\beta_m)\right)^p x^{p-1}}{(p-1)!} \end{aligned} \tag{4.12}$$

It follows that for $j < p-1$, $f^{(j)}(0) = 0$. This is because of that term x^{p-1}. If $j \geq p$, $f^{(j)}(0)$ is an integer multiple of p. Here is why. The terms in this derivative which are nonzero involve taking $p-1$ derivatives of x^{p-1} and this introduces a $(p-1)!$ which cancels out the denominator. Then there are some other derivatives of the product of the $(x-\beta_i)$ raised to the power p. By the chain rule, these all involve a multiple of p. Thus this j^{th} derivative is of the form

$$pg(x, v\beta_1, \cdots, v\beta_m), \tag{4.13}$$

where $g(x, v\beta_1, \cdots, v\beta_m)$ is a polynomial in x with coefficients which are symmetric polynomials in $\{v\beta_1, \cdots, v\beta_m\}$ having integer coefficients. Then derivatives of g with respect to x also yield polynomials in x which have coefficients which are symmetric polynomials in $\{v\beta_1, \cdots, v\beta_m\}$ having integer coefficients. Evaluating g at $x = 0$ must therefore yield a polynomial which is symmetric in the $\{v\beta_1, \cdots, v\beta_m\}$ with integer coefficients. Since the $\{\beta_1, \cdots, \beta_m\}$ are the roots of a polynomial having integer coefficients with leading coefficient v, it follows from Theorem D.1.7 that this last polynomial is an integer and so the j^{th} derivative of f given by 4.13 when evaluated at $x = 0$ yields an integer times p. Now consider the case of the $(p-1)$ derivative of f. The only nonzero term of $f^{(j)}(0)$ is the one which comes from taking $p-1$ derivatives of x^{p-1} and so it reduces to

$$v^{mp}(-1)^{mp}(\beta_1\beta_2\cdots\beta_m)^p$$

Now $Q(0) = v(-1)^m(\beta_1\beta_2\cdots\beta_m) = u$ and so $v^p(-1)^{mp}(\beta_1\beta_2\cdots\beta_m)^p = u^p$ which yields

$$f^{(p-1)}(0) = v^{mp}u^pv^{-p} = v^{p(m-1)}u^p$$

Note this is not necessarily a multiple of p and in fact will not be so if $p > u, v$ because p is a prime number. It follows

$$\sum_{j=0}^{n} f^{(j)}(0) = v^{p(m-1)}u^p + m(p)\,p$$

where $m(p)$ is some integer.

Now consider the other sum in 4.11,

$$c\sum_{i=1}^{m}\sum_{j=0}^{n} f^{(j)}(\beta_i)$$

Using the formula in 4.12 it follows that for $j < p$, $f^{(j)}(\beta_i) = 0$. This is because for such derivatives, each term will have that product of the $(x-\beta_i)$ in it. Next consider the case where $j \geq p$. In this case, the nonzero terms must involve at least p derivatives of the expression

$$((x-\beta_1)(x-\beta_2)\cdots(x-\beta_m))^p$$

since otherwise, when evaluated at any β_k the result would be 0. Hence the $(p-1)!$ will vanish from the denominator and so all coefficients of the polynomials in the β_j and x will be integers and in fact, there will be an extra factor of p left over. Thus the j^{th} derivatives for $j \geq p$ involve taking the k^{th} derivative, $k \geq 0$ with respect to x of

$$pv^{mp}g(x, \beta_1, \cdots, \beta_m)$$

where $g(x, \beta_1, \cdots, \beta_m)$ is a polynomial in x having coefficients which are integers times symmetric polynomials in the $\{\beta_1, \cdots, \beta_m\}$. It follows that the k^{th} derivative for $k \geq 0$

is also a polynomial in x having the same properties. Therefore, taking the k^{th} derivative where k corresponds to $j \geq p$ and adding, yields

$$\sum_{i=1}^{m} pv^{mp} g_{,k}\left(\beta_i, \beta_1, \cdots, \beta_m\right) = \sum_{i=1}^{m} f^{(j)}\left(\beta_i\right) \tag{4.14}$$

where $g_{,k}$ denotes the k^{th} derivative of g taken with respect to x. Now

$$\sum_{i=1}^{m} g_{,k}\left(\beta_i, \beta_1, \cdots, \beta_m\right)$$

is a symmetric polynomial in the $\{\beta_1, \cdots, \beta_m\}$ with no term having degree more than mp and[1] so by Corollary D.1.6 this is of the form

$$\sum_{i=1}^{m} g_{,k}\left(\beta_i, \beta_1, \cdots, \beta_m\right) = \sum_{k_1, \cdots, k_m} a_{k_1 \cdots k_m} p_1^{k_1} \cdots p_m^{k_m}$$

where the $a_{k_1 \cdots k_m}$ are integers and the p_k are the elementary symmetric polynomials in $\{\beta_1, \cdots, \beta_m\}$. Recall these were roots of $vx^m + \cdots + u$ and so from the definition of the elementary symmetric polynomials given in Definition D.1.2, these p_k are each an integer divided by v, the integers being the coefficients of $Q(x)$. Therefore, from 4.14

$$\begin{aligned} \sum_{i=1}^{m} f^{(j)}\left(\beta_i\right) &= pv^{mp} \sum_{i=1}^{m} g_{,k}\left(\beta_i, \beta_1, \cdots, \beta_m\right) \\ &= pv^{mp} \sum_{k_1, \cdots, k_m} a_{k_1 \cdots k_m} p_1^{k_1} \cdots p_m^{k_m} \end{aligned}$$

which is pv^{mp} times an expression which consists of integers times products of coefficients of $Q(x)$ divided by v raised to various powers, the sum of which is always no more than mp. Therefore, it reduces to an integer multiple of p and so the same is true of

$$c \sum_{i=1}^{m} \sum_{j=0}^{n} f^{(j)}\left(\beta_i\right)$$

which just involves adding up these integer multiples of p. Therefore, 4.11 is of the form

$$Kv^{p(m-1)} u^p + M(p)\, p$$

for some integer $M(p)$. Summarizing, it follows

$$c \sum_{i=1}^{m} I\left(\beta_i\right) = \left(K + c \sum_{i=1}^{m} e^{\beta_i}\right) \sum_{j=0}^{n} f^{(j)}(0) + Kv^{p(m-1)} u^p + M(p)\, p$$

where the left side is very small whenever p is large enough. Let p be larger than $\max(K, v, u)$. Since p is prime, it follows it cannot divide $Kv^{p(m-1)} u^p$ and so the last two terms must sum to a nonzero integer and so the equation cannot hold unless

$$K + c \sum_{i=1}^{m} e^{\beta_i} \neq 0$$

[1] Note the claim about this being a symmetric polynomial is about the sum, not an individual term.

This proves the lemma.

Note this shows π is irrational. If $\pi = k/m$ where k, m are integers, then both $i\pi$ and $-i\pi$ are roots of the polynomial with integer coefficients,

$$m^2x^2 + k^2$$

which would require from what was just shown that

$$0 \neq 2 + e^{i\pi} + e^{-i\pi}$$

which is not the case since the sum on the right equals 0.

The following corollary follows from this.

Corollary D.3.3 *Let K and c_i for $i = 1, \cdots, n$ be nonzero integers. For each k between 1 and n let $\{\beta(k)_i\}_{i=1}^{m(k)}$ be the roots of a polynomial with integer coefficients,*

$$Q_k(x) \equiv v_k x^{m_k} + \cdots + u_k$$

where $v_k, u_k \neq 0$. Then

$$K + c_1\left(\sum_{j=1}^{m_1} e^{\beta(1)_j}\right) + c_2\left(\sum_{j=1}^{m_2} e^{\beta(2)_j}\right) + \cdots + c_n\left(\sum_{j=1}^{m_n} e^{\beta(n)_j}\right) \neq 0.$$

Proof: Defining $f_k(x)$ and $I_k(s)$ as in Lemma D.3.2, it follows from Lemma D.3.2 that for each $k = 1, \cdots, n$,

$$\begin{aligned} c_k \sum_{i=1}^{m_k} I_k(\beta(k)_i) &= \left(K_k + c_k \sum_{i=1}^{m_k} e^{\beta(k)_i}\right) \sum_{j=0}^{\deg(f_k)} f_k^{(j)}(0) \\ &\quad -K_k \sum_{j=0}^{\deg(f_k)} f_k^{(j)}(0) - c_k \sum_{i=1}^{m_k} \sum_{j=0}^{\deg(f_k)} f_k^{(j)}(\beta(k)_i) \end{aligned}$$

This is exactly the same computation as in the beginning of that lemma except one adds and subtracts $K_k \sum_{j=0}^{\deg(f_k)} f_k^{(j)}(0)$ rather than $K \sum_{j=0}^{\deg(f_k)} f_k^{(j)}(0)$ where the K_k are chosen such that their sum equals K. By Lemma D.3.2,

$$\begin{aligned} c_k \sum_{i=1}^{m_k} I_k(\beta(k)_i) &= \left(K_k + c_k \sum_{i=1}^{m_k} e^{\beta(k)_i}\right)\left(v_k^{(m_k-1)p} u_k^p + N_k p\right) \\ &\quad -K_k\left(v_k^{(m_k-1)p} u_k^p + N_k p\right) - c_k N_k' p \end{aligned}$$

and so

$$\begin{aligned} c_k \sum_{i=1}^{m_k} I_k(\beta(k)_i) &= \left(K_k + c_k \sum_{i=1}^{m_k} e^{\beta(k)_i}\right)\left(v_k^{(m_k-1)p} u_k^p + N_k p\right) \\ &\quad -K_k v_k^{(m_k-1)p} u_k^p + M_k p \end{aligned}$$

for some integer M_k. By multiplying each $Q_k(x)$ by a suitable constant, it can be assumed without loss of generality that all the $v_k^{m_k-1}u_k$ are equal to a constant integer U. Then the above equals

$$c_k \sum_{i=1}^{m_k} I_k(\beta(k)_i) = \left(K_k + c_k \sum_{i=1}^{m_k} e^{\beta(k)_i}\right)(U^p + N_k p)$$

$$-K_k U^p + M_k p$$

Adding these for all k gives

$$\sum_{k=1}^{n} c_k \sum_{i=1}^{m_k} I_k \left(\beta\left(k\right)_i\right) = U^p \left(K + \sum_{k=1}^{n} c_k \sum_{i=1}^{m_k} e^{\beta(k)_i} \right) - KU^p + Mp$$

$$+ \sum_{k=1}^{n} N_k p \left(K_k + c_k \sum_{i=1}^{m_k} e^{\beta(k)_i} \right) \tag{4.15}$$

For large p it follows from Lemma D.3.2 that the left side is very small. If

$$K + \sum_{k=1}^{n} c_k \sum_{i=1}^{m_k} e^{\beta(k)_i} = 0$$

then $\sum_{k=1}^{n} c_k \sum_{i=1}^{m_k} e^{\beta(k)_i}$ is an integer and so the last term in 4.15 is an integer times p. Thus for large p it reduces to

$$\text{small number} = -KU^p + Ip$$

where I is an integer. Picking prime $p > \max\left(U, K\right)$ it follows $-KU^p + Ip$ is a nonzero integer and this contradicts the left side being a small number less than 1 in absolute value. This proves the corollary.

Next is an even more interesting Lemma which follows from the above corollary.

Lemma D.3.4 *If $b_0, b_1, \cdots, b_n$ are non zero integers, and $\gamma_1, \cdots, \gamma_n$ are distinct algebraic numbers, then*

$$b_0 e^{\gamma_0} + b_1 e^{\gamma_1} + \cdots + b_n e^{\gamma_n} \neq 0$$

Proof: Assume

$$b_0 e^{\gamma_0} + b_1 e^{\gamma_1} + \cdots + b_n e^{\gamma_n} = 0 \tag{4.16}$$

Divide by e^{γ_0} and letting $K = b_0$,

$$K + b_1 e^{\alpha(1)} + \cdots + b_n e^{\alpha(n)} = 0 \tag{4.17}$$

where $\alpha\left(k\right) = \gamma_k - \gamma_0$. These are still distinct algebraic numbers none of which is 0 thanks to Theorem 8.3.28. Therefore, $\alpha\left(k\right)$ is a root of a polynomial

$$v_k x^{m_k} + \cdots + u_k \tag{4.18}$$

having integer coefficients, $v_k, u_k \neq 0$. Recall algebraic numbers were defined as roots of polynomial equations having rational coefficients. Just multiply by the denominators to get one with integer coefficients. Let the roots of this polynomial equation be

$$\left\{\alpha\left(k\right)_1, \cdots, \alpha\left(k\right)_{m_k}\right\}$$

and suppose they are listed in such a way that $\alpha\left(k\right)_1 = \alpha\left(k\right)$. Letting i_k be an integer in $\{1, \cdots, m_k\}$ it follows from the assumption 4.16 that

$$\prod_{\substack{\left(i_1, \cdots, i_n\right) \\ i_k \in \{1, \cdots, m_k\}}} \left(K + b_1 e^{\alpha(1)_{i_1}} + b_2 e^{\alpha(2)_{i_2}} + \cdots + b_n e^{\alpha(n)_{i_n}} \right) = 0 \tag{4.19}$$

This is because one of the factors is the one occuring in 4.17 when $i_k = 1$ for every k. The product is taken over all distinct ordered lists $(i_1, \cdots, i_n)$ where i_k is as indicated. Expand this possibly huge product. This will yield something like the following.

$$K' + c_1 \left(e^{\beta(1)_1} + \cdots + e^{\beta(1)_{\mu(1)}} \right) + c_2 \left(e^{\beta(2)_1} + \cdots + e^{\beta(2)_{\mu(2)}} \right) + \cdots +$$

$$c_N \left(e^{\beta(N)_1} + \cdots + e^{\beta(N)_{\mu(N)}} \right) = 0 \tag{4.20}$$

These integers c_j come from products of the b_i and K. The $\beta(i)_j$ are the distinct exponents which result. Note that a typical term in this product 4.19 would be something like

$$\overbrace{K^{p+1} b_{k_1} \cdots b_{k_{n-p}}}^{\text{integer}} e^{\overbrace{\alpha(k_1)_{i_1} + \alpha(k_2)_{i_2} \cdots + \alpha(k_{n-p})_{i_{n-p}}}^{\beta(j)_r}}$$

the k_j possibly not distinct and each $i_k \in \{1, \cdots, m_{i_k}\}$. Other terms of this sort are

$$K^{p+1} b_{k_1} \cdots b_{k_{n-p}} e^{\alpha(k_1)_{i'_1} + \alpha(k_2)_{i'_2} \cdots + \alpha(k_{n-p})_{i'_{n-p}}},$$
$$K^{p+1} b_{k_1} \cdots b_{k_{n-p}} e^{\alpha(k_1)_1 + \alpha(k_2)_1 \cdots + \alpha(k_{n-p})_1}$$

where each i'_k is another index in $\{1, \cdots, m_{i_k}\}$ and so forth. A given j in the sum of 4.20 corresponds to such a choice of $\left\{b_{k_1}, \cdots, b_{k_{n-p}}\right\}$ which leads to $K^{p+1} b_{k_1} \cdots b_{k_{n-p}}$ times a sum of exponentials like those just described. Since the product in 4.19 is taken over all choices $i_k \in \{1, \cdots, m_k\}$, it follows that if you switch $\alpha(r)_i$ and $\alpha(r)_j$, two of the roots of the polynomial

$$v_r x^{m_r} + \cdots + u_r$$

mentioned above, the result in 4.20 would be the same except for permuting the

$$\beta(s)_1, \beta(s)_2, \cdots, \beta(s)_{\mu(s)}.$$

Thus a symmetric polynomial in

$$\beta(s)_1, \beta(s)_2, \cdots, \beta(s)_{\mu(s)}$$

is also a symmetric polynomial in the $\alpha(k)_1, \alpha(k)_2, \cdots, \alpha(k)_{m_k}$ for each k. Thus for a given $r, \beta(r)_1, \cdots, \beta(r)_{\mu(r)}$ are roots of the polynomial

$$(x - \beta(r)_1)(x - \beta(r)_2) \cdots \left(x - \beta(r)_{\mu(r)}\right)$$

whose coefficients are symmetric polynomials in the $\beta(r)_j$ which is a symmetric polynomial in the $\alpha(k)_j, j = 1, \cdots, m_k$ for each k. Letting g be one of these symmetric polynomials and writing it in terms of the $\alpha(k)_i$ you would have

$$\sum_{l_1, \cdots, l_n} A_{l_1 \cdots l_n} \alpha(n)_1^{l_1} \alpha(n)_2^{l_2} \cdots \alpha(n)_{m_n}^{l_n}$$

where $A_{l_1 \cdots l_n}$ is a symmetric polynomial in $\alpha(k)_j, j = 1, \cdots, m_k$ for each $k \leq n-1$. These coefficients are in the field (Proposition 8.3.27) $\mathbb{Q}[A(1), \cdots, A(n-1)]$ where $A(k)$ denotes

$$\left\{\alpha(k)_1, \cdots, \alpha(k)_{m_k}\right\}$$

and so from Proposition D.1.4, the above symmetric polynomial is of the form

$$\sum_{(k_1 \cdots k_{m_n})} B_{k_1 \cdots k_{m_n}} p_1^{k_1} \left(\alpha(n)_1, \cdots, \alpha(n)_{m_n}\right) \cdots p_{m_n}^{k_{m_n}} \left(\alpha(n)_1, \cdots, \alpha(n)_{m_n}\right)$$

where $B_{k_1\cdots k_{m_n}}$ is a symmetric polynomial in $\alpha(k)_j, j=1,\cdots,m_k$ for each $k \leq n-1$. Now do for each $B_{k_1\cdots k_{m_n}}$ what was just done for g featuring this time

$$\left\{\alpha(n-1)_1,\cdots,\alpha(n-1)_{m_{n-1}}\right\}$$

and continuing this way, it must be the case that eventually you have a sum of integer multiples of products of elementary symmetric polynomials in $\alpha(k)_j, j=1,\cdots,m_k$ for each $k \leq n$. By Theorem D.1.7, these are each rational numbers. Therefore, each such g is a rational number and so the $\beta(r)_j$ are algebraic. Now 4.20 contradicts Corollary D.3.3. This proves the lemma.

Note this lemma is sufficient to prove Lindeman's theorem that π is transcendental. Here is why. If π is algebraic, then so is $i\pi$ and so from this lemma, $e^0+e^{i\pi}\neq 0$ but this is not the case because $e^{i\pi}=-1$.

The next theorem is the main result, the Lindemann Weierstrass theorem.

Theorem D.3.5 *Suppose $a(1),\cdots,a(n)$ are nonzero algebraic numbers and suppose*

$$\alpha(1),\cdots,\alpha(n)$$

are distinct algebraic numbers. Then

$$a(1)e^{\alpha(1)}+a(2)e^{\alpha(2)}+\cdots+a(n)e^{\alpha(n)}\neq 0$$

Proof: Suppose $a(j)\equiv a(j)_1$ is a root of the polynomial

$$v_j x^{m_j}+\cdots+u_j$$

where $v_j,u_j\neq 0$. Let the roots of this polynomial be $a(j)_1,\cdots,a(j)_{m_j}$. Suppose to the contrary that

$$a(1)_1 e^{\alpha(1)}+a(2)_1 e^{\alpha(2)}+\cdots+a(n)_1 e^{\alpha(n)}=0$$

Then consider the big product

$$\prod_{\substack{(i_1,\cdots,i_n)\\ i_k\in\{1,\cdots,m_k\}}}\left(a(1)_{i_1}e^{\alpha(1)}+a(2)_{i_2}e^{\alpha(2)}+\cdots+a(n)_{i_n}e^{\alpha(n)}\right) \tag{4.21}$$

the product taken over all ordered lists $(i_1,\cdots,i_n)$. This product equals

$$0=b_1e^{\beta(1)}+b_2e^{\beta(2)}+\cdots+b_Ne^{\beta(N)} \tag{4.22}$$

where the $\beta(j)$ are the distinct exponents which result. The $\beta(i)$ are clearly algebraic because they are the sum of the $\alpha(i)$. Since the product in 4.21 is taken for all ordered lists as described above, it follows that for a given k,if $\alpha(k)_i$ is switched with $\alpha(k)_j$, that is, two of the roots of $v_kx^{m_k}+\cdots+u_k$ are switched, then the product is unchanged and so 4.22 is also unchanged. Thus each b_k is a symmetric polynomial in the $a(k)_j, j=1,\cdots,m_k$ for each k. It follows

$$b_k=\sum_{(j_1,\cdots,j_{m_n})}A_{j_1,\cdots,j_{m_n}}a(n)_1^{j_1}\cdots a(n)_{m_n}^{j_{m_n}}$$

and this is symmetric in the $\left\{a(n)_1,\cdots,a(n)_{m_n}\right\}$ the coefficients $A_{j_1,\cdots,j_{m_n}}$ being in the field (Proposition 8.3.27) $\mathbb{Q}[A(1),\cdots,A(n-1)]$ where $A(k)$ denotes

$$a(k)_1,\cdots,a(k)_{m_k}$$

and so from Proposition D.1.4,

$$b_k = \sum_{(j_1,\cdots,j_{m_n})} B_{j_1,\cdots,j_{m_n}} p_1^{j_1}\left(a(n)_1 \cdots a(n)_{m_n}\right)\cdots p_{m_n}^{j_{m_n}}\left(a(n)_1 \cdots a(n)_{m_n}\right)$$

where the $B_{j_1,\cdots,j_{m_n}}$ are symmetric in $\left\{a(k)_j\right\}_{j=1}^{m_k}$ for each $k \leq n-1$. Now doing to $B_{j_1,\cdots,j_{m_n}}$ what was just done to b_k and continuing this way, it follows b_k is a finite sum of integers times elementary polynomials in the various $\left\{a(k)_j\right\}_{j=1}^{m_k}$ for $k \leq n$. By Theorem D.1.7 this is a rational number. Thus b_k is a rational number. Multiplying by the product of all the denominators, it follows there exist integers c_i such that

$$0 = c_1 e^{\beta(1)} + c_2 e^{\beta(2)} + \cdots + c_N e^{\beta(N)}$$

which contradicts Lemma D.3.4. This proves the theorem.

This theorem is sufficient to show e is transcendental. If it were algebraic, then

$$ee^{-1} + (-1)\,e^0 \neq 0$$

but this is not the case. If $a \neq 1$ is algebraic, then $\ln(a)$ is transcendental. To see this, note that

$$1e^{\ln(a)} + (-1)\,ae^0 = 0$$

which cannot happen according to the above theorem. If a is algebraic and $\sin(a) \neq 0$, then $\sin(a)$ is transcendental because

$$\frac{1}{2i}e^{ia} - \frac{1}{2i}e^{-ia} + (-1)\sin(a)\,e^0 = 0$$

which cannot occur if $\sin(a)$ is algebraic. There are doubtless other examples of numbers which are transcendental by this amazing theorem.

Bibliography

[1] **Apostol T.,** *Calculus Volume II Second edition,* Wiley *1969.*

[2] **Baker, Roger**, *Linear Algebra*, Rinton Press 2001.

[3] **Baker, A.** *Transcendental Number Theory*, Cambridge University Press 1975.

[4] **Chahal J.S.,** *Historical Perspective of Mathematics 2000 B.C. - 2000 A.D. Kendrick Press, Inc. (2007)*

[5] **Coddington and Levinson**, *Theory of Ordinary Differential Equations* McGraw Hill 1955.

[6] **Davis H. and Snider A.,** *Vector Analysis* Wm. C. Brown 1995.

[7] **Edwards C.H.,** *Advanced Calculus of several Variables,* Dover 1994.

[8] **Friedberg S. Insel A. and Spence L.**, *Linear Algebra*, Prentice Hall, 2003.

[9] **Gurtin M.,** *An introduction to continuum mechanics,* Academic press 1981.

[10] **Hardy G.,** *A Course Of Pure Mathematics, Tenth edition,* Cambridge University Press 1992.

[11] **Horn R. and Johnson C.,** *matrix Analysis,* Cambridge University Press, 1985.

[12] **Jacobsen N.** *Basic Algebra* Freeman 1974.

[13] **Karlin S. and Taylor H.,** *A First Course in Stochastic Processes,* Academic Press, 1975.

[14] **Nobel B. and Daniel J.,** *Applied Linear Algebra, Prentice Hall, 1977.*

[15] **E. J. Putzer,** American Mathematical Monthly, Vol. 73 (1966), pp. 2-7.

[16] **Rudin W.,** *Principles of Mathematical Analysis*, McGraw Hill, 1976.

[17] **Rudin W.,** *Functional Analysis, McGraw Hill, 1991.*

[18] **Salas S. and Hille E.,** *Calculus One and Several Variables,* Wiley 1990.

[19] **Strang Gilbert**, *Linear Algebra and its Applications,* Harcourt Brace Jovanovich 1980.

[20] **Wilkinson, J.H.,** *The Algebraic Eigenvalue Problem, Clarendon Press Oxford 1965.*

Index

A invariant, 253
Abel's formula, 101, 265
adjugate, 78, 91
almost linear, 447
almost linear system, 448
Archimedian property, 18
assymptotically stable, 447
augmented matrix, 24
autonomous, 447

basic feasible solution, 138
basic variables, 138
basis, 204
block matrix, 96
block multiplication, 96
bounded linear transformations, 350

Cauchy Schwarz inequality, 296, 347
Cauchy sequence, 310, 347
Cayley Hamilton theorem, 95, 280
centrifugal acceleration, 62
centripetal acceleration, 62
characteristic and minimal polynomial, 245
characteristic equation, 159
characteristic polynomial, 95, 243
characteristic value, 159
cofactor, 74, 88
column rank, 92, 111
companion matrix, 271, 397
complete, 371
completeness axiom, 16
complex conjugate, 12
complex numbers, 12
composition of linear transformations, 236
condition number, 358
conformable, 39
conjugate linear, 301
convex combination, 246
convex hull, 246
Coordinates, 27
Coriolis acceleration, 62
Coriolis acceleration
    earth, 64
Coriolis force, 62
Courant Fischer theorem, 323
Cramer's rule, 79, 91

damped vibration, 444
defective, 165
dense, 19
density of rationals, 18
determinant, 84
    product, 87
    product of eigenvalules, 196
    transpose, 85
diagonalizable, 172, 174, 233
differentiable matrix, 59
differential equations
    first order systems, 199
digraph, 40
dimension of vector space, 206
direct sum, 71, 250
discrete Fourier transform, 344
Dolittle's method, 124
dot product, 29
dyadics, 228
dynamical system, 175

eigenspace, 161
eigenvalue, 72, 159
eigenvalues, 95, 192, 243
eigenvector, 72, 159
elementary matrices, 105
equality of mixed partial derivatives, 187
equilibrium point, 447
equivalence class, 213, 232
equivalence of norms, 350
equivalence relation, 231
Euclidean algorithm, 19
exchange theorem, 54

field
    splitting, 215
field axioms, 10
field extensions, 215

Foucalt pendulum, 65
Fourier series, 309
Fredholm alternative, 118, 306
free variable, 25
Frobinius norm, 343
fundamental matrix, 377, 440
fundamental theorem of algebra, 457
fundamental theorem of arithmetic, 22

gambler's ruin, 290
Gauss Jordan method for inverses, 46
Gauss Seidel method, 367
Gelfand, 360
generalized eigenspace, 72
Gerschgorin's theorem, 190
Gram Schmidt procedure, 135, 176, 298
Gramm Schmidt process, 176, 298
greatest common divisor, 19, 209
greatest lower bound, 16
Gronwall's inequality, 379, 438

Hermitian, 181
    positive definite, 326
Hermitian matrix
    positive part, 430
Hilbert space, 321
Holder's inequality, 353
homogeneous
    polynomial, 459

idempotent, 69
inconsistent, 25
initial value problem
    existence, 377, 433
    linear system, 435
    local solutions, 436
    uniqueness, 379, 433
inner product, 29, 295
inner product space, 295
integers mod a prime, 224
integral
    operator valued function, 378
    vector valued function, 378
invariant, 318
inverses and determinants, 77, 90
invertible, 45
irreducible, 209

Jocobi method, 365
Jordan block, 255, 260

ker, 115

Laplace expansion, 74, 88
least squares, 121, 305
least upper bound, 16
Lindemann Weierstrass theorem, 474
linear combination, 53, 87
linear transformation, 227
linearly dependent, 53
linearly independent, 53, 204
Lipschitz condition, 433

main diagonal, 75
Markov chain, 287, 288
Markov matrix, 283
    steady state, 283
mathematical induction, 17
matrix, 33
    inverse, 45
    left inverse, 91
    lower triangular, 75, 91
    non defective, 181
    normal, 181
    right inverse, 91
    self adjoint, 174
    symmetric, 172
    upper triangular, 75, 91
matrix of linear transformation, 229
migration matrix, 287
minimal polynomial, 71, 243, 252
    finding it, 264
minor, 74, 88
mixed partial derivatives, 186
monic, 209
monomial, 459
Moore Penrose inverse, 339
moving coordinate system, 59
    acceleration , 62

normal, 332
null and rank, 310
nullity, 115

operator norm, 350
orthogonal matrix, 72, 81, 82, 130, 179
orthonormal polynomials, 307

permutation matrices, 105
perp, 117
Perron's theorem, 420
pivot column, 114
polar decomposition
    left, 332
    right, 330

polar form complex number, 13
polynomial, 208, 459
prime number, 19
prime numbers
    infinity of primes, 224
principle directions, 167
product rule
    matrices, 59
projection map
    convex set, 310
    subspace, 309
Putzer's method, 441

QR algorithm, 195, 402
    convergence, 404
    nonconvergence, 196
QR factorization, 132
quotient space, 225

random variables, 287
rank, 112
rank of a matrix, 92, 111
rank one transformation, 303
real numbers, 9
real Schur form, 179
regression line, 305
regular Sturm Liouville problem, 308
relatively prime, 19
Riesz representation theorem, 300
right Cauchy Green strain tensor, 330
ring, 462
row equivalent, 114
row operations, 24, 105
row rank, 92, 111
row reduced echelon form, 113

scalar product, 29
scalars, 15, 27, 33
Schur's theorem, 318
second derivative test, 190
self adjoint, 181
similar matrices, 80, 101, 231
similarity transformation, 231
simplex tableau, 140
simultaneous corrections, 365
simultaneously diagonalizable, 316
singular value decomposition, 335
singular values, 335
skew symmetric, 44, 171
slack variables, 138, 140
span, 53, 87
spectral mapping theorem, 429
spectral norm, 352
spectral radius, 358, 360
spectrum, 159
stable, 447
stable manifold, 455
stationary transition probabilities, 288
Stochastic matrix, 288
subspace, 53, 204
Sylvester, 70, 249
symmetric, 44, 171
    polynomial, 459

tensor product, 303
trace, 184
    sum of eigenvalues, 196
triangle inequality, 30
trivial, 53

Vandermonde determinant, 102
variation of constants formula, 201, 443
variational inequality, 310
vector space, 34, 203

well ordered, 17
Wronskian, 101, 265, 443
Wronskian alternative, 200, 443